# MOTION OF A CHARGED PARTICLE IN A STRAIGHT HELICAL-CUBIC MAGNETIC FIELD

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A solution is obtained for the equations of motion of a particle in a straight helical-cubic magnetic field. The method of solution can be applied to the study of an arbitrary helical nonlinear field.

THE motion of a particle in a helical nonlinear field, in which the direction of the field gradient (which is equal to zero along the axis of the motion) rotates about the z axis with some pitch $l$ possesses a number of interesting features. We consider here a helical field with cubic nonlinearity, a so-called four-cusp straight field. Such a field possesses better focusing properties than a triple field, and can evidently be employed in accelerator technology. It is easy to carry over the method of solution of the equations given below to the case of a helical field of another type.

We shall first seek a particular solution of the equations of motion which contain two free parameters—the amplitude and the phase of the fundammental. This solution in represented in the form of a Fourier series with frequencies dependent on the amplitudes. All the frequencies and amplitudes can be expressed by recurrence relations in terms of the complex amplitude $a_1$ of the fundamental; such a series converges well if $a_1$ is small enough.

Such a particular solution can be taken as the zeroth approximation for finding the general solution by means of the usual perturbation theory, with linearization in terms of small departures from the particular solution.

The equations of motion in the four-cusp and, in general, n-cusp-straight field are well known from stellarator theory. For compactness of presentation, we introduce them again, starting out from the assumptions (r = horizontal and z = vertical displacement from the axis of motion)

$$r/l \ll 1, \quad z/l \ll 1. \tag{1}$$

We introduce the "helical" coordinates u, v, s:

$$u = r\cos\alpha x + z\sin\alpha x,\ v = -r\sin\alpha x + z\cos\alpha x,$$
$$s = x, \qquad \alpha = 2\pi/l. \tag{2}$$

By finding the metric tensor

$$g_{\alpha\beta} = \frac{\partial x_\gamma}{\partial x'_\alpha}\frac{\partial x_\gamma}{\partial x'_\beta} \quad (x'_1 = u, \quad x'_2 = v, \quad x'_3 = s),$$

we obtain the Lagrangian of the free particle:

$$\mathcal{L} = -mc^2 [1 - c^{-2}\{\dot u^2 + \dot v^2 + [\alpha^2(u^2+v^2)$$
$$+ 1]\dot s^2 - 2\alpha v\dot u\dot s + 2\alpha u\dot v\dot s\}]^{1/2}. \tag{3}$$

The determinant $|g_{ik}| = -1$ $(i = 1, 2, 3, 4)$; therefore, the equations of the field in the helical system are written in the form

$$\partial F_{ik}/\partial x'^l + \partial F_{li}/\partial x'^k + \partial F_{kl}/\partial x'^i = 0, \quad \partial F^{ik}/\partial x'^k = 0. \tag{4}$$

It is not difficult to prove that, with accuracy up to cubic terms (discarding terms of the order $\alpha u \ll 1$, $\alpha v \ll 1$ relative to the fundamental), we have

$$F^{23} = F_{23} = H_u = {}^1/_6\,(\partial^3 H_u/\partial v^3)_0\,(v^3 - 3vu^2),$$
$$F^{13} = F_{13} = H_v = -{}^1/_6\,(\partial^3 H_u/\partial v^3)_0\,(u^3 - 3uv^2). \tag{5}$$

In other words, the field can be regarded as strictly cubic in each plane x = const, if the length of the helix period is much larger than the transverse displacements of the particle. In this approximation, we get equations with constant coefficients as is customary, by neglecting the terms $(\dot u/\dot s)^2$, $(\dot v/\dot s)^2$, $\ddot s/\dot s^2$:

$$u'' - 2\alpha v' - \alpha^2 v + \gamma(u^3 - 3uv^2) = 0, \tag{6}$$

$$v'' + 2\alpha u' - \alpha^2 v + \gamma(v^3 - 3vu^2) = 0;$$
$$u' = du/ds, \quad \gamma = {}^1/_6\,(\partial^3 H_u/\partial v^3)_0/HR. \tag{7}$$

