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TYPICAL STRUCTURE OF A TOROIDAL MAGNETIC FIELD

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The forms of the elementary structures of a toroidal magnetic field are obtained and the problem of the structure of the field as a whole is discussed.

If the lines of force of a magnetic field occupy a certain toroidal region V_i, do not have any singularities within it, and do not leave it, then the magnetic field containing such a region is said to be toroidal. The field can consist of only the region V_i; such is, for example, the field within an arbitrarily deformed toroidal perfectly-conducting tube over the surface of which $H_n = 0$. However, the region V_i may represent only a part of a more general field (the field can also contain several toroidal regions). Then we shall refer to the boundary of the region considered above as the separating surface, and to the region V_i as the internal region of a toroidal magnetic field.

For the time being we shall suppose that the separating surface is a closed toroidal surface without gaps. Among toroidal fields we should also include "straight" fields periodic along the z axis. It is evident that a single period L of such a field is topologically equivalent to a torus.

We have shown earlier [1] that if a straight field has a symmetry, in the most general case a screw symmetry, then the equations for the lines of force have an integral—an equation for the magnetic surfaces:

$$\Psi = A_z(r,\theta) + \alpha r A_\varphi(r,\theta) = \text{const}. \qquad (1)$$

Here A_z and A_φ are components of the vector potential, and $\theta = \varphi - \alpha z$, $\alpha = 2\pi/L$, where L is the period of the field.

We assume that among the magnetic surfaces of such a field there are some magnetic surfaces whose diameter is equal to zero. Such a degenerate surface is a helical line of force which can be found from the following conditions

$$\partial\Psi/\partial r = 0, \qquad \partial\Psi/\partial\theta = 0, \qquad (2a)$$

$$(\partial^2\Psi/\partial r^2)(\partial^2\Psi/\partial\theta^2) - (\partial^2\Psi/\partial r\partial\theta)^2 > 0. \qquad (2b)$$

Such a line which closes on itself at the end of a single period is called a magnetic axis. The number of such magnetic axes in a magnetic field can have quite different values. Surfaces which separate regions with different magnetic axes have ridges.

If we take the simplest irrotational so-called triple ($l = 3$) magnetic field specified by the scalar potential

$$\Phi = H_0 z + h_3 I_3(3\alpha r)\sin 3(\varphi - \alpha z), \qquad (3)$$

then such a field has one magnetic axis—the z axis—and three ridges of the separating surface, which are helical lines of pitch $L = 2\pi/\alpha$. The intersection of the plane z = const with the magnetic surfaces (1) of such field is shown in Fig. 1.

The lines of force lying in magnetic surfaces encircle the magnetic axis [2] in spite of the condition curl $\mathbf{H} = 0$. This encirclement can be characterized by the torsional parameter $\omega(\Psi)$ whose reciprocal $1/\omega$ is equal to the average number of periods required for an encirclement of the magnetic axis. If ω is a rational number m/n, then the magnetic surface consists of lines of force which are closed (in the topological sense) after n periods [1)]. But if a line of force does not close upon itself then it covers the surface densely everywhere.

If we violate the symmetry of the field, for example, by superimposing a perturbation having a different symmetry [2)] or bend a straight field into a torus, then the equations for the lines of force will no longer have a general integral analogous to (1), and, consequently, generally speaking there will no longer be any uniquely defined magnetic surfaces. However, if a "straight" field has a constant z-component H_0 which is much larger than the periodic part of the field $\mathbf{h}(r, \varphi; z)$, or if a field folded into a torus has a component $H_\varphi = H_0 a/r$ which is much larger than the com-

1)Exceptions to this are separating surfaces which contain only a finite number of closed lines of force to which the other lines of force in the surface tend asymptotically.

2)As an example we can consider the perturbation of a helical field (3) by an axially symmetric corrugated field.

