QUASICLASSICAL SCATTERING IN A CENTRALLY SYMMETRIC FIELD

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The amplitude is obtained for quasiclassical scattering in a centrally symmetric field into classically unattainable angles. The position of the Regge poles and the asymptotic behavior of the scattering matrix in the complex angular momentum plane are investigated for a certain class of potentials $U(r)$ at energies E satisfying the conditions: $E \gg |U(r)|$ along the real axis and $\lambda \ll a$ (λ is the wavelength of the incident particle).

1. FORMULATION OF THE PROBLEM

When a classical particle is scattered by a weak potential field ($U \ll E$) there exists a limiting scattering angle $\theta_0(E)$. From the classical point of view the particle cannot be deflected by an angle greater than $\theta_0(E)$. In quantum mechanics there can arise no restrictions on the scattering angles, and there always exists a small, but finite, probability of scattering into the range of nonclassical angles $\theta > \theta_0(E)$.

The very formulation of the problem of scattering into the range $\theta > \theta_0$ makes sense only for quasiclassical particles whose wavelength λ is considerably smaller than the characteristic dimensions of the potential a. Indeed, the quantum uncertainty in the scattering angle is of order λ/a, and if this quantity is not small compared to θ_0 there is no sense in speaking of a limiting angle of scattering.

At very high energies (conditions for this will be found in greater detail later) the Born formula can be used for the scattering amplitude $f(\theta)$

$$f(\theta) = \frac{m}{2\pi}\int e^{i\mathbf{qr}} U(r)\, dv, \tag{1.1}$$

where $\mathbf{q} = \mathbf{k}_2 - \mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_1$ are the momenta of the particle before and after scattering, m is the mass of the particle. Of interest is that range of energies in which the Born approximation is not applicable. However, certain important conclusions can be drawn from the Born approximation. In the spherically symmetric case formula (1.1) assumes the form

$$f(\theta) = 2m\int_0^\infty U(r)\,\frac{\sin qr}{q} r\,dr. \tag{1.2}$$

For large q the scattering amplitude falls off in accordance with a power law as q increases if $U(r)$ has singularities on the real axis or is odd. But if $U(r)$ is an even analytic function without singularities on the real axis, then $f(\theta)$ is an exponentially small quantity for not too small values of θ.

In the region in which the Born approximation is not applicable it has been customary to use adiabatic perturbation theory [1,2]. In the case of a nonanalytic or an odd potential the first nonvanishing approximation of this theory apparently yields the correct result. However, in the case of an analytic even potential it turns out that all the approximations of adiabatic perturbation theory give contributions to $f(\theta)$ of the same order. Here a situation arises analogous to that investigated in the case of the problem of reflection by a one-dimensional potential at energies above the barrier [3].

In this paper we investigate scattering into the range of nonclassical angles by an even analytic potential $U(r)$. As will be shown later, in the case of a potential which falls off sufficiently rapidly at infinity the principal role is played by the singularities of the potential closest to the real axis. We shall assume that these are simple poles. In virtue of the fact that $U(r)$ is real along the real r axis and is even, there will be four such poles at points $\pm r_0$, $\pm r_0^*$. We shall denote by R the residue at the point r_0 ($0 < \arg r_0 < \pi/2$).

As our starting point we use the well known Faxén-Holtsmark formula

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\, e^{2i\delta l} P_l(\cos\theta). \tag{1.3}$$

With the aid of Watson's transformation [4], which has recently found wide application in the physics of high energy particles, we transform the sum (1.3) into the integral:

$$f(\theta) = -\frac{1}{2k}\int\limits_{\Gamma} \nu S(\nu) P_{\nu-1/2}(-\cos\theta)\frac{d\nu}{\cos\nu\pi}, \qquad (1.4)$$

where the contour Γ encompasses the real semi-axis of ν as shown in Fig. 1, while $S(\nu)$ denotes the function $\exp(2i\delta_{\nu-\frac{1}{2}})$ continued analytically from half-integer positive values of ν to the whole complex plane.

Without making any claim as to rigor, we describe the general idea of the calculation. We begin with the range of the classical scattering angles:

$$\lambda/a \ll \theta < \theta_0. \qquad (1.5)$$

In this range after the contour of integration has been suitably transformed the integral (1.4) must be evaluated by the saddle-point method [5]. For each scattering angle θ there will be found two real saddle points $\nu_1(\theta)$, $\nu_2(\theta)$ —two real impact parameters corresponding to the given scattering angle (we recall that particles passing through the center of the potential and at infinitely great distances are not deflected). At $\theta = \theta_0$ the two impact parameters coincide and $\nu_1 = \nu_2$. As the angle is increased further the saddle points move into the complex ν plane and $f(\theta)$ becomes exponentially small.

Thus, the calculation of $f(\theta)$ must be preceded by an investigation of the analytic properties of $S(\nu)$ in the complex ν plane. In an earlier paper [6] 1) a method was developed for investigating the asymptotic behavior of $S(\nu)$ in the quasiclassical case based on a study of the behavior of the solutions of Schrödinger's equations in the complex r plane. We shall investigate the analytic properties of $S(\nu)$ with the aid of this method.

For convenience we shall write out the formulas relating $S(\nu)$ to the coefficients A_ν, B_ν and a_ν, b_ν:

$$S(\nu) = ie^{i\nu\pi}A_\nu/B_\nu = a_\nu^{-1} + ie^{i\nu\pi}b_\nu/a_\nu. \qquad (1.6)$$

The coefficients A_ν, B_ν, a_ν, b_ν define the asymptotic behavior of the functions j_ν, $h_\nu^{(1)}$:

$$j_\nu = \begin{cases} 2^{-\nu}\sqrt{2\pi}r^{\nu+1/2}/\Gamma(\nu+1) & r\to 0 \\ A_\nu e^{ikr} + B_\nu e^{-ikr} & r\to +\infty \end{cases}, \qquad (1.7)$$

$$h_\nu^{(1)} = \begin{cases} e^{ikr} & r\to +\infty \\ a_\nu e^{ikr} + b_\nu e^{-ikr} & r\to -\infty \end{cases}. \qquad (1.8)$$

All the definitions and notation which are not explained in the text of the present communication have been taken over from [6].

