THE ENERGY OF ACCIDENTAL MOTIONS IN THE EXPANDING UNIVERSE

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A differential equation is set up which connects the kinetic energy of accidental motions with a suitably defined change of the gravitational energy due to inhomogeneities of the density. An inequality is obtained which gives an upper limit on the energy of accidental motions developed owing to gravitational instability.

1. STATEMENT OF THE PROBLEM

The observed universe differs from the isotropic and homogeneous cosmological model developed by A. A. Fridman in two respects: 1) the actual density distribution is not uniform, $\rho(r, t) \neq \bar{\rho}(t)$, and 2) the actual velocity distribution of the matter differs from Hubble's law $u = Hr$. The velocity of accidental (or peculiar) motion is defined as the difference $v = u - Hr$.

It is well known that in the Fridman model the momentum of an individual particle which has the accidental velocity v decreases with time[1)] in inverse proportion to the increase of the radius R of the universe: $|p| = \text{const}/R$. This theorem is sometimes applied to the accidental motions of galaxies.[1] Present accidental velocities are of the order of 200 km/sec. It is assumed that at the time t_0 of the formation of the galaxies the mean density of matter in the universe was equal to the present mean intragalactic density, $\bar{\rho}(t_0) \approx 10^{-25}$ g cm^{-3}, whereas at the present time $\bar{\rho}(t_1) \approx 10^{-30}$ g cm^{-3}. Consequently

$$R(t_0) = R(t_1)\,[\bar{\rho}(t_1)/\bar{\rho}(t_0)]^{1/3} = 0.02R(t_1).$$

The conclusion is drawn from this that at the time of formation of the galaxies the accidental velocity was 200/0.02 = 10,000 km/sec. Such an accidental velocity is clearly too large; this estimate is regarded by Hoyle as a difficulty in the application of the Fridman theory to the actually observed universe.

An obvious error in the argument is that the change of the accidental velocity is treated as if each individual particle moves in the gravitational field of matter uniformly distributed in space. Actually the nonuniform distribution of density has a decided effect on the law according to which the velocity changes.

As a popular example we can consider the revolution of the earth around the sun; obviously the expansion of the universe does not diminish this velocity. The same applies to any stationary system of bodies in which the mean distances remain constant. Another and more instructive example is that of small perturbations of the density and velocity imposed on the homogeneous isotropic model. In this case, as E. M. Lifshitz has shown,[2] there are two solutions: an increasing disturbance,

$$\delta\rho/\bar{\rho} \sim t^{2/3}, \qquad v \sim t^{1/3},$$

and a decreasing disturbance,

$$\delta\rho/\bar{\rho} \sim t^{-1}, \qquad v \sim t^{-4/3}.$$

Consequently, in the expanding universe decreasing velocities of accidental motion are not the only possibility.

For a definite relation between the phase of the accidental velocity and that of the density perturbation the velocity increases during the expansion. An obvious cause of this is the gravitational instability of homogeneous matter expanding strictly according to Hubble's law. A deviation from homogeneity is accompanied by a decrease of the gravitational energy, which is partly converted into kinetic energy of the random motions.

The problem of the present paper is to obtain a general relation between the kinetic energy of the random motions and a quantity which is to characterize the deviation of the actual density distribution from uniformity, without confining ourselves to small perturbations. Such a relation can be of interest in connection with the hypothesis that at an early stage of evolution the universe exactly satisfied the equations of the Fridman model. Suppose that then some small perturbations (thermodynamic

[1)]The red shift of the spectra of distant objects can be regarded as a special case in which the theorem is applied to light quanta.

fluctuations, or phenomena at phase transitions,[3][2]) or some other causes) produced small deviations from uniformity and caused small accidental velocities. We assume that after this all processes developed through the action of gravitational forces alone. Then there must be a definite relation between the present energy of random motions and the inhomogeneities of the density distribution which are now observed. Unfortunately, the present state of the observational data scarcely makes it possible to come to definite conclusions. Moreover, besides the gravitational forces, the release of thermonuclear energy in stars may also have some effect. Therefore, making a sober estimate of the significance of our results, we must suppose that they are mainly of a methodological and negative character: the only thing shown clearly is that the observed accidental velocities cannot be regarded as a refutation of the applicability of the Fridman model in the past.

