INTERACTING SPIN 1 FIELDS AND SYMMETRY PROPERTIES

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A deep connection is established between the conservation laws for baryon number, strangeness, isotopic spin, and also electric charge, and the space-time property of vector fields, that of having a definite spin (interacting fields with definite spin are described by our class-A theories [1]). Thus the existence of truly neutral fields with spin 1 (for example, the photon or the ω meson) brings with it invariances which correspond to the conservation of additive quantum numbers (for example, electric charge or strangeness). The existence of charged fields with spin 1 (for example, the ρ meson) leads to invariances of the type of isotopic invariance, and so on. The proof is obtained by an analysis of the general local relativistically invariant Lagrangian for an arbitrary system of any number of interacting fields with spins 1, 1/2, and 0. In theories of class A each interacting massive vector field must satisfy a Lorentz condition—a condition which singles out the spin 1 and is closely connected with the inhomogeneous Lorentz group. For vector fields with mass 0 this is replaced by the requirement that the four-divergence be arbitrary. A consequence is that the matrices formed from the coupling constants necessarily form a Lie algebra, and this leads to the indicated symmetry properties. The structure coefficients of the algebra are the constants for the interactions between the vector fields.

1. INTRODUCTION

1. For strongly interacting elementary particles there is conservation of quantities such as isotopic spin, strangeness, and baryon number. At first glance this group of conservation laws and the corresponding invariances is not connected in any way with the properties of the Minkowski space-time. At the same time other conservation laws—energy-momentum, angular momentum—are obviously connected with properties of space-time; homogeneity and isotropy.

The assertion made in the abstract, that there is a close connection between the first group of conservation laws and a space-time property of vector fields, that of having a definite spin, has an unexpected sound, and we shall try to explain it. In theories of class A[1] the equations of motion for vector fields

$$\Box b_\mu^i - \partial_\mu\partial_\nu b_\nu^i - m_i^2 b_\mu^i = -j_\mu^i \tag{1}$$

(the currents j_μ^i are certain combinations of the fields, including the fields b_μ^i) must have the consequence that the spin of each field is 1, i.e., that the alternate conditions (cf.[1-3])

$$\partial_\mu b_\mu^i = \begin{cases} 0, & \text{if } m_i^2 \neq 0, \\ \text{arbitrary}, & \text{if } m_i^2 = 0 \end{cases} \tag{2}$$

must apply. Then the currents must be conserved, $\partial_\mu j_\mu^i = 0$, and this means that the theory is invariant under certain phase transformations.

2. Precisely what invariances are possible, and what structure of interactions, is a question which we shall investigate by means of a direct analysis of the field equations in the framework of the Lagrangian formalism. The Lagrangian is written down with undetermined coefficients (coupling constants), and then owing to the requirements (2) certain algebraic relations arise which these coefficients must satisfy. These relations mean that definite matrices formed from the coupling constants must constitute a Lie algebra.

We start from the general local relativistically invariant Lagrangian for an arbitrary system of any number of interacting fields with spins 1, $^1/_2$, and 0, with conservation of the number of spinor particles. The only restriction we adopt in this work is that all coupling constants are dimensionless (in units $\hbar = c = 1$). This simply means that at this stage of the investigation we confine ourselves to the first term of the expansion of the Lagrangian with respect to the dimensions of the coupling constants. There are reasons to suppose that all important conclusions relating to invariance properties will remain unchanged even after terms

	In the papers by authors who give gauge-invariant theories this is	In the present paper this is
Existence of vector fields	derived (actually postulated, see our criticism [1])	postulated
Invariance under transformations with phases independent of χ (for example, isotopic invariance)	postulated	derived
The basic principle	gauge invariance with phases depending on χ	singling out of spin 1: $\partial\mu b\mu = \begin{cases} 0 \text{ for massive vector field} \\ \text{arbitrary for case of mass } 0 \end{cases}$
Mass of vector field	0	any value
Properties of vector fields	derived	derived
Form of the interaction	derived	derived

with the dimensions of the coupling constants are included in the Lagrangian. A natural name for interactions of class A with dimensionless coupling constants is minimal interactions.[2]

3. In this paper we prove that interactions for which the spins of the vector fields are always 1 (theories of class A) can belong only to the following cases:

a) Those in which the vector fields are neutral and interact through conserved currents, for example through currents corresponding to the conservation of strangeness or of baryon number. In such a theory there will of course be invariance under the corresponding phase transformations.

b) Those in which three vector fields form an isotopic triplet (the b meson of Yang and Mills[4]), and the entire theory is isotopically invariant. This possibility is the simplest one for the description of a charged vector field.

c) Those in which the vector fields form richer multiplets, and the interactions have higher symmetries, corresponding to the classical transformation groups: SU(ν) (ν = 3, 4, ...), O(ν) (ν = 5, 6, ...), Sp(ν/2) (ν = 4, 6, ...), and the five exceptional groups G_2, F_4, E_6, E_7, E_8 (see the review articles[5,6] and the literature cited there). Field theories in which such symmetries are postulated have been discussed by Glashow and Gell-Mann.[7]

4. Thus the conservation laws of the first group are actually connected with space-time properties: the isotopic invariance with the spin 1 of the ρ meson, and the conservation of strangeness and of baryon number with the spin 1 of two neutral vector mesons (evidently the ω meson and the recently discovered φ meson). The existence of an octuplet of vector mesons with equal masses corresponds to invariance under the group SU(3) (eightfold way), and so on.