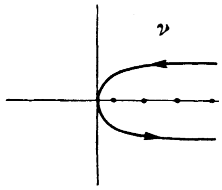

FIG. 1

2. THE POLES OF $S(\nu)$ (REGGE POLES)

The poles of $S(\nu)$ coincide with the zeros of the integral functions B_ν and a_ν in consequence of formula (1.6). Of course, not all the zeros of a_ν are at the same time poles of $S(\nu)$; if $a_{\nu_0} = 0$ then the pole is either at the point ν_0 or at $-\nu_0$ (cf., [6]). First of all we note that B_ν certainly does not vanish if there do not exist at least two singularities lying on or near a line of a given level. Indeed, let us consider the line belonging to the lowest level which passes through a singular point in the upper half-plane and $+\infty$ (the lines passing through r_1 in Fig. 2).2) On one of the branches of this curve j_ν is represented by a single exponential. As a result of going around a singular point and passing onto a branch leading to $+\infty$ an exponential increasing in the upward direction always appears.

From this argument it follows also that B_ν vanishes only in the case when two singular points lie on the line belonging to the lowest level which passes through the singular points and goes towards $+\infty$. If ν lies in the upper half-plane then the zeros of B_ν occur when r_1 and r_2 lie on a line of the same level. In this case two different configurations are possible: r_2 lies on a branch of the line of the $[0, r_1]$ level (Fig. 3) or on a branch of the line of the $[+\infty, r_1]$ level (Fig. 4).

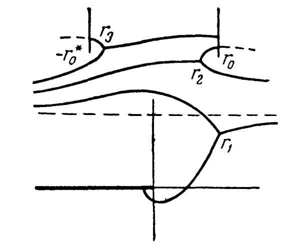

FIG. 2

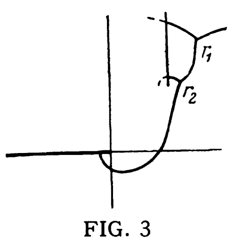

FIG. 3

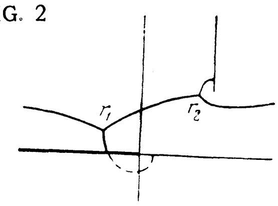

FIG. 4

1)We shall hereafter denote references to formulas from this paper by the numeral I preceding the number of the formula.

2)We recall the definitions of the complex turning points [6]:

$$r_1 = \nu/k, \quad r_2 = r_0 + \frac{R}{k^2 - \nu^2/r_0^2}, \quad r_3 = -\overset{*}{r_0} - \frac{R^*}{k^2 - \nu^2/\overset{*}{r_0}{}^2}.$$

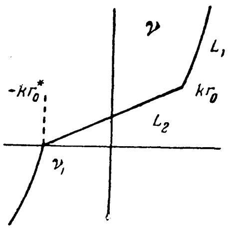

FIG. 5

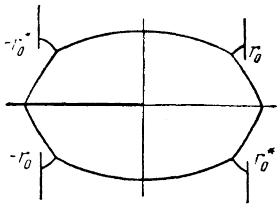

FIG. 6

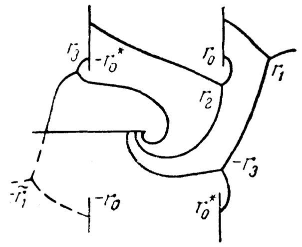

FIG. 7

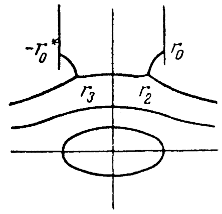

FIG. 8

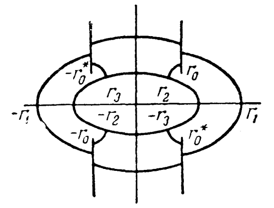

FIG. 9

The former configuration corresponds to poles lying on the line L_1 in the ν plane (Fig. 5). The line L_1 is defined by the equation

$$\operatorname{Im}(r_2, r_1) = 0. \qquad (2.1)$$

It leaves the neighborhood of the point[3] $\nu = kr_0$ and proceeds upwards being for large values of ν asymptotically parallel to the imaginary axis. The latter configuration (Fig. 4) corresponds to poles lying on the line L_2 in the ν plane (Fig. 5). The curve L_2 is defined by the same equation (2.1). It goes from the point $\nu = kr_0$ downwards and to the left approaching the real axis at values of ν close to ν_1, where ν_1 is the negative real value of ν for which the boundary of the "eye" (cf. [6], Sec. 3) passes through $\pm r_0$ and $\pm r_0^*$ (Fig. 6). Beyond the point ν_1 (2.1) formally yields solutions lying in the lower half-plane of ν. Actually B_ν has no zeros at such values of ν.

If ν lies in the lower half-plane, then a level line which passes through r_2 necessarily intersects the cut associated with the pole $-r_0^*$ (Fig. 7). Therefore, for $\operatorname{Im}\nu < 0$ ($\operatorname{Re}\nu < 0$) we must require that the points $\tilde{r}_1$ and r_3 (or, what is the same thing, $-\tilde{r}_1$, $-r_3$) should lie on the same level line (Fig. 7)

$$\operatorname{Im}(r_3, \tilde{r}_1) = 0. \qquad (2.2)$$

The poles corresponding to the configuration of Fig. 7 lie on the line L_3 (Fig. 5) defined by (2.2). The line L_3 intersects the real axis at the point ν_1.

We have indicated the lines of poles in the complex ν plane. However, we have not as yet investigated the real ν axis. For real values of ν the points r_2, r_3 always lie on a line of the same level. But for $|\nu| < |\nu_1|$ this level line goes from $+\infty$ to $-\infty$ (Fig. 8). Therefore, the coefficient a_ν of the exponential Z_+ which increases downwards is not altered. From this it follows that $a_\nu \neq 0$ and $S(\nu)$ has no poles.

The configuration for real $|\nu| > |\nu_1|$ is shown in Fig. 9. If $\nu > 0$, then j_ν increases with increasing r and at the upper boundary of the "eye" is represented by a single exponential increasing upwards. This exponential also does not vanish for $\operatorname{Re} r \to +\infty$. Consequently, also in this case $B_\nu \neq 0$. But if $\nu < 0$, then j_ν decreases with increasing r, and the arguments used by us fail. Thus, Regge poles can also lie near the real axis for $\operatorname{Re}\nu < \nu_1$. In this case it is more convenient to investigate the position of the zeros of a_ν. Since there are no poles for $\nu > 0$, then poles of $S(\nu)$ correspond to all the zeros of a_ν near the negative semi-axis.