2. THE RESULT, AND LIMITING CASES

Our main result can be written in the following form:

$$\frac{1}{R}\frac{d}{dt}[R(T+F)] = -T\frac{1}{R}\frac{dR}{dt}. \tag{1}$$

Here R is the radius of the universe and T is the kinetic energy of the accidental motions:

$$T = \int \tfrac{1}{2}\rho v^2\, dV.$$

The quantity F is given by the expression

$$F = -\frac{\varkappa}{2}\iint \frac{(\rho_1-\bar\rho)(\rho_2-\bar\rho)}{r_{12}}\, dV_1\, dV_2,$$

$$\rho_1 = \rho(r_1, t) \qquad \rho_2 = \rho(r_2, t), \qquad \bar\rho = \bar\rho(t),$$

$$r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|, \quad dV_1 = d^3\mathbf{r}_1, \qquad dV_2 = d^3\mathbf{r}_2,$$

where κ is the Newtonian gravitational constant.

This result refers to a state which is homogeneous and isotropic, but only on the average, on a sufficiently large scale L. Therefore when the integration is over a volume $V \gg L^3$ the quantities T and F are both proportional to V. For the quantity T this is immediately obvious; for the quantity F we must note that at distances $r_{12} > L$ there is no correlation between $\rho_1 - \bar\rho$ and $\rho_2 - \bar\rho$. Fixing the point r_1, we note that for $r_{12} > L$ the quantity $\rho_2 - \bar\rho$ is of varying sign and is zero over a large volume, so that there is no contribution to the integral from $r_{12} > L$. Consequently, F is in order of magnitude

$$F \approx \varkappa \int (\rho_1 - \bar\rho)^2 L^2\, dV_1,$$

and it is seen that F is also proportional to the volume of integration V.

In considering the change of T and F with time, we suppose that the volume of integration always contains the same quantity of matter, $M = \rho V$, so that it follows that $V = \text{const}\cdot R^3$, and the volume V expands along with the general Hubble expansion. In this sense we can speak of the specific values (per unit mass) $T_1 = T/M$ and $F_1 = F/M$, which differ from T and F only by a constant factor.

The quantities T_1 and F_1 are "local" quantities on scales larger than L. The quantity R is conceptually connected with ideas about the curvature of space, which are characteristic features of the general theory of relativity. Actually, however, our theorem is local in nature, and we can eliminate R. To do so we note that $d\ln R/dt = H$, where H, the Hubble constant, is a quantity definable locally in terms of the distribution of the mean velocity near the volume considered. Thus we can rewrite the theorem in terms of local quantities:

$$d(T_1 + F_1)/dt + H(T_1 + F_1) = -HT_1.$$

Let us consider various limiting cases.

1) Uniform density, $\rho = \bar\rho$, $F = 0$. We find

$$dT_1/dt = -2HT_1 = -2T_1 d\ln R/dt.$$

From this we have at once

$$T_1 = \text{const}/R^2,$$

corresponding to the fact that in this case the accidental velocity falls off as 1/R.

2) The matter has divided into clusters of constant volume which occupy a small part of space and do not expand with the general expansion (only the distance between clusters increases). In this case $F_1 \approx \text{const} \approx -m/r$, where m is the mass of a cluster and r is its radius;

$$dT_1/dt = -2HT_1 - HF_1.$$

T_1 tends asymptotically to the limit

$$T_1 = -\tfrac{1}{2}F_1.$$

This result is the virial theorem for the individual clusters.

3) The theorem holds in the theory of small perturbations. The formulas given in Section 1 refer to a flat world, i.e., to the case $R \sim t^{2/3}$, $H = 2/3t$, where t is measured from the instant of infinite density. The mean density is $\bar\rho = (6\pi\kappa t^2)^{-1}$. In the increasing disturbances

2)We note that according to subsequent calculations we have made the phase transitions have less effect than was assumed in [3].

$$v \sim t^{1/3}, \qquad T \sim t^{2/3},$$
$$\delta\rho/\bar{\rho} \sim t^{2/3}, \qquad \delta\rho \sim t^{2/3}t^{-2} = t^{-4/3},$$
$$L \sim t^{2/3}, \qquad F \sim (\delta\rho)^2 L^2 V \sim t^{-8/3}t^{4/3}t^2 = t^{2/3}.$$

Substituting in Eq. (1), we get

$$T_1 = -{}^2/{}_3 F_1.$$

In a similar way we have for the decreasing disturbances

$$T_1 \sim F_1 \sim t^{-8/3}, \qquad T_1 = -{}^3/{}_2 F_1.$$

4) When perturbations appear rapidly

$$T_1 + F_1 = \text{const.}$$

Thus the theorem includes all of the special cases.