Conversely, when there is any particular conservation law there is a place for a particle that causes it to exist, and one can raise the question of searching for such a particle. On a wider scale this applies not only to vector conservation laws, but also to others, for example laws of the conservation of the four-momentum, angular momentum, and so on.

5. The form we have obtained for the interactions of vector fields is basically the same as found by the authors of numerous papers which treat the vector fields on the basis of the so called "gauge" principle[4,7,8] (for other references see[2,9]). This resemblance is not accidental, and is due to the fact that those authors, postulating the symmetry properties (for example, isotopic invariance) from the very beginning and setting the mass of the vector fields equal to zero, further require "local" symmetry properties, which assumption is equivalent to requiring that $\partial_\mu b_\mu$ be arbitrary and singles out the spin 1.[2] Meanwhile we do not assume any symmetries—we derive them. The ways in which our approach and results are different in principle can be seen clearly from the comparison shown in the table.

6. Section 2 contains a brief discussion of the method used. In Section 3 we consider an interaction among an arbitrary number of vector fields ("self-action"). The interaction of a system of fields with spins 1, ½, and 0 is investigated in Sections 4 and 5. The last section is devoted to a discussion of the results.

2. REMARKS ON THE GENERAL APPROACH

In the investigation of the interaction of an arbitrary number of vector, spinor, and scalar fields we shall be guided by the following simple considerations, which are clear from the previously considered[3] simple example of the interaction of a neutral vector field with a spinor field:

1) The equations of motion must not give any extra restrictions on the number of degrees of freedom of the fields (beyond what are necessary), so that they must have as a consequence either that the four-divergences of the vector fields are zero, or else that they are arbitrary. Therefore in the supplementary conditions obtained by taking the four-divergence of the equations of motion of vector fields it will always be necessary and sufficient to make the terms of independent structure equal to zero. As before,[3] the necessity follows from the fact that otherwise one could not prescribe the required initial conditions in the Cauchy problem in an arbitrary way (or, in quantum theory, set up noncontradictory equal-time commutation relations).

2) The restrictions on the coupling constants can be obtained successively: first one examines the interaction between the vector fields themselves, then proceeds to study the interaction with the spinor fields, and finally to the analysis of the interactions of the vector and spinor fields with the scalar fields. Each successive stage will give new relations, without changing those obtained earlier. This can be traced most simply in the simple example considered previously[3]: in that case we could first consider the self-action of the vector field and convince ourselves that $\alpha = \beta = 0$, and then get the remaining relations by studying the interaction with the spinor field.

3. THE INTERACTIONS AMONG THE VECTOR FIELDS

1. In accordance with what we said in the preceding section, we begin with the interaction of the vector fields. Let there be n such fields b_μ^i ($i = 1, \ldots, n$). The most general Lagrangian, which describes all conceivable interactions with dimensionless coupling constants, can be written in the form

$$\mathcal{L}_1(x) = -{}^1/_4 f^i_{\mu\nu} f^i_{\mu\nu} - {}^1/_2 (m^2)_{ij} b^i_\mu b^j_\mu + \alpha_{ijk}\, \partial_\nu b^i_\mu b^j_\mu b^k_\nu$$
$$+ \beta_{ijkl} b^i_\mu b^j_\mu b^k_\nu b^l_\nu + \varepsilon_{\mu\nu\lambda\rho} (\gamma_{ijk}\, \partial_\mu b^i_\nu b^j_\lambda b^k_\rho + \delta_{ijkl} b^i_\mu b^j_\nu b^k_\lambda b^l_\rho), \tag{3}$$

where $f^i_{\mu\nu} = \partial_\mu b^i_\nu - \partial_\nu b^i_\mu$, and α_{ijk}, β_{ijkl}, γ_{ijk}, and δ_{ijkl} are real (to assure that the Lagrangian is Hermitian) numerical coefficients, the coupling constants; there is always a summation over repeated indices. It is natural to take the symmetric matrix $\|(m^2)_{ij}\|$ to be diagonal with nonnegative elements. At the same time, it is not initially assumed that this matrix is a multiple of the unit matrix; therefore the masses of the various fields can be either equal or different.[1)] The free part of the Lagrangian (3) is from the beginning written as a diagonal quadratic form, in order that in the absence of interactions each field will satisfy the usual equation for a vector field with spin 1. We also note that the terms in γ and δ allow for the possibility of parity nonconservation. From the very definitions of the nonlinear terms it follows that they have the following properties

$$\beta_{ijkl} = \beta_{jikl} = \beta_{ijlk} = \beta_{klij}, \tag{4}$$

$$\gamma_{ijk} = -\gamma_{ikj}, \tag{5}$$

δ_{ijkl} is completely antisymmetric in all indices. (6)