We now derive the equations which determine the position of the Regge poles. We assume that $\sigma = (\nu - kr_0)/kr_0$ is sufficiently great:

$$\sigma \gg \sqrt{U_0/E} \text{ for } (U_0/E)^{3/4} kr_0 \gg 1,$$

$$\sigma \gg (kr_0)^{2/3} \text{ for } (U_0/E)^{3/4} kr_0 \ll 1.$$

Then, as shown in Sec. 5 of [6], the points r_1 and r_2 are far from each other. For the sake of definiteness we consider the configuration of Fig. 3. The function j_ν is represented on the segment $[0, r_2]$ in the form

$$j_\nu(r) = N(\nu) Z_+(r, r_1). \qquad (2.3)$$

Further, after going around r_2 we have on the segment $[r_2, r_1]$

$$j_\nu = N(\nu) e^{i(r_2, r_1)} (Z_+(r, r_2) - F(\xi_2) Z_-(r, r_2)). \qquad (2.4)$$

Finally, after going around r_1 we have on the segment $[r_1, +\infty]$

$$j_\nu = N(\nu)(Z_+(r, r_1) + iZ_-(r, r_1) - e^{2i(r_2, r_1)} F(\xi_2) Z_-(r, r_1)). \qquad (2.5)$$

The condition $B_\nu = 0$ assumes the form

$$e^{2i(r_2, r_1)} F(\xi_2) = i. \qquad (2.6)$$

Equation (2.6) can be conveniently written in the following form

$$(r_2, r_1) = n\pi + \tfrac{1}{4}\pi + \tfrac{1}{2} i \ln F(\xi_2). \qquad (2.7)$$

[3]Near $\nu = kr_0$ all three points r_0, r_1, r_2 can be near one another (cf., [6], Sec. 5).

In the case of not too high energies $(r_2, r_0) \gg 1$, for example, the case $U_0 k r_0/E \gg 1$. In this case $|\xi_2| \gg 1$ and $F(\xi_2) = -i$. Then equation (2.7) formally coincides with Bohr's condition. In the opposite limiting case $|(r_2, r_0)| \ll 1$, when the Born approximation is applicable to radial functions, it follows from (I.4.4):

$$(r_2, r_1) = \left(n + \tfrac{1}{2}\right)\pi + \tfrac{1}{2} i \ln (2\pi\xi_2). \qquad (2.8)$$

From formula (2.8) it follows that for high ordinal numbers the poles logarithmically approach each other ($\delta\nu \sim 1/\ln n$) and Re ν increases like Im $\nu/\ln \nu$.

The series of poles lying near the curve L_2 (Fig. 10) is associated with the configuration of Fig. 4. The equation which determines the poles of this series is derived in a manner analogous to (2.7) and has the form

$$(r_2, r_1) = n\pi + \tfrac{1}{4}\pi - \tfrac{1}{2} i \ln F(\xi_2). \qquad (2.9)$$

There exists some arbitrariness in the choice of the sign of p_ν. In connection with this we can assume that both series are numbered by positive values of n. Another possibility consists of ascribing positive numbers to one of the series and negative numbers to the other. The last possibility is convenient at high energies. In this case all the solutions of (2.8) correspond to $|\xi_2| \gg 1$. Therefore, formula (2.8) describes continuously both series of poles in going over from positive to negative values of n.

It is natural to call the series of poles situated near the curve L_2 the physical series, since at negative energies in the case of a potential well these poles correspond to bound states (an analogous situation is investigated in detail in [7]). We shall call the series of poles situated along L_1 the first unphysical series, and we shall call the poles associated with L_3 the second unphysical series. The equation for the poles of the second unphysical series has the form

$$(r_3, \tilde{r}_1) = \pi n - \tfrac{1}{4}\pi - \tfrac{1}{2} i \ln F(\xi_2). \qquad (2.10)$$

The point $\tilde{r}_1$ is situated on the unphysical sheet below the cut from $r = 0$.

For the investigation of the poles situated near the real axis for Re $\nu < \nu_1$ we shall find the quantity a_ν. The solution $h_\nu^{(1)}$ increases as we move from Re $r \to +\infty$ to the segment of the line of the level $[r_2, -r_3]$ (cf., Fig. 9). Along $[r_2, -r_3]$ the function $h_\nu^{(1)}$ has the form

$$h_\nu^{(1)} \approx \sqrt{k}\, e^{i(+\infty, r_2)} Z_+ (r, r_2). \qquad (2.11)$$

In going around first the point r_2, and then r_3, we obtain along the segment $[r_3, -r_2]$:

$$h_\nu^{(1)} \approx \sqrt{k}\, e^{-i(+\infty, r_2)} \{ e^{i(r_2, r_3)} Z_+ (r, r_3) + [F(\xi_3)\, e^{i(r_2, r_3)} + F(\xi_2)\, e^{-i(r_2, r_3)}] Z_-(r, r_3)\}. \qquad (2.12)$$

We now go on to the boundary of the "eye" above the cut. In this case $Z_-(r, r_3)$ increases, while $Z_+(r, r_3)$ diminishes, so that at the boundary of the "eye" there remains only the function $Z_-(r, r_3)$, if its coefficient is not equal to zero. In going around the point $r_1 \approx \nu/k$ we again obtain two exponentials, so that for $r < r_1$ along the real axis $h_\nu^{(1)}$ has the form

$$h_\nu^{(1)} \approx \sqrt{k}\, e^{-i(+\infty, r_2) - i(r_2, r_3)} (F(\xi_3)\, e^{i(r_2, r_3)} + F(\xi_2)\, e^{-i(r_2, r_3)}) \times (Z_-(r, r_1) - i Z_+(r, r_1)). \qquad (2.13)$$

It can be seen from (2.13) that in order for a_ν to vanish it is necessary that the following equation be satisfied

$$F(\xi_3)\, e^{i(r_2, r_3)} + F(\xi_2)\, e^{-i(r_2, r_3)} = 0, \qquad (2.14)$$

and this can be rewritten in the equivalent form:

$$(r_3, r_2) = \left(n + \tfrac{1}{2}\right)\pi - \tfrac{1}{2} i \ln [F(\xi_2)/F(\xi_3)]. \qquad (2.15)$$

For $|\nu| \gg |\nu_1| \sim |kr_0|$ we obtain from (2.15)

$$\nu_n \approx -\left(n + \tfrac{1}{2}\right)\pi/(\pi - 2\varphi_0), \qquad (2.16)$$

where $\varphi_0 = \arg r_0$. The distance between neighboring poles is equal to $\pi/(\pi - 2\varphi_0)$. We note that it tends to infinity as the poles of $U(r)$ approach the imaginary axis. In the region $|\nu| \sim |\nu_1|$ the distance between the poles increases somewhat.