We introduce the complex variable $\varphi = u + iv$, for which

$$\varphi'' + 2i\alpha\varphi' - \alpha^2\varphi + \gamma\varphi^{*3} = 0. \tag{8}$$

If we introduce

$$f = r + iz = e^{i\alpha x}\varphi = e^{i\alpha x}\ (u + iv), \tag{9}$$

then the equation takes the form

$$f'' + \gamma e^{4i\alpha x} f^{*3} = 0. \tag{10}$$

Whatever might be the character of the stability, the stable solution for r and z can have only an oscillatory character with frequencies dependent on the amplitude of the oscillations. Such a particular solution of (8), with two arbitrary parameters, has the form of an expansion in a Fourier series:

$$\varphi_0 = \sum_k a_{2k+1}(\nu) \exp[(-i)^{2k+1}(2k+1)(1+\nu)\alpha x], \quad (11)$$

where it is convenient to regard $\nu \ll 1$ as a given number, and the amplitudes $a_{2k+1}(\nu)$ are functions of $\nu$, in which $a_{2k+1} \to 0$ as $\nu \to 0$.

The amplitudes are found from the recurrence relations:

$$\alpha^2\nu^2 a_1 = \gamma(3a_1^{*2}a_3^* + 3a_3^{*2}a_5^* + 6a_1^* a_5^* a_7^* + \ldots),$$
$$\alpha^2(4+3\nu)^2 a_3 = \gamma(a_1^{*3} + 6a_1^* a_3^* a_5^* + 3a_5^{*2}a_7^* + \ldots),$$
$$\alpha^2(4+5\nu)^2 a_5 = \gamma(3a_1^* a_3^{*2} + 3a_1^{*2}a_7^* + 6a_3^* a_5^* a_7^* + \ldots),$$
$$\alpha^2(8+7\nu)^2 a_7 = \gamma(3a_1^{*2}a_5^* + 3a_3^* a_5^{*2} + \ldots),$$
$$9(2k+1)^2\alpha^2(1+\nu)^2 a_{3(2k+1)} \approx \gamma a_{2k+1}^{*3}, \quad k \gg 1. \quad (12)$$

We then find

$$|a_1|^4 = \frac{\alpha^4\nu^2(4+3\nu)^2}{3\gamma^2}\left[1 - \frac{21\gamma^4|a_1|^8}{\alpha^8(4+3\nu)^6(4+5\nu)^2} + \ldots\right], \quad (13)$$

$$a_3 = \frac{\gamma a_1^{*3}}{\alpha^2(4+3\nu)^2}\left[1 + \frac{18\gamma^4|a_1|^8}{\alpha^8(4+3\nu)^6(4+5\nu)^2} + \ldots\right], \quad (14)$$

$$a_5 = \frac{3\gamma^3|a_1|^2 a_1^5}{\alpha^6(4+3\nu)^4(4+5\nu)^2} + \ldots \quad (15)$$

etc. The phase of $a_1$ is arbitrary. In accord with (13) and (9),

$$\nu \approx \pm {}^1/_4\sqrt{3}\,\gamma|a_1|^2/\alpha^2; \quad (16)$$

$$r_0 + iz_0 \approx a_1 \exp(\pm {}^1/_4\sqrt{3}\,\gamma|a_1|^2 x/\alpha^2),$$
$$r_0^2 + z_0^2 \approx |a_1|^2, \; (r_0')^2 + (z_0')^2 \approx {}^3/_{16}\gamma|a_1|^6/\alpha^2. \quad (17)$$

It will become clear below that the general solution for $\varphi$ contains either $\nu > 0$ or $\nu < 0$; superposition of these two fundamental frequencies is not possible. It is seen from the recurrence formulas that all the series are satisfactorily convergent, regardless of the sign of $\nu$ for $\gamma|a_1|^2/\alpha^2 \lesssim 1$.