When these properties are taken into account we get from the Lagrangian (3) the equations of motion

$$\Box b^i_\mu - \partial_\mu\partial_\nu b^i_\nu - (m^2)_{ij} b^j_\mu - \alpha_{ijk}\partial_\nu (b^j_\mu b^k_\nu) + \alpha_{jik}\partial_\nu b^j_\mu \cdot b^k_\nu$$
$$+ \alpha_{jki}\partial_\mu b^j_\nu b^k_\nu + 4\beta_{ijkl} b^j_\mu b^k_\nu b^l_\nu + 2(\gamma_{ijk} - \gamma_{jki})\, \varepsilon_{\mu\nu\lambda\rho}\partial_\nu b^j_\lambda b^k_\rho$$
$$+ 4\delta_{ijkl}\varepsilon_{\mu\nu\lambda\rho} b^j_\nu b^k_\lambda b^l_\rho = 0 \tag{7}$$

Taking the four-divergence of this equation and eliminating $\Box b^j_\mu$ from the resulting relation by means of Eq. (7), we find

$$-(m^2)_{ij}\partial_\mu b^j_\mu$$
$$- \alpha_{ijk}(\partial_\mu\partial_\nu b^j_\mu \cdot b^k_\nu + b^j_\mu\partial_\mu\partial_\nu b^k_\nu + \partial_\mu b^j_\mu \cdot \partial_\nu b^k_\nu + \partial_\nu b^j_\mu \cdot \partial_\mu b^k_\nu)$$
$$+ \alpha_{jik}(\partial_\mu\partial_\nu b^j_\mu \cdot b^k_\nu + \partial_\nu b^j_\mu \cdot \partial_\mu b^k_\nu) + \alpha_{mni} b^n_\mu \{\partial_\mu\partial_\nu b^m_\nu + (m^2)_{mj} b^j_\mu$$
$$+ \alpha_{mjk}(\partial_\nu b^j_\mu \cdot b^k_\nu + b^j_\mu \partial_\nu b^k_\nu) - \alpha_{jmk}\partial_\nu b^j_\mu \cdot b^k_\nu - \alpha_{jkm}\partial_\mu b^j_\nu \cdot b^k_\nu$$
$$- 4\beta_{mjkl} b^j_\mu b^k_\nu b^l_\nu - 2(\gamma_{mjk} - \gamma_{jkm})\, \varepsilon_{\mu\nu\lambda\rho}\partial_\nu b^j_\lambda b^k_\rho$$
$$- 4\delta_{mjkl}\varepsilon_{\mu\nu\lambda\rho} b^j_\nu b^k_\lambda b^l_\rho\} + \alpha_{jki}\partial_\mu b^j_\nu \cdot \partial_\mu b^k_\nu$$
$$+ 4\beta_{ijkl}(\partial_\mu b^j_\mu \cdot b^k_\nu b^l_\nu + 2 b^j_\mu b^k_\nu \cdot \partial_\mu b^l_\nu) - 2\gamma_{jki}\varepsilon_{\mu\nu\lambda\rho}\partial_\nu b^j_\lambda \cdot \partial_\mu b^k_\rho$$
$$+ 12\delta_{ijkl}\varepsilon_{\mu\nu\lambda\rho}\partial_\mu b^j_\nu \cdot b^k_\lambda b^l_\rho = 0. \tag{8}$$

2. We require that the alternative condition (2) be satisfied. Just as before,[3] and as formulated in Section 2, the necessary and sufficient condition for this is that each combination of terms of the same structure be equal to zero. From this we find the properties of the coefficients α_{ijk}, β_{ijkl}, γ_{ijk}, and δ_{ijkl}.

[1)]In general it is sufficient that the mass matrix be reducible to diagonal form.

Let us first consider the terms that do not contain $\partial_\mu b^i_\mu$. There is only one term of the form $\alpha_{jki}\,\partial_\mu b^j_\nu \cdot \partial_\mu b^k_\nu$. Setting it equal to zero, we find that

$$\alpha_{jki} = -\alpha_{kji}. \tag{9}$$

Next, equating the combination

$$-\alpha_{ijk}\partial_\nu b^j_\mu \cdot \partial_\mu b^k_\nu + \alpha_{jik}\partial_\nu b^j_\mu \cdot \partial_\mu b^k_\nu$$

to zero and using Eq. (9), we get

$$\alpha_{ijk} = -\alpha_{ikj}, \tag{10}$$

and this and Eq. (9) mean that the coefficients α_{ijk} are completely antisymmetric. Equating to zero the four terms of the third degree in the field which have the structure $\partial_\nu b_\mu \cdot b_\mu b_\nu$ and using Eq. (9), we find that

$$8\beta_{ijkl} + 2\alpha_{mki}\alpha_{mlj} - \alpha_{mji}\alpha_{lkm} = 0. \tag{11}$$

Symmetrizing and antisymmetrizing this equation with respect to i and j and taking into account the symmetry property (4) of the coefficients β and the complete antisymmetry of γ which we have proved, we find

$$8\beta_{ijkl} + \alpha_{mki}\alpha_{mlj} + \alpha_{mkj}\alpha_{mli} = 0, \tag{12}$$

$$\alpha_{mij}\alpha_{klm} + \alpha_{mki}\alpha_{jlm} + \alpha_{mjk}\alpha_{ilm} = 0. \tag{13}$$

When we equate to zero the only term of fourth degree in the field b_μ, but without the factor $\epsilon_{\mu\nu\lambda\rho}$, we get the condition

$$\alpha_{mni}\beta_{mjkl} + \alpha_{mji}\beta_{mnkl} + \alpha_{mki}\beta_{mlnj} + \alpha_{mli}\beta_{mknj} = 0, \tag{14}$$

which, as can be verified, is automatically satisfied as a consequence of the relations (12), (13), (9), and (10).