Within the limits of its accuracy formula (2.15) yields purely real values of ν. But it was shown in Sec. 2 of [6] that the poles cannot be real. In actual fact the poles near the real axis do have small imaginary parts, but in order to find them we must improve the accuracy of the method, and as a result of this it becomes more complicated. Since we shall not subsequently need to know the exact position of the poles, we shall investigate them only qualitatively in order to elucidate the relation between the poles of the physical and of the second unphysical series and the poles lying on the real axis.

We denote by $n_1(E)$ the number of poles of the physical series from kr_0 to ν_1. Up to quasiclassical accuracy this number coincides with the ordinal number of the pole nearest to ν_1:

$$n_1(E) \approx \int_{\nu_1/k}^{r_2} p_\nu \, dr \qquad (2.17)$$

(here we approximately replace r_1 by ν_1/k). Further, we denote by $n_2(E)$ and $n_3(E)$ the numbers of poles of the second unphysical series and

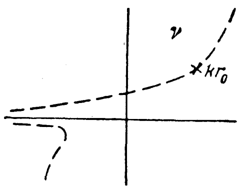

FIG. 10

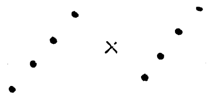

FIG. 11

of the series near to the real axis which are closest to ν_1. From (2.10) and (2.15) we obtain approximately

$$n_2(E) \approx \int_{\nu_1/k}^{r_1} p_\nu\, dr, \quad n_3(E) \approx \int_{r_3}^{r_2} p_\nu\, dr. \qquad (2.18)$$

From (2.17) and (2.18) it follows that

$$n_1 = n_2 + n_3. \qquad (2.19)$$

Relation (2.19) is an expression of the "law of conservation" of the number of poles.

The general picture of the motion of the poles appears in the following form. The poles situated near the real axis and above it move away from the real axis as the energy is increased and move upwards following the point kr_0. The breaking away occurs at the moment when the point ν_1 approaches the pole. The poles which are far away from ν_1 are all situated above the real axis, and almost do not move at all until the point ν_1 reaches them. The poles of the second unphysical series move in the direction towards the real axis as the energy increases. After each successive pole of this series has approached the real axis (to within a distance of the order of unity) it begins to move along the real axis and below it in the direction of $|\nu|$.

Schematically the position of the poles is shown in Fig. 10. It is natural to include all the poles lying near the real axis and above it in the physical series and all those lying below the real axis in the second unphysical series.

We now consider the neighborhood of the point $\nu = kr_0$. Here we shall require more specific assumptions about the relations between the parameters. We consider two cases (cf., Sec. 5 of [6]).

A. $(U_0/E)^{3/4} kr_0 \gg 1$. In this case any two singularities are far from one another. If these distant points are turning points, then the equation which determines the poles has the previous form (2.7). But if the turning points are close, then in accordance with (I.4.5)—(I.4.7) the poles of $S(\nu)$ are determined by the equation

$$1/\Gamma\left(\tfrac{1}{2} - \eta\right) = 0 \qquad (2.20)$$

or

$$\pi\eta = (r_2, r_1) = \left(n + \tfrac{1}{2}\right)\pi. \qquad (2.21)$$

Condition (2.21) is joined onto (2.7) for large η. The turning points become close when ν is close to the points $kr_0(1 \pm i\sqrt{2U_0/E})$. Consequently, the physical and the first unphysical series originate in these particular points. The first poles will be at a distance of the order of unity from the indicated points and from each other. The position of the poles at the beginning of the series is shown schematically in Fig. 11.

B. $(U_0/E)^{3/4} kr_0 \lesssim 1$. In this case inside a circle of radius $(kr_0)^{2/3}$ about the point $\nu = kr_0$ all three singular points are close to each other, and we cannot indicate any poles lying inside this circle.

We evaluate the residues at the poles of $S(\nu)$. In order to do this it is necessary to evaluate separately the coefficients A_ν, B_ν and then to find the value of the residue

$$\operatorname{Res} S(\nu) = A_{\nu_n} / (dB_{\nu_n} / d\nu). \qquad (2.22)$$

Without reproducing the calculations we quote the results. For the poles of the first unphysical series and for the complex poles of the physical series we have

$$\operatorname{Res} S(\nu_n) = -ie^{2i(+\infty, r_1)} \left[2\nu \int_{r_1}^{r_2} \frac{dr}{r^2 p_\nu}\right]^{-1} e^{i\nu\pi}. \qquad (2.23)$$

For the poles of the second unphysical series we have

$$\operatorname{Res} S(\nu_n) = -ie^{2i(-\infty, r_1)} \left[2\nu \int_{r_1}^{r_3} \frac{dr}{r^2 p_\nu}\right]^{-1} e^{i\nu\pi}. \qquad (2.23')$$

For $\nu \sim kr_0$ the absolute values of all the residues are of order of magnitude unity and do not contain exponentially small or exponentially large factors.

The zeros of a_ν can be easily obtained if we show on the same diagram the poles of $S(\nu)$ and $S(-\nu)$ (Fig. 12). In order to evaluate the residues of b_ν/a_ν we utilize formula (I.2.16).

$$\tfrac{1}{2}\left[S(\nu) - S(-\nu)\right] = -\sin\nu\pi\,(b_\nu/a_\nu). \qquad (2.24)$$

Since the residues of $S(\nu)$ are of order of magnitude unity, then for $|\operatorname{Im}\nu| \gg 1$ the residues of b_ν/a_ν are exponentially small.