The general solution of Eq. (8) can be sought by means of perturbation theory:

$$\varphi = \varphi_0 + \Delta\varphi, \qquad \Delta\varphi \ll \varphi_0. \quad (18)$$

This solution can be represented in the form

$$\varphi = a_1 e^{-i\alpha(1+\nu)x} + a_3 e^{3i\alpha(1+\nu)x} + \ldots + \eta\,(b_{11}e^{-i\alpha(1+\mu)x}$$
$$+ c_{11}e^{-i\alpha(1+2\nu-\mu)x}) + \eta\,(b_{31}e^{i\alpha(3+2\nu+\mu)x} + c_{31}e^{i\alpha(3+4\nu-\mu)x}$$
$$+ \ldots + \eta^2(b_{12}e^{-i\alpha(1-\nu+2\mu)x} + c_{12}e^{-i\alpha(1+3\nu-2\mu)x})$$
$$+ \eta^2(b_{32}e^{i\alpha(3+\nu+2\mu)x} + c_{32}e^{i\alpha(3+5\nu-2\mu)x}) + \ldots, \quad (19)$$

where $\eta \ll 1$ is the smallness parameter, which is of the order of $|\Delta\varphi/\varphi_0|$; the expansion in terms of it is carried out in a series of all amplitudes in (19) (among these are the fundamental amplitudes $a_1$, $a_3$, etc.). All the $b_{1i}$ and $c_{1i}$ are of the order of $a_1$; all the $b_{3i}$ and $c_{3i}$ are of the order of $a_3$, etc; $\varphi$ has four independent parameters in the two complex amplitudes $a_1$ and $b_{11}$. All the remaining amplitudes can be expressed in terms of these two.

Substituting (19) in (8), we get in the first approximation in $\eta$ equations that connect $b_{11}$, $c_{11}$, $b_{31}$, and $c_{31}$:

$$-\mu^2 b_{11} + 6\gamma a_1^* a_3^* c_{11}^* + 3\gamma a_1^{*2} b_{31}^* = 0,$$
$$-(2\nu-\mu)^2 c_{11} + 6\gamma a_1^* a_3^* b_{11}^* + 3\gamma a_1^{*2} b_{32}^* = 0;$$
$$b_{31} = 3\gamma a_1^{*2} b_{11}^*/(4+2\nu+\mu)^2; \quad b_{32} = 3\gamma a_1^{*2} c_{11}^*/(4+4\nu-\mu)^2. \quad (20)$$

This gives an equation for $\mu$:

$$\left[-\mu^2 + \frac{9\gamma^2|a_1|^4}{(4+2\nu+\mu)^2}\right]\left[-(2\nu-\mu)^2 + \frac{9\gamma^2|a_1|^4}{(4+4\nu-\mu)^2}\right]$$
$$-\frac{36\gamma^4|a_1|^8}{(4+3\nu)^4} = 0. \quad (21)$$

If $\mu_1$ is a root, then $\mu_2 = (2\nu - \mu_1)$ is also a root; both enter in (19). Except for $\mu_1 \neq \mu_2$ we have a multiple root $\mu = \nu$ which does not affect the general form of (19).[1)]

Denoting $\mu = k\gamma|a_1|^2/\alpha^2$, $\nu \approx {}^1/_4\sqrt{3}\,\gamma|a_1|^2/\alpha^2$, we get an approximate equation for $k$ for the case $\nu > 0$:

$$k^4 - \sqrt{3}k^3 - \tfrac{3}{8}k^2 + \tfrac{9}{16}\sqrt{3}\,k - \tfrac{63}{256} = 0. \quad (22)$$

Here $k_1^+ = 1.659$, $k_2^+ = (\sqrt{3}/2) - k_1^+ = 0.791$. For $\nu \approx {}^1/_4\sqrt{3}\,\gamma|a_1|^2/\alpha^2$, we get other values of $k_1^-$.