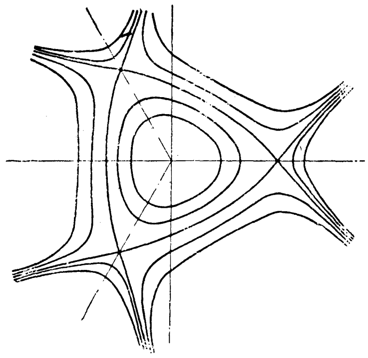

FIG. 1

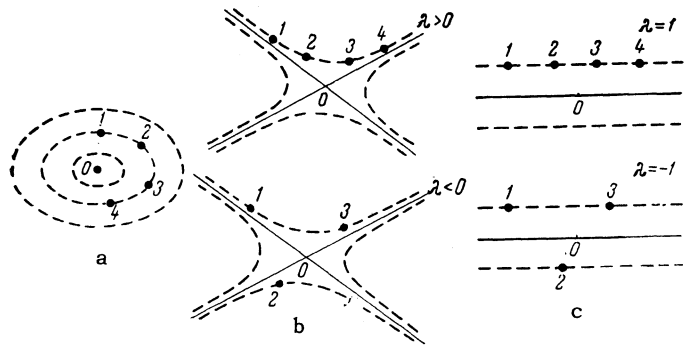

FIG. 2

ponent of $\mathbf{h}(r, z; \varphi)$ which is periodic in φ, then, as shown in [3], the equations for the lines of force averaged respectively over z or over φ do have an integral. In the second case it is of the form

$$\psi = \overline{rA_\varphi} + H_0^{-1} r^2 \overline{\tilde{h}_z \hat{h}_r} = \text{const.} \tag{4}$$

Here the bar indicates averaging over φ, while $\hat{h}_r$ denotes the variable part of the integral $\int^{\varphi} \tilde{h}_r \, d\varphi$ involving the variable part $\tilde{h}_r \equiv h_r - \bar{h}_r$.[0]

Thus, also in this case there exist magnetic surfaces, but in contrast to the former exact surfaces these are approximate or adiabatic surfaces. In actual stellarators the magnetic surfaces observed by means of electron beams belong to this class.

As the amplitude of the periodic part of the field and the degree of asymmetry both increase[3)] the structure of the field characterized by adiabatic surfaces inserted into one another must more and more clearly differ from the true structure of the field[4]. However, even in this case we can form an idea of the structure of the field.

Indeed, if ω is variable for small $\mathbf{h}$, then there exists everywhere a dense set of adiabatic surfaces on which $\omega = m/n$ is a rational number. But this means that on these surfaces there exist lines of force which close upon themselves after n turns (however, now, as we shall see later, such lines do not cover the whole surface). As $\mathbf{h}$ increases and the adiabatic surfaces are progressively destroyed the closed lines of force will be conserved as before. This follows, if lines of force do not leave V_i, from the fixed point theorem which states that for any continuous mapping of a closed singly connected region upon itself there exists at least one fixed point[5]. Moreover, if the disruption of the magnetic surfaces is sufficiently small then the closed lines will lie near those surfaces on which ω is rational. Thus, for example, it has been shown in [6,7] by means of a numerical calculation that a strongly perturbed field does indeed contain a system of closed lines of force which close upon themselves after n = 1, 12, 23 etc. periods, and that these lines of force lie close to those surfaces upon which they should lie according to the data of the method of averaging. However, the number of lines which close upon themselves once after n turns turns out to be not infinite but finite, and in particular [7] equal to two.

In the neighborhood of the line of force $r = r_0(\varphi)$, $z = z_0(\varphi)$ which closes upon itself after n turns around the torus, the structure of the field can be easily determined if the equations for the lines of force can be linearized. Indeed, upon substituting

$$z = z_0(\varphi) + \xi, \qquad r = r_0(\varphi) + \eta \tag{A}$$

into the equations for the line of force, and on restricting ourselves to terms of order ξ and η, we shall obtain a linear system with coefficients whose period is equal to $2\pi n$, where 2π is the period of the torus.

If ξ_1, η_1 are the coordinates of the representative point of the line of force, i.e., of the point of intersection of the line of force with the surface φ = const, then after one period these coordinates will be equal to

$$\begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}, \tag{B}$$

where the matrix (α_{ik}) in virtue of the conservation of flux div $\mathbf{H} = 0$ has a determinant equal to unity[4)]. The eigenvalues (λ_1, λ_2) of the matrix (α_{ik}) are either complex conjugate or real numbers satisfying the relation $\lambda_1 \lambda_2 = 1$, and therefore the following three cases are possible:

[3)]The degree of asymmetry should be interpreted as the ratio of the amplitudes of fields having different degrees of symmetry.

[4)]Since we are restricting ourselves to an investigation of the immediate neighbourhood of the closed line we can take the quantity H_z to be constant on a given line of force for $z = 0$ and $z = 2\pi$.

1) $\lambda_1, \lambda_2 = e^{\pm i\mu}$, μ — real
2) $\lambda_1 = \lambda$, $\lambda_2 = 1/\lambda$, $\lambda \gtrless 0$, λ — real (C)
3) $\lambda_1 = \lambda_2 = \pm 1$.