3. BEHAVIOR OF $S(\nu)$ IN THE COMPLEX PLANE AND THE CHOICE OF THE CONTOUR OF INTEGRATION

It is necessary to investigate the behavior of $S(\nu)$ only in the upper half-plane of ν, since the unitarity condition (I.2.17) enables us to continue $S(\nu)$ into the lower half-plane of ν. We now obtain the quasiclassical asymptotic behavior of $S(\nu)$. This asymptotic behavior turns out to be different

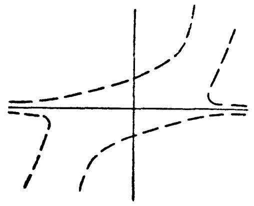

FIG. 12

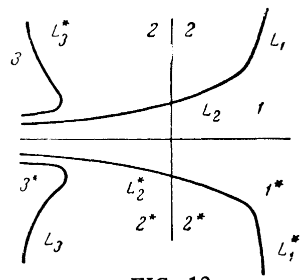

FIG. 13

in different regions of the upper half-plane. The boundaries of the regions are formed by lines of poles and of zeros (we recall that the lines of zeros are symmetric to the lines of poles with respect to the real axis). The picture of the regions in the ν plane having different asymptotic behavior of $S(\nu)$ is shown in Fig. 13.

In region 1 the usual quasiclassical expression (I.6.16) for $S(\nu)$ holds:

$$S(\nu) = e^{2i(+\infty,\, r_1)+i\nu\pi}. \tag{3.1}$$

In region 2 we obtain by methods indicated in Sec. 6 of [6]

$$S(\nu) = -iF(\xi_2)\, e^{2i(+\infty,\, r_2)+i\nu\pi}. \tag{3.2}$$

Finally, in region 3 we have

$$S(\nu) = e^{2i(+\infty,\, -r_1)+i\nu\pi}. \tag{3.3}$$

In region 1 the function $S(\nu) \to 1$ as $|\nu| \to \infty$. In region 2 as $|\nu| \to \infty$ $S(\nu)$ falls off at different rates in different directions. If we move along the line L_1 the magnitude of $S(\nu)$ does not decrease at all. Along the line L_3^* it decreases like $\exp(i\nu^*\pi)$. In region 3 as $|\nu| \to \infty$ $S(\nu)$ everywhere falls off like $\exp(i\nu\pi)$. Correspondingly in the lower half-plane in region 1* the function $S(\nu) \to 1$; in region 2* it increases differently in different directions (from 1 along L_1^* to $\exp(-i\nu\pi)$ along L_3) and in region 3* it increases like $\exp(-i\nu^*\pi)$.

The asymptotic behavior of the even part $S_s(\nu)$ and of the odd part $S_a(\nu)$ of $S(\nu)$ can be easily obtained if we know the asymptotic behavior of $S(\nu)$. The boundaries for the applicability of the different asymptotic forms for $S_s(\nu)$ and $S_a(\nu)$ can be obtained if we show on the same diagram the boundaries for the different asymptotic forms of $S(\nu)$ and $S(-\nu)$. In each of these regions the asymptotic form of $S_s(\nu)$, $S_a(\nu)$ coincides with the asymptotic form of the larger of the quantities $\frac{1}{2}S(\nu)$; $\pm\frac{1}{2}S(-\nu)$. We note that the magnitude of b_ν/a_ν does not increase as $|\nu| \to \infty$. Indeed, $b_\nu/a_\nu = -S_a(\nu)/\sin\nu\pi$, and the function $S_a(\nu)$ does not increase faster than $\sin\nu\pi$.

The knowledge of the asymptotic behavior of $S(\nu)$ enables us to determine how we can deform the contour of integration Γ in the formula (1.4) (Fig. 1). For convenience we write out once again Watson's integral:

$$f(\theta) = -\frac{1}{2k}\int_\Gamma \nu\, S(\nu)\, P_{\nu-1/2}(-\cos\theta)\,\frac{d\nu}{\cos\nu\pi}. \tag{3.4}$$

We quote here well known formulas which determine the asymptotic behavior of Legendre polynomials for large complex values of ν:

$$P_{\nu-1/2}(-\cos\theta) = \sqrt{2/\pi\nu\sin\theta}\,\cos[\nu(\pi-\theta)-\pi/4]. \tag{3.5}$$

We shall show that by suitable deformation it is always possible to make the contour Γ (Fig. 1) symmetric with respect to the point $\nu = 0$. Indeed, in the upper half-plane of ν the function $S(\nu)$ does not increase. The convergence of the integral is guaranteed by the factor $\exp(i\nu\theta)$, and the contour Γ can be arbitrarily deformed in the upper half-plane. Of course, in doing this it is necessary to take into account the residues from the poles which cut across the contour in the course of the deformation. In the lower half-plane the contour cannot be placed arbitrarily, since $S(\nu)$ increases in regions 2*, 3*. The contour can be pulled in the direction L_3 (Fig. 13) only as long as $S(\nu)$ grows not faster than $\exp(-i\nu\theta)$. But such a curve can always be found between L_1^* and L_3. Therefore, it is always possible to deform the contour Γ into the contour Γ_1' symmetric with respect to $\nu = 0$, so that in the course of deformation it will only cut across poles belonging to the first unphysical series (Fig. 14).

In the integral over Γ_1 we can replace $S(\nu)$ by the odd part $S_a(\nu)$ in virtue of the parity properties of the integrand. Thus, our transformations have led to the following expression for $f(\theta)$:

$$f(\theta) = -\frac{1}{2k}\int_{\Gamma_1} \nu S_a(\nu)\, P_{\nu-1/2}(-\cos\theta)\,\frac{d\nu}{\cos\nu\pi} - \frac{\pi i}{k}\sum\frac{\nu_n}{\cos\nu_n\pi}\,\mathrm{Res}\, S(\nu_n)\, P_{\nu_n-1/2}(-\cos\theta), \tag{3.6}$$

where the summation is taken over the poles of the first unphysical series.