The series (19) is so constructed that the roots $\mu_1$ and $\mu_2$ are contained in it simultaneously.

Thus there are two independent groups of solutions (19), corresponding to stable r and z oscillations. These groups can be classified according to the frequencies:

$$(\nu, \mu_1^+), \quad (-\nu, \mu_1^-).$$

Evidently there are no other types of solutions, in any case with small amplitudes and with $\mu$, $\nu \ll 1$. Attempts at construction of solutions with $\nu$ or $\mu \sim 1$ (for example, $\nu \approx -{}^4/_3, -{}^4/_5, -{}^8/_7, \ldots$) lead to contradictions if $\gamma|a_1|^2/\alpha^2 \ll 1$.

Stable small oscillations are close to motion along the helix (17). For $\gamma|a_1|^2/\alpha^2 \lesssim 1$ and $|\Delta\varphi| \lesssim {}^1/_2|\varphi_0|$, the total phase volume of the transverse oscillations is

$$\Omega = \Delta r\,\Delta z\,\Delta\vartheta_R\,\Delta\vartheta_z \lesssim \alpha^2|a_1|^4\ \text{cm}^2\,\text{sr} \quad (23)$$

If we denote by B the distance from the equilibrium orbit to that point where the field is

[1)]The author is grateful to V. C. Zakharov, who corrected an error and who pointed out the existence of the multiple root $\mu = \nu$.

doubled:

$$\tfrac{1}{6}\,(\partial^3 H_u/\partial v^3)_0\, B^3/H_0 = 1, \quad \gamma = 1/RB^3, \tag{24}$$

then, for $\gamma|a_1|^2/\alpha^2 \sim 1$ ($\nu \sim 1/2$) we have $\alpha^2 \sim |a_1|^2/RB^2$. Therefore, assuming $|a_1| \lesssim B$, $\alpha \sim 1/\sqrt{RB}$, $l = 2\pi/\sqrt{RB}$, we have

$$\Omega < |a_1|^6/RB^3 \leqslant B^3/R \text{ cm}^2\text{ sr}. \tag{25}$$

We note that if the period of rotation of the octupole lens producing the cubic field, is $l$, then the real period of the cubic nonlinearity will obviously be $l/4$ [this is the very reason why the frequency 4a appears in (10)]. The quantity $l/4$ must be compared with the length of the period in linear alternating focusing, which (for $\nu \sim 1/4$) is of the order of $\pi\sqrt{Ra}$, where $a = H/(\partial H/\partial R)$.

For a given value of the energy of transverse oscillations

$$\mathscr{H} = |\varphi'|^2 - \alpha^2|\varphi|^2 + \frac{\gamma}{4}(\varphi^4 + \varphi^{*4}), \tag{26}$$

in addition to stable small oscillations, an asymptotic increase of amplitude is possible (beyond the hump of the dynamic potential well). This growth takes place according to the law

$$\varphi = Ae^{i\psi}, \quad \psi = \frac{\pi - \beta}{4};$$

$$\beta = \frac{4\sqrt{2}\,\alpha}{3\sqrt{\gamma}}\frac{1}{A} + \frac{64\sqrt{2}}{81}\frac{\alpha^3}{\gamma\sqrt{\gamma}}\frac{1}{A^3} + \ldots, \tag{27}$$

$$A' = \sqrt{\gamma}A^2 + \left(\frac{\mathscr{H}}{\sqrt{2\gamma}} - \frac{32\sqrt{2}}{27}\frac{\alpha^4}{\gamma\sqrt{\gamma}}\right)\frac{1}{A^2} + \ldots, \quad A \to \infty. \tag{28}$$

The amplitude goes to infinity after a finite time, as is always the case in nonlinear problems.

The analysis given here does not take into account many real effects, which can appear in an accelerator with a helical field; such as toroidal character, centripetal force, or perturbation of the magnetic field. If these effects are small, then the resultant solution can be used as a zeroth approximation.

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