Let us now consider the parity-nonconserving terms. The equating to zero of the term of the structure $\epsilon_{\mu\nu\lambda\rho}\,\partial_\nu b_\lambda \cdot \partial_\mu b_\rho$ leads to the relation

$$\gamma_{jki} = -\gamma_{kji}. \tag{15}$$

The relations (5) and (15) show that the coefficients γ_{ijk} are completely antisymmetric. As a result of this the terms in γ_{ijk} drop out of the equation of motion (7). This is indeed understandable, since when γ_{ijk} is completely antisymmetric the term in the Lagrangian that contains it is a four-divergence and consequently has no significance.

After the terms in γ_{ijk} have been dropped there remains only one term with the structure $\epsilon_{\mu\nu\lambda\rho}\,\partial_\mu b_\nu \cdot b_\lambda b_\rho$, and the requirement that it be equal to zero gives

$$\delta_{ijkl} = 0. \tag{16}$$

This removes the need to examine the other terms containing δ_{ijkl}.

Thus it has been proved that the parity-nonconserving terms are incompatible with the Lorentz condition.

It is easily verified that because of the properties (9), (10), and (12) of the coupling coefficients the terms containing $\partial_\mu b^i_\mu$ are identically zero (except the term $(m^2)_{ij}\,\partial_\mu b^j_\mu$); this is essential for the proof of sufficiency, and for zero masses, of necessity as well.

Finally, we still have to consider the term $\alpha_{mni}\, b^n_\mu (m^2)_{mj}\, b^j_\mu$. Equating it to zero, we get the last condition, which determines the choice of the masses:

$$\alpha_{mni}(m^2)_{mj} = -\alpha_{mji}(m^2)_{mn}. \tag{17}$$

3. Thus it has been proved that a theory of the interaction between vector fields with dimensionless coupling constants will be a theory of class A if and only if

1) the Lagrangian is of the form

$$\mathcal{L}(x) = -{}^1\!/_4 G^i_{\mu\nu}G^i_{\mu\nu} - {}^1\!/_2 (m^2)_{ij} b^i_\mu b^j_\mu, \tag{18}$$

where we have used the fact that β_{ijkl} is expressed in terms of α_{ijk} by Eq. (12) and have introduced the compact notation

$$G^i_{\mu\nu} = \partial_\mu b^i_\nu - \partial_\nu b^i_\mu + \alpha_{ijk} b^j_\mu b^k_\nu; \tag{19}$$

2) α_{ijk} is completely antisymmetric; (20)

3) α_{ijk} satisfies the structural relation (13), which, when we introduce matrices $(\alpha_i)_{jk} = \alpha_{ijk}$, can be written in the matrix form

$$[\alpha_i, \alpha_j] = -\alpha_{ijk}\alpha_k; \tag{21}$$

4) The masses are restricted by the condition (17), which can conveniently be written in the form

$$[\alpha_i, m^2] = 0. \tag{22}$$

4. Consequences. a) The relations (20) and (21) mean that the matrices α_i form the regular representation of a Lie algebra with the structure constants α_{ijk} (the regular representation of a Lie algebra is the representation by the matrices whose matrix elements are the structure constants of the algebra).

b) From this it is clear, and can easily be verified, that a theory of this kind is invariant under the infinitesimal transformations with the parameters ω_j:

$$b^{i\prime}_\mu = b^i_\mu + \alpha_{ijk}\omega_j b^k_\mu \tag{23}$$

(the quantity $G^i_{\mu\nu}$ transforms according to the same law).

c) Any representation breaks up into a direct product of irreducible representations, and the vector fields break up into direct sums of the corresponding multiplets. An expression of the form (23) transforms the fields contained in a particular multiplet only into each other. If the representation is irreducible, then by Schur's lemma the fact that the matrix m^2 commutes with all of the matrices α_i [Eq. (22)] means that m^2 is a multiple of the unit matrix. Therefore within each multiplet the masses of the fields are equal.

d) Parity is conserved in the interaction, and the vector fields are to be assigned quantum numbers 1^-.

4. THE INTERACTION OF FIELDS WITH SPINS 1 AND ½

1. As the next step we consider the interaction of vector fields b_μ^i with spinor fields ψ_r ($r = 1, \ldots, m$), which can be conveniently combined into a column

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_m \end{pmatrix},$$

where each ψ_r is an ordinary four-component Dirac spinor. On the assumption that the coupling constants are dimensionless and that the number of spinor particles is conserved, the most general Lagrangian is

$$\mathscr{L}_{1/2,1} = \mathscr{L}_1 - \bar{\psi}(\gamma\partial + M)\psi + i\bar{\psi}\gamma_\mu(T_j^{(1)} + \gamma_5 T_j^{(2)})\psi b_\mu^j. \quad (24)$$

In this expression $\mathscr{L}_1$ is the Lagrangian (3) of the interaction of the vector fields; M is the mass operator, which assigns to each spinor ψ_r its particular mass, and which is represented by a diagonal matrix with nonnegative elements[2]; $T_j^{(1)}$ and $T_j^{(2)}$ are Hermitian matrices made up of the coupling constants. They mix up the spinors ψ_r, but do not act on the components of individual spinors.