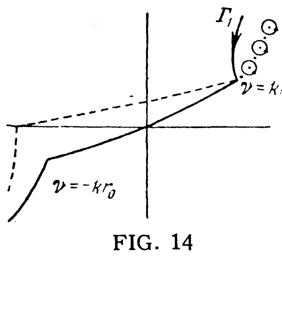

FIG. 14

4. THE FUNCTION $\tilde{h}_{\nu}^{(1)}$ AND QUANTITIES ASSOCIATED WITH IT

We begin by calculating the integral I over the contour Γ_1 occurring in the first part of formula (3.6):

$$I = -\frac{1}{2k}\int_{\Gamma_1} \nu S_a(\nu)\, P_{\nu-1/2}(-\cos\theta)\,\frac{d\nu}{\cos\nu\pi}$$
$$= \frac{1}{2k}\int \nu \,\mathrm{tg}\, \nu\pi \frac{b_\nu}{a_\nu} P_{\nu-1/2}(-\cos\theta)\, d\nu. \qquad (4.1)^*$$

If we make use of the quasiclassical asymptotic behavior of $S_a(\nu)$ obtained in Sec. 3, then we can show that the integrand in (4.1) does not have a saddle point.

The absence of a saddle point is associated with the fact that in region 2 (Fig. 15) the quantity $\gamma_\nu = b_\nu/a_\nu$ has the asymptotic behavior

$$\gamma_\nu \sim e^{2i(r_3, -\infty)}, \qquad (4.2)$$

while in region 3 in the same figure we have

$$\gamma_\nu \sim e^{2i(r_2, -\infty)}. \qquad (4.3)$$

If we formally obtain the saddle point ν_{st} in the integral (4.1) with the asymptotic behavior (4.2), then it turns out that $\nu_{st} = kr_3\cos(\theta/2)$. Consequently, ν_{st} lies in region 3 of Fig. 15, but in this region the asymptotic behavior (4.2) is not applicable. And if we obtain the saddle point by utilizing the asymptotic behavior (4.3), then it turns out to be equal to $kr_2\cos(\theta/2)$, and lies in region 2 where the asymptotic behavior (4.3) does not hold.

In order to overcome this difficulty it is necessary to break up γ_ν into two terms $\gamma_{2\nu}$, $\gamma_{3\nu}$, so that $\gamma_{2\nu}$ would have the asymptotic behavior (4.2) in region 2, while $\gamma_{3\nu}$ would have the asymptotic behavior (4.3) in region 3. We emphasize that this decomposition must be exact, and not approximate, since one of the terms is exponentially greater than the other one, so that any inaccuracy in the determination of the greater one exceeds the value of the smaller term.

In order to carry out this program we define the solution $\tilde{h}_{\nu}^{(1)}$ of the Schrödinger equation in such a manner that in the first nonphysical sheet below the cut associated with the point r_0 this solution should behave like $h_{\nu}^{(1)}$ in the physical sheet. Then in the physical sheet the asymptotic behavior of $\tilde{h}_{\nu}^{(1)}$ will be the following:

$$\tilde{h}_{\nu}^{(1)} = \begin{cases} \alpha_\nu e^{ikr} + \beta_\nu e^{-ikr} & r \to +\infty \\ \tilde{a}_\nu e^{ikr} + \tilde{b}_\nu e^{-ikr} & r \to -\infty \end{cases}. \qquad (4.4)$$

From (2) follows the connection between $\tilde{h}_{\nu}^{(1)}$, $h_{\nu}^{(1)}$, $h_{\nu}^{(2)}$:

$$\tilde{h}_{\nu}^{(1)} = \alpha_\nu h_{\nu}^{(1)} + \beta_\nu h_{\nu}^{(2)}. \qquad (4.5)$$

Between the coefficients which determine the behavior of $h_{\nu}^{(1)}$, $h_{\nu}^{(2)}$, $\tilde{h}_{\nu}^{(1)}$ at infinity [cf. definition (I.2.7)] there exist the following relations

$$\alpha = \tilde{a}d - \tilde{b}c, \qquad (4.6)$$
$$\beta = a\tilde{b} - b\tilde{a} \qquad (4.7)$$

(here we have utilized the fact that the Wronskian is constant). From (4.5) and (I.2.7) we obtain on utilizing (4.6)

$$\gamma_\nu = \gamma_{2\nu} + \gamma_{3\nu}, \qquad \gamma_{3\nu} = \tilde{b}_\nu/\tilde{a}_\nu, \qquad \gamma_{2\nu} = -\alpha\beta/\tilde{a}(\tilde{a} - \beta c). \qquad (4.8)$$

Asymptotic investigation of the individual terms on the right hand side of (4.8) leads to the following results.

In regions 1, 2, 3 in Fig. 16 we have

$$\gamma_{3\nu} \approx F(\xi_3)\, e^{2i(r_3, -\infty)}. \qquad (4.9)$$

In region 4 on the same diagram we have

$$\gamma_{3\nu} \approx i e^{2i(r_1, -\infty)}. \qquad (4.10)$$

The boundaries of regions having different asymptotic behavior in Figs. 16 and 17 are formed by lines in the ν plane whose equations have the following form

$$\mathrm{Im}\int_{r_1}^{r_2} p_\nu\, dr = 0, \qquad \mathrm{Im}\int_{r_1}^{r_3} p_\nu\, dr = 0.$$

They are lines of poles and zeros of the quantities $S(\nu)$, or γ_ν, or $\gamma_{3\nu}$.

We recall that $\tilde{b}_\nu$ and $\tilde{a}_\nu$ are even functions of ν. In regions 1, 2, 4 in Fig. 16 the difference between the quantities γ_ν and $\gamma_{3\nu}$ is exponentially small in comparison with either of them. Nevertheless, we shall obtain this difference with the required accuracy. In region 3 the quantity γ_ν is exponentially greater than the quantity $\gamma_{3\nu}$.