In the first case the representative point moves around the fixed point—the trace of the closed line of force—along ellipses (Fig. 2a), and in the second case along hyperbolas, [moreover, if $\lambda > 0$ then the representative point moves all the time along the same branch, while for $\lambda < 0$ it jumps from one branch to the other (Fig. 2b)]. Finally, in the last case for $\lambda = \pm 1$ there exists a surface which passes through the fixed point upon which all the lines of force are closed, while the representative points in its neighborhood move (either without or with jumping) along neighboring surfaces (Fig. 2c). We shall refer to points which do not jump as points of the first kind, and we shall refer to points which do jump as points of the second kind.

Evidently the general structure of the field is a peculiar addition of neighborhoods of closed lines of force. Moreover, for different n we shall, generally speaking, obtain in the representative plane different pictures which acquire increasingly complex structure as $n \to \infty$.

Fields corresponding to the degenerate case $\lambda = \pm 1$ are fields with exact magnetic surfaces. This degeneracy is, generally speaking, removed if the field becomes asymmetric. Although at present there are no detailed data on the dynamics of the disruption of asymmetric fields, nevertheless, one can suppose [7] that in the case of small asymmetry the surfaces with $\omega = 1/n$ on splitting assume the form shown in Fig. 3, i.e., around one of the lines of force there arises a filament which is joined along another line of force which in the plane z = const is represented by hyperbolic points. This explains the appearance of 2n fixed points in the layer with $\omega = 1/n$. Thus, around every "elliptical" closed magnetic line of force which is a magnetic axis of order n there appear magnetic surfaces belonging to it which are formed by lines of force encircling this n-th order axis.

As the degree of nonadiabaticity increases the rate of rotation about n-th order axes increases [7], and when it becomes comparable to the rate of rotation about O—the original magnetic axis of the first order, the structure of the field undergoes another change.

Thus, in the example given in [7] the "stable" elliptical points are converted for $x < x^{**}$ into hyperbolic points of the second kind, and, consequently, all the fixed points become hyperbolic. We have assumed above that the lines of force do not leave the region V_i. In Mel'nikov's papers [8] it was shown that when symmetric fields are perturbed the separating surface, generally speaking, acquires gaps. In this case closed lines of force may completely disappear. However, if $d\omega/d\psi \neq 0$ and the gaps are not too great, then, most probably, the picture in typical cases will remain the same, as is indicated by the already cited example [7].

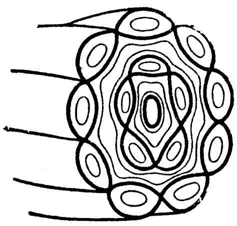

FIG. 3

Finally, when the system has still greater degree of nonadiabaticity, the role played by the gaps becomes, most probably, dominant, and the toroidal field practically disappears.

In summary, we can say that magnetic surfaces exist only as a result of degeneracy of more general field structures. In the case of fields close to symmetric and weakly nonadiabatic the field structure is filamentary of the type shown in Fig. 3. In the case of fields with a sufficiently great degree of nonadiabaticity the lines of force begin to be strongly intertwined, since all or, at any rate, most neighborhoods of closed lines of force become hyperbolic of the first or of the second kind. Finally, as the degree of nonadiabaticity increases further the region disappears completely because of the growth of gaps in the separating surface.

Naturally, in specific cases we can observe all these stages simultaneously, with the region of greatest nonadiabaticity lying in the neighborhood of the separating surface [6,7].

It should be emphasized that all the arguments given in the present paper have utilized only the equation of continuity div **H** = 0 and are not associated with the irrotational nature of the field.

It might be supposed that the observed "quantization" of plasma pinches [9], the filamentary nature of astrophysical formations of magnetic plasmas are associated with the aforementioned characteristic feature of the nondegenerate magnetic field. This assertion is supported by the fact that instabilities in toroidal plasma configurations consisting of inserted surfaces develop in the neighborhood of closed lines of force (instabilities of Shafranov-Kruskal and Suydam, convective and current-convective instability). These instabilities can be regarded as a tendency towards the removal of degeneracy which was mentioned previously.

The filamentary structure of a plasma pinch of the type shown in Fig. 3 formed in the course of this may turn out to be on the whole sufficiently stable, since the plasma pinch will consist of intertwined plasma fibers surrounding the ring axis of the toroid 0, 1, 2 etc. times. These fibers may either be at rest or in a state of some stationary motion. As a result of the work of Mercier [10] one might suppose that the separate fibers will also be sufficiently stable.

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