The equations of motion that follow from the Lagrangian (24) are

$$\text{LHS } (7) + \bar{\psi}\gamma_\mu(T_i^{(1)} + \gamma_5 T_i^{(2)})\psi = 0, \quad (25)$$

$$-(\gamma\partial + M)\psi + i\gamma_\mu(T_j^{(1)} + \gamma_5 T_j^{(2)})\psi b_\mu^j = 0. \quad (26)$$

Here and in what follows LHS means "the left-hand member of the equation."

Now we: 1) take the four-divergence of Eq. (25), 2) eliminate $\Box b_\mu^i$ from the resulting equation by means of Eq. (25), and finally 3) get rid of the derivatives of the spinor field by means of Eq. (26). The result is

$$\text{LHS } (8) - i\alpha_{kni} b_\mu^n \bar{\psi}\gamma_\mu (T_k^{(1)} + \gamma_5 T_k^{(2)})\psi - i\bar{\psi}[T_i^{(1)}, M]\psi + i\bar{\psi}\gamma_5[T_i^{(2)}, M]_+\psi - \bar{\psi}\gamma_\mu \times [T_i^{(1)} + \gamma_5 T_i^{(2)}, T_j^{(1)} + \gamma_5 T_j^{(2)}]\psi b_\mu^j = 0 \quad (27)$$

([,] denotes the commutator, and $[\,,]_+$ the anticommutator).

2. In order for the vector fields to have only the spin 1 (theory of class A),[1] they must satisfy the condition (2). A necessary and sufficient condition for this is that in the expression (27) the sums of terms of the same structure be identically equal to zero (cf.[3]).

It is not hard to see that there are no changes in the relations for the self-action constants of the vector fields which we found in the preceding section by an analysis of the LHS of Eq. (8). Analyzing the new terms, we get

$$[T_i^{(1)}, M] = 0, \quad (28)$$

$$[T_i^{(2)}, M]_+ = 0, \quad (29)$$

$$[T_i^{(1)} + \gamma_5 T_i^{(2)}, T_j^{(1)} + \gamma_5 T_j^{(2)}] = i\alpha_{ijk}(T_k^{(1)} + \gamma_5 T_k^{(2)}). \quad (30)$$

The relations (28)—(30) exhaust the restrictions on the matrices T and M which follow from the requirement (2). Equation (30) means that the matrices $T_k^{(1)} + \gamma_5 T_k^{(2)}$ form a representation of the Lie algebra with the structure coefficients α_{ijk}. Owing to this the theory is invariant under the group of transformations with infinitesimal transformations of the form

$$b_\mu^{i'} = b_\mu^i + \alpha_{ijk}\omega_j b_\mu^k, \qquad \psi' = \psi - i\omega_j(T_j^{(1)} + \gamma_5 T_j^{(2)})\psi \quad (31)$$

(the ω_j are infinitesimal parameters).

3. We shall now show that in theories of class A there are no direct interactions between spinor fields without mass and spinor fields with mass. In fact, if both kinds of fields are present, the mass matrix can be represented in block form:

$$M = \begin{pmatrix} M' & 0 \\ 0 & 0 \end{pmatrix}, \quad (32)$$

where the matrix M′ can be reduced to diagonal form with positive, non-zero, elements. Putting the matrices $T^{(1)}$ and $T^{(2)}$ in the same block form and analyzing their commutation relations with the mass matrix M, Eqs. (28) and (29), we easily see that they must be of the forms

$$T_i^{(1)} = \begin{pmatrix} A_i^{(1)} & 0 \\ 0 & B_i^{(1)} \end{pmatrix}; \quad T_i^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & B_i^{(2)} \end{pmatrix}. \quad (33)$$

This means that the spinor fields with mass do not interact directly with the spinor fields without mass.

[2]Generally speaking it is sufficient that the matrix M be reducible to diagonal form. The masses of all spinor fields can be taken to be nonnegative, since this can always be achieved by making replacements $\psi_r \to \gamma_5\psi_r$.

These two kinds of fields can be treated separately:

a) for the fields with zero mass ($M = 0$) the relations (30) and the transformation law (31) are valid in their most general forms;

b) if the masses of all spinor fields are different from zero, then when we use the fact that the matrix M can be diagonalized it follows from Eq. (29) that

$$T_i^{(2)} = 0, \tag{34}$$

and then the relations (30) and the law of transformation for ψ take simpler forms:

$$[T_i^{(1)}, T_j^{(1)}] = i\alpha_{ijk} T_k^{(1)}, \tag{35}$$

$$\psi' = \psi - i\omega_j T_j^{(1)}\psi. \tag{36}$$

The conclusions drawn in points a) and b) can be seen directly from Eq. (33).