The asymptotic behavior of $\gamma_{2\nu}$ in the region crosshatched in Fig. 17 is given by the formula

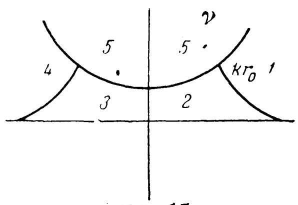

FIG. 15

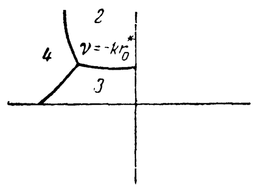

FIG. 16

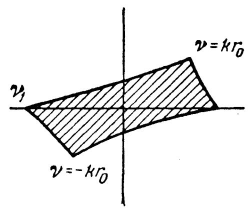

FIG. 17

*tg = tan.

$$\gamma_{2\nu} \approx F(\xi_2)\, e^{2i(r_2, -\infty)}. \qquad (4.11)$$

In regions 1, 2, 4 and those symmetric to them with respect to zero in Fig. 16 where γ_ν and $\gamma_{3\nu}$ almost coincide, $\gamma_{2\nu}$ is small in comparison with these quantities. Since γ_ν does not increase as $|\nu| \to \infty$, it follows from this that $\gamma_{2\nu}$ also has the same property.

5. THE SCATTERING AMPLITUDE IN THE DOMAIN OF NONCLASSICAL ANGLES

We return to the integral I (4.1) and decompose it into two terms:

$$I = I_2 + I_3,$$

$$I_n = \frac{1}{2k} \int_{\Gamma_1} \nu \operatorname{tg} \nu\pi P_{\nu-1/2}(-\cos\theta)\, \gamma_{n\nu}\, d\nu,$$

$$(n = 2, 3). \qquad (5.1)$$

The Legendre function $P_{\nu-\frac{1}{2}}(-\cos\theta)$ is an integral even function. For $|\nu| \gg 1$ the asymptotic behavior of $P_{\nu-\frac{1}{2}}$ has the form (3.5).

We consider first the integral I_2. The integrand in I_2 is proportional to the small exponential $\exp(2i\Phi_2(\nu))$ for values of ν having a large modulus but not lying near the real axis. The function $\Phi_2(\nu)$ has the following form in the region cross-hatched in Fig. 17:

$$\Phi_2(\nu) = (r_2, -\infty) \mp \nu(\pi - \theta)/2. \qquad (5.2)$$

The minus sign corresponds to the upper half-plane of ν and the plus sign corresponds to the lower half-plane. The saddle points $\pm\nu_2$ are determined by the equation:

$$\partial\Phi/\partial\nu\,|_{\nu_2} = 0. \qquad (5.3)$$

In a rough approximation we replace in (5.2) r_2 by r_0 and neglect the potential in p_ν. In this approximation equation (5.3) yields

$$\nu_2 = kr_0 \cos(\theta/2). \qquad (5.4)$$

In the same approximation we obtain

$$\Phi_2(\pm\nu_2) = kr_0 \sin(\theta/2). \qquad (5.5)$$

The effective size of the region of integration is of the order

$$|(\Phi''(\nu_2))^{-1/2}| = |kr_0 \sin(\theta/2)|^{1/2}. \qquad (5.6)$$

The path $\mathscr{P}_2$ over the saddlepoint is determined by the equation

$$\operatorname{Re}[\Phi_2(\nu) - \Phi_2(\nu_2)] = 0. \qquad (5.7)$$

The path $\mathscr{P}_2$ over the saddlepoint has the form shown in Fig. 18 in which for convenience we have also shown the lines of poles of the quantity $\gamma_{2\nu}$. The line P_2 passes through the point ν_2 at an angle $3\pi/4 - \varphi_0/2$ with respect to the real axis ($\varphi_0 = \arg r_0$). In moving from the initial position of Γ_1 (Fig. 14) towards $\mathscr{P}_2$ (Fig. 18) the contour of integration cuts across the poles of the function $\gamma_{2\nu}$ in the complex ν plane. The contributions of these poles are exponentially small compared to the contribution from the neighborhood of the saddlepoint.

Indeed, the poles of the integrand are of the same order of magnitude as the asymptotic form of the integrand near the line of poles (at a distance of order of magnitude unity from it). But along the contour of integration the integrand falls off exponentially with distance from the saddlepoint. From this it follows that the contribution of the pole nearest to the contour is exponentially small in comparison with the contribution from the saddlepoint. The contribution from the other poles is even smaller.

Along the line of steepest descent (5.7) it is possible to approach the real ν axis sufficiently closely. However, in crossing the axis the asymptotic behavior of $P_{\nu-\frac{1}{2}}(-\cos\theta)$ changes. We connect the lines of steepest descent passing through the saddlepoints ν_2 and $-\nu_2$ by the segment MN lying near the real axis and passing through the point $\nu = 0$. Although the segment MN does not satsify the condition of steepest descent (5.7) the values of the integrand along it are everywhere exponentially small in comparison with the value of this function at the saddlepoint.[4)]

The discussion given above does not hold for θ close to π, since in this case the two saddlepoints $\pm\nu_2$ become close. The condition under which the contributions from $\pm\nu_2$ are still independent has the form

$$\pi - \theta \gg |kr_0|^{-1/2}. \qquad (5.8)$$

In the region $\pi - \theta \lesssim |kr_0|^{-1/2}$ the results obtained below are accurate only up to a factor multiplying the exponential.

Evaluation of the integral I_3 is carried out in an analogous fashion, and in the same rough approximation yields the value $\sim \exp(-2ikr_0^* \sin(\theta/2))$ which is of the same order as I_2.

We now evaluate the series of residues from the poles of $S(\nu)$ in formula (3.6). The contribution from the first pole of the unphysical series ν_1 is of order of magnitude $\exp(i\nu_1\theta)$. But $\nu_1 \approx kr_0 \pm i\sqrt{2U_0/E}$, if $(U_0/E)^{3/4} kr_0 \gg 1$. For $(U_0/E)^{3/4} \times kr_0 \gg 1$ the angular range of applicability of for-

4)For example, at the point $\nu = 0$ this function can be easily estimated. It is equal to $\exp(2ikr_0)$ with an accuracy up to a factor multiplying the exponential.