4. The system of fields ψ can be broken up into irreducible multiplets which transform according to irreducible representations of the corresponding groups. Within each irreducible multiplet the masses of the fields ψ_r are necessarily equal. In the case of nonzero masses this is a consequence of the fact that the matrix M commutes with all of the matrices of the irreducible representation $T_m^{(1)}$ (Schur's lemma).

5. THE INTERACTION OF FIELDS WITH SPINS 0, ½, AND 1

Finally, let us include interactions with fields with spin 0—scalar fields $\varphi^a (a = 1, \ldots, l)$. Under the conditions that the coupling constants are dimensionless and the number of spinor particles is conserved the most complete local Lagrangian, taking in all possible interactions of fields with spins 0, ½, and 1 is of the form

$$\mathscr{L}_{0,\,1/2,\,1} = \mathscr{L}_{1/2,\,1} - {}^1/_2\partial_\mu\varphi^a\cdot\partial_\mu\varphi^a - {}^1/_2\varphi^a(\mu^2)_{ab}\varphi^b + \xi_{abcd}\varphi^a\varphi^b\varphi^c\varphi^d + \eta^i_{ab}\varphi^a\partial_\mu\varphi^b\cdot b^i_\mu + \zeta^{ij}_{ab}\varphi^a\varphi^b b^i_\mu b^j_\mu + \bar\psi\,(G_a^{(1)} + i\gamma_5 G_a^{(2)})\,\psi\varphi^a, \tag{37}$$

where $\mathscr{L}_{1/2,\,1}$ is the Lagrangian for the fields with spins 1 and ½, Eq. (24), and $||(\mu^2)_{ab}||$ is the matrix of the squares of the masses of the scalar fields. We can regard it as diagonal with nonnegative elements (or as reducible to this form).

The coupling coefficients have the following obvious properties:

$$\left.\begin{array}{c}\xi_{abcd}\text{ is completely symmetric in all its indices}\\ \zeta^{ij}_{ab} = \zeta^{ij}_{ba} = \zeta^{ji}_{ab},\\ G_a^{(1)},\ G_a^{(2)}\text{— are Hermitian matrices}\end{array}\right\}. \tag{38}$$

The equations of motion for the fields b^i_μ and ψ are modified in the following ways,

$$\text{LHS}\,(25) + \eta^i_{ab}\varphi^a\partial_\mu\varphi^b + 2\zeta^{ij}_{ab}\varphi^a\varphi^b b^j_\mu = 0, \tag{39}$$

$$\text{LHS}\ (26) + \{G_a^{(1)} + i\gamma_5 G_a^{(2)}\}\,\psi\varphi^a = 0, \tag{40}$$

and the equations for the fields φ^a are

$$\Box\,\varphi^a - (\mu^2)_{ab}\varphi^b + 4\xi_{abcd}\varphi^b\varphi^c\varphi^d + \eta^i_{ab}\partial_\mu\varphi^b\cdot b^i_\mu - \eta^i_{ba}\partial_\mu(\varphi^b b^i_\mu) + 2\zeta^{ij}_{ab}\varphi^b b^i_\mu b^j_\mu + \bar\psi\,(G_a^{(1)} + i\gamma_5 G_a^{(2)})\,\psi = 0. \tag{41}$$

Differentiation of the equation (39) (followed by elimination of $\Box b^i_\mu$, $\Box\varphi^a$, and the derivatives of ψ by means of the equations of motion) leads to a relation which will not be written out because it is so cumbersome. An analysis of the independent terms appearing in this relation shows that it is consistent with the condition (2) for the theory to belong to class A if and only if the following structure relations hold:

$$\tilde\eta^i = -\eta^i, \tag{42}$$

$$[\eta^i, \mu^2] = 0, \tag{43}$$

$$[\eta^i, \eta^j] = \alpha_{ijk}\eta^k, \tag{44}$$

$$\eta^i_{aa'}\xi_{a'bcd} + \eta^i_{bb'}\xi_{ab'cd} + \eta^i_{cc'}\xi_{abc'd} + \eta^i_{dd'}\xi_{abcd'} = 0, \tag{45}$$

$$(T_j^{(1)} - \gamma_5 T_j^{(2)})\,(G_a^{(1)} + i\gamma_5 G_a^{(2)}) - (G_a^{(1)} + i\gamma_5 G_a^{(2)})\,(T_j^{(1)} + \gamma_5 T_j^{(2)}) = -\,i\eta^j_{ab}\,(G_b^{(1)} + i\gamma_5 G_b^{(2)}), \tag{46}$$

$$\zeta^{ij} = {}^1/_4\,[\eta^i, \eta^j]_+, \tag{47}$$

where we have introduced matrices η^i and ζ^{ij} with the matrix elements $(\eta^i)_{ab} = \eta^i_{ab}$ and $(\zeta^{ij})_{ab} = \zeta^{ij}_{ab}$ [the sign ~ in Eq. (42) means transposition].

It again turns out that the coupling matrices η^i realize a representation of the Lie algebra with the structure coefficients α_{ijk}. As a result the theory is invariant under the group of transformations (31) if the scalar fields are simultaneously transformed according to the law

$$\varphi^{a'} = \varphi^a + \omega_j\eta^j_{ab}\varphi^b. \tag{48}$$

The relations (42)—(47) assure that all of the terms of the Lagrangian are invariant under such a transformation. The scalar fields can also be broken up into irreducible multiplets, and by Eq. (43) the masses are equal within each multiplet. We note that in the most important case—when the masses of all the spinor fields are different from zero—the invariance relation (46) can be written especially simply:

$$[T_j^{(1)}, G_a^{(1)} + i\gamma_5 G_a^{(2)}] = -\,i\eta^j_{ab}\,(G_b^{(1)} + i\gamma_5 G_b^{(2)}). \tag{49}$$

6. SUMMARY

1. We see that on our assumptions—namely that: 1) the formalism is a Lagrangian one, 2) the coupling constants are dimensionless, and 3) the number of spinor particles is conserved—the most general theory of class A[1] (theory of fields with definite spins) for a system of fields with spins 1, $^1/_2$, and 0 is described by the Lagrangian

$$\mathscr{L}_{0,\,^1/_2,\,1} = -\,^1/_4 G_{\mu\nu}G_{\mu\nu} - \,^1/_2 b_\mu m^2 b_\mu - \bar{\psi}\{\gamma_\mu(\partial_\mu - iT_j^{(1)}b_\mu^j) + M\}\psi - \,^1/_2(\partial_\mu - \eta^j b_\mu^j)\varphi\cdot(\partial_\mu - \eta^j b_\mu^j)\varphi - \,^1/_2\varphi\mu^2\varphi + \xi_{abcd}\varphi^a\varphi^b\varphi^c\varphi^d + \bar{\psi}\{G_a^{(1)} + i\gamma_5 G_a^{(2)}\}\psi\varphi^a \quad (50)$$

under the condition that the masses of all spinor fields are different from zero. When there are zero masses the matrices $T_j^{(1)}$ in Eq. (50) must be replaced by matrices $T_j^{(1)} + \gamma_5 T_j^{(2)}$. A compact matrix notation has been used wherever possible in Eq. (50). The "vector field tensor" $G_{\mu\nu}$ is given by Eq. (19). The coupling matrices α, T, η, ξ, and G and the mass matrices m^2, M, and μ^2 must satisfy the relations (20)—(22), (28), and (35) [when the masses of the spinor fields are not zero, or Eqs. (28)—(30) in the general case], (38), and (42)—(46).

2. The relations (21), (30) or (35), and (44), mean that the matrices made up of coupling constants necessarily form a Lie algebra whose structure coefficients are the self-action constants of the vector fields. Owing to this the theory is invariant under the group of phase transformations:

$$b_\mu^{i'} = b_\mu^i + \alpha_{ijk}\omega_j b_\mu^k,$$

$$\psi' = \psi - i\omega_j T_j^{(1)}\psi, \qquad \varphi^{a'} = \varphi^a + \omega_j \eta_{ab}^j \varphi^b \quad (51)$$

[when there are zero masses among the spinor fields we must write $T_j^{(1)} + \gamma_5 T_j^{(2)}$ instead of $T_j^{(1)}$].

Lie algebras can be broken up into direct sums of simple algebras. Corresponding to this, all of the fields φ^a, ψ^r, b_μ^i can be broken up into multiplets which transform according to irreducible representations, the vector fields always transforming according to the regular representation. The masses of the fields within each multiplet are necessarily equal; this is a consequence of the relations (22), (28), and (43) and application of Schur's lemma.

3. We emphasize that the vector fields are the source of the symmetries. For example, if there are one or more truly neutral vector fields with spin 1 (for example, the ω meson or the vecton of Kobzarev and Okun'[10]), then $\alpha_{ijk} = 0$. The coupling matrices T_i then commute with each other [see Eq. (35) or Eq. (30)], and the same is true of the matrices η [see Eq. (43)]. This means that they can all be simultaneously reduced to diagonal form, and this is the true manifestation of the neutral character of the vector field. The corresponding symmetry property is invariance under a transformation of the type $\psi \to e^{i\Lambda}\psi$.

The next simplest possibility is a triplet of vector fields (for example, the ρ meson). Then the assumption that the triplet has spin 1 gives us $\alpha_{ijk} = g\epsilon_{ijk}$ (ijk = 1, 2, 3; ϵ_{ijk} is the antisymmetric unit tensor, and g is the coupling constant); the theory is isotopically invariant, with the vector fields forming an isotopic vector and the other fields transforming according to particular representations of the group of isotopic rotations. In this case we arrive at the Yang-Mills theory, but the mass of the vector fields can have any value.

Richer multiplets of vector fields lead to higher symmetries. The multiplets of the various fields transform according to the irreducible representations of the classical groups mentioned in the introduction. Such theories are generalizations of the Yang-Mills theory,[7] but without any restrictions on the masses of the vector fields. Next after the triplet comes the octuplet b_μ^i,[3)] which corresponds to the group SU(3).

4. It is clear from what has been said that the symmetries of the strong interactions find their natural explanation in theories of class A. It is remarkable that in theories of class A concepts as far from ordinary space as, for example, the baryon and hyperon charges and isotopic spin, and the corresponding conservation laws, are generated by a space-time property of the vector fields—the property of having a definite spin.

5. The inclusion of electromagnetic interactions breaks both the isotopic invariance and the condition (2) for charged vector fields, which led to this invariance. These two facts are closely connected.