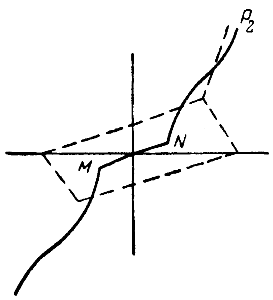

FIG. 18

mulas obtained by the saddle-point method is restricted by the requirement that the interval of integration (5.6) should not contain any poles. From this we obtain

$$\theta \gg (U_0 / E)^{1/4}. \qquad (5.9)$$

In the opposite case the exact location of the poles is unknown. If we assume $\nu_1 \approx kr_0$, then the contributions from the poles of $S(\nu)$ and of the saddle-point integral are of comparable magnitude for

$$\theta \sim (kr_0)^{-1/3}. \qquad (5.10)$$

In the angular region restricted by the inequalities (5.8), (5.9) in the case $(U_0/E)^{3/4} kr_0 \gg 1$, and by (5.8), (5.10) in the case $(U_0/E)^{3/4} kr_0 \lesssim 1$ it is sufficient to take into account only the contribution from the saddle points. We write it out exactly:

$$f(\theta) = \tfrac{1}{2} r_0 F(-iR / k \sin(\theta/2)) \times \exp\{2i(r_2, -\infty) - \nu_2(\pi - \theta)\} - \tfrac{1}{2} r_0^* F(iR^* / k \sin(\theta/2)) \times \exp\{2i(r_3, -\infty) - \nu_3(\pi - \theta)\}. \qquad (5.11)$$

In the limiting case $U_0|kr_0|/E \ll 1$ formula (5.11) goes over into the result of the Born approximation (1.1):

$$f(\theta) \approx \pi q^{-1}(r_0 R e^{iqr_0} + r_0^* R^* e^{-iqr_0^*}). \qquad (5.12)$$

In the other limiting case $U_0|kr_0|/E \gg 1$ it is necessary to substitute into formula (5.11) the values of kr_2, kr_3, ν_2, ν_3, evaluated with an accuracy such that the error in the index of the exponentials would be considerably less than unity.

In the case $U_0 kr_0/E \sim 1$ the quantities r_2, r_3, ν_2, ν_3 can be found with the required accuracy explicitly, and formula (5.11) assumes in this case the following form:

$$f(\theta) = \frac{r_0}{2} F\left(-\frac{iR}{k\sin(\theta/2)}\right) + \exp\left\{2ikr_0 \sin\frac{\theta}{2} - i\int_{+\infty}^{r_2} \frac{U\,dr}{\sqrt{1 - r_0^2\cos^2(\theta/2)/r^2}} + \frac{iR}{k\sin(\theta/2)}\left(\frac{1}{2} + \ln 2\right)\right\} - \frac{r_0^*}{2} F\left(\frac{iR}{k\sin(\theta/2)}\right)\exp\left\{-2ikr_0^*\sin\frac{\theta}{2} - i\int_{-\infty}^{r_3}\frac{U\,dr}{\sqrt{1 - r_0^{*2}\cos^2(\theta/2)/r^2}} - \frac{iR^*}{k\sin(\theta/2)}\left(\frac{1}{2} + \ln 2\right)\right\}. \qquad (5.13)$$

In the derivation of formula (5.13) some simplifying circumstances have been utilized. First of all, the accuracy in the determination of ν_2, ν_3 need not exceed the magnitude of $\sqrt{kr_0}$. Therefore, we can restrict ourselves to the zero order approximation (5.4). Secondly,

$$\int_{-\infty}^{r_3} \frac{U\,dr}{\sqrt{1 - \nu_3^2/r^2}} = \int_{+\infty}^{r_3} \frac{U\,dr}{\sqrt{1 - \nu_3^2/r^2}} \qquad (5.14)$$

in view of the parity of $U(r)$ and of the relation

$$\int_{-r}^{r} \frac{U\,dr}{\sqrt{1 - \nu^2/r^2}} = 0. \qquad (5.15)$$

We note that the integral (5.14) is of order $U_0 E^{-1} r_0 \ln(U_0/E)$. We can easily give a physical interpretation of the first two terms in the indices of the exponentials in (5.13). For this it is sufficient to note that

$$2kr_0 \sin(\theta/2) = qr_0, \qquad (5.16)$$

$$\int_{+\infty}^{r_2} \frac{U\,dr}{\sqrt{1 - r_0^2\cos^2(\theta/2)/r^2}} = \int_{-\infty}^{z_2} U\,dz, \qquad (5.17)$$

where $q = |\mathbf{k}_2 - \mathbf{k}_1|$. The integral on the right hand side of (5.17) is taken along a straight line parallel to the z axis (direction of motion of the incident beam) in three-dimensional space. Thus, these terms represent the first terms of a formal expansion in powers of U/E of action corresponding to motion along a rectilinear trajectory. Formula (5.13) shows that in the nonclassical region scattering occurs as if it were caused by a hard sphere of complex radius r_0.

From (5.13) it can be seen that for the calculation of the factor in front of the exponential it is not possible to restrict oneself, as has been done in the paper by Schiff and Saxon[1], to the first powers in the expansion of action in powers of U/E, since in doing this no account is taken of the terms $iR(\frac{1}{2} + \ln 2)/k\sin(\theta/2)$, and the function $F(\xi)$ is replaced by its argument. This is valid only in the limiting Born case $|Ukr_0/E| \ll 1$.

We quote the formula for the effective scattering cross section for the case of the Fermi potential:

$$U(r) = U_0\left[(e^{(r-\rho)/a} + 1)^{-1} + (e^{-(r+\rho)/a} + 1)^{-1}\right]. \qquad (5.18)$$

Here ρ defines the size of the scattering center, a is the size of the region over which the boundary is smeared out. The calculations have been carried out on the assumption $\rho \gg a$. The condition for a quasiclassic situation has in the present case the form $ka \gg 1$. The result has the form

$$d\sigma = 2\rho^2 e^{-4\pi ka \sin(\theta/2)+\pi y} \operatorname{sh}^2(\pi y/2) \times \left\{1 + e^{\pi y}\left(4k\rho - 2y \ln\frac{ka}{e^2 \sin(\theta/2)} + 2\varphi(y)\right)\right\}, \quad (5.19)^*$$

where

$$y = U_0 ka/E \sin(\theta/2), \qquad \varphi(y) = \arg\Gamma(iy/2). \quad (5.20)$$

The authors consider it their pleasant duty to express their sincere gratitude to L. D. Landau for numerous discussions during 1958—1959.

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*sh = sinh.

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