At the same time theories of class A are very similar to electrodynamics, since because of gauge invariance the latter is always a theory of class A in regard to the electromagnetic field. In particular, the similarity is manifested in the fact that all interactions with the vector fields come in only through the "covariant derivatives"

$$\partial_\mu - \,^1/_2\alpha_j b_\mu^j, \qquad \partial_\mu - iT_j b_\mu^j, \qquad \partial_\mu - \eta^j b_\mu^j$$

applied to the respective fields b_μ, ψ, and φ.

6. A point worth attention is the universality of the constant for the interaction with each irreduci-

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3)In the Appendix we give an illustration of how an arbitrary quadruplet of fields is decomposed into irreducible parts – a singlet and a triplet.

ble multiplet of the vector fields—the self-action constant comes in everywhere. This can be seen at once from the structure relations of the Lie algebra, Eqs. (35) and (44). For example, if $\alpha_{ijk} = g\epsilon_{ijk}$ (isotopic invariance), then the matrices $T_j^{(1)}$ for the coupling with the isospinor field are realized in the form $T_i^{(1)} = \frac{1}{2} g\tau_i$.

7. Furthermore, in theories of class A the vector mesons must have the quantum numbers 1^- (must be vectors, not axial vectors), and parity must be conserved in interactions with them.

8. Finally, let us give some attention to the assumptions which were formulated at the beginning of this section. They are not all of equal weight. The first relates to dynamics, and the authors can only express the hope that the connection between the symmetry properties and the spin 1 can also be established outside the framework of the Lagrangian formalism. As for the assumption 2), this connection still holds after one has included in the Lagrangian, besides the terms with dimensionless coupling constants, terms with the subsequent dimensionalities for the coupling constants. Finally, assumption 3) completely ceases to be any restriction if, for example, there are no bosons with baryon charge. Generally speaking, this assumption can be eliminated easily, and we shall do this in another paper (see also [11]).

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APPENDIX

RESOLUTION OF A QUADRUPLET OF FIELDS INTO REDUCIBLE PARTS

In the case of four vector fields we can always write

$$\alpha_{ijk} = \varepsilon_{ijkl} g_l \qquad (i, j, k, l = 1, 2, 3, 4) \qquad \text{(A.1)}$$

(ϵ_{ijkl} is the unit absolute antisymmetric "tensor"), i.e., α_{ijk} is determined by the coupling constants g_1, g_2, g_3, g_4. If we make an orthogonal transformation $b_\mu^{i'} = r_{ij} b_\mu^j$, $r_{ij} r_{ik} = \delta_{jk}$, then

$$\alpha'_{ijk} = r_{ii'} r_{jj'} r_{kk'} \alpha_{i'j'k'}. \qquad \text{(A.2)}$$

Contracting Eq. (A.2) with $r_{ll'} \epsilon_{ijkl}$ and using Eq. (A.1), we easily find that

$$g'_l = r_{lj} g_j,$$

i.e., the g_l undergo a rotation in four-dimensional Euclidean space. For arbitrary initial g_l we can always find a rotation $\| r_{ij} \|$ after which, for example, $g_1' = g_2' = g_3' = 0$, and $g_4' = (g_1^2 + g_2^2 + g_3^2 + g_4^2)^{1/2}$, so that α'_{ijk} breaks up into two blocks:

$$\alpha'_{ijk} = \begin{cases} g'_4 \varepsilon_{ijk}, & \text{if } i, j, k \neq 4, \\ 0 & \text{in all other cases} \end{cases}.$$

This means that $b_\mu^{1'}$, $b_\mu^{2'}$, and $b_\mu^{3'}$ form a triplet of interacting fields and $b_\mu^{4'}$ is a singlet which does not interact with the triplet.

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[1] V. I. Ogievetskiĭ and I. V. Polubarinov, Preprint E-1170, Joint Institute for Nuclear Research, 1963; JETP **45**, 237 (1963), Soviet Phys. JETP **18**, 166 (1964).

[2] V. I. Ogievetskiĭ and I. V. Polubarinov, JETP **41**, 247 (1961), Soviet Phys. JETP **14**, 179 (1962); Nuovo cimento **23**, 173 (1962); Proc. 1962 International Conf. on High Energy Physics at CERN, page 666.

[3] V. I. Ogievetskiĭ and I. V. Polubarinov, JETP **45**, 709 (1963), Soviet Phys. JETP **18**, 487 (1964).

[4] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[5] E. B. Dynkin, Uspekhi Mat. Nauk **2**, 59 (1947).

[6] Behrends, Dreitlein, Fronsdahl, and Lee, Revs. Modern Phys. **34**, 1 (1962).

[7] S. L. Glashow and M. Gell-Mann, Ann. Phys. **15**, 437 (1961).

[8] J. Sakurai, Ann. Phys. **11**, 1 (1960).

[9] B. d'Espagnat, Proc. 1962 International Conf. on High Energy Physics at CERN, page 917.

[10] I. Yu. Kobzarev and L. B. Okun', JETP **41**, 499 (1961), Soviet Phys. JETP **14**, 358 (1962).

[11] V. I. Ogievetskiĭ and I. V. Polubarinov, Preprint R-1241, Joint Institute for Nuclear Research.

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