The main cosmological conclusion drawn from this is: if there was a uniform state with $T = 0$, $F = 0$, and then some small influences disturbed the uniformity, the only state that can arise under the action of gravitation alone is one in which the kinetic energy does not exceed a definite limit: $T_1 \leq -F_1$. If an analysis of the observations shows that this inequality is not satisfied, and it is shown that the large kinetic energy did not come from the nuclear energy of reactions in stars—then and only then will it be necessary to renounce the Fridman model for the early stage of the evolution of the universe.

3. PROOF

The proof given below is based on the consideration of a large ($r \gg L$) spherical volume in the framework of classical mechanics and the Newtonian theory of gravitation.

As is well known, this statement of the problem gives correctly all of the local properties of the Fridman model; the results for $\bar{\rho}(t)$ and $H(t)$ are exactly equal to the results of the general theory of relativity. As Bonnor has shown,[4] this is also true for the development of perturbations, which in this model agree with the calculations of E. M. Lifshitz. With this statement of the problem there is no trace of the so-called gravitational paradox,[3)] and when we let the radius of the sphere become infinite, $r \to \infty$, we get a quite definite, finite answer for all local quantities, and in particular for our quantities F_1 and T_1.

[3)]In dealing with this it is precisely necessary and sufficient to consider a sphere; the case of a body of arbitrary shape does not satisfy the requirement that the tensor of the second derivatives of the gravitational potential be isotropic, $\partial^2\phi/\partial x^i \partial x^k = \text{const} \cdot \delta_{ik}$ (in rectangular coordinates).

We note that with the classical approach the kinetic energy of the Hubble motion and the potential energy of the entire sphere, referred to unit mass, increase as r^2 and do not approach a definite limit for $r \to \infty$. Nevertheless, against this background we can obtain a relation between the kinetic energy of the accidental motions and the potential energy of the deviation from uniformity.

Let $\mathbf{r}_i$ and m_i be the radius vectors and the masses of the individual material points which make up the system (galaxies, say), and let $R(t)$ be the characteristic dimension of the system. We introduce relative (fractional) coordinates ξ_i by means of the formulas

$$\mathbf{r}_i = R(t)\,\xi_i,$$
$$\frac{d\mathbf{r}_i}{dt} = R\frac{d\xi_i}{dt} + \frac{dR}{dt}\xi_i = R\frac{d\xi_i}{dt} + HR\xi_i = R\frac{d\xi_i}{dt} + H\mathbf{r}_i,$$

so that the accidental velocity is written

$$\mathbf{v}_i = d\mathbf{r}_i/dt - H\mathbf{r}_i = R\,d\xi_i/dt.$$

The equations of motion are

$$\frac{d^2\mathbf{r}_i}{dt^2} = R\frac{d^2\xi_i}{dt^2} + 2\frac{dR}{dt}\frac{d\xi_i}{dt} + \frac{d^2R}{dt^2}\xi_i = \operatorname{grad}_{\mathbf{r}_i}\sum_{j \neq i}\frac{\varkappa m_j}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

The last term on the left side is equal to the acceleration of the point in the pure Fridman motion, i.e., under the action of the attraction of a sphere of radius R and the uniform density $\bar{\rho}(t)$. Transposing it to the right side, we have

$$R\frac{d^2\xi_i}{dt^2} + 2\frac{dR}{dt}\frac{d\xi_i}{dt} = \operatorname{grad}_{\mathbf{r}_i}\Big(\sum_{j \neq i}\frac{\varkappa m_j}{|\mathbf{r}_i - \mathbf{r}_j|} - \int_0^R \frac{\varkappa\bar{\rho}\,d^3\bar{\mathbf{r}}}{|\mathbf{r}_i - \bar{\mathbf{r}}|}\Big).$$

When we now introduce the relative coordinates on the right side, we get

$$\frac{1}{R^2}\operatorname{grad}_{\xi_i}\Big(\sum_{j \neq i}\frac{\varkappa m_j}{|\xi_i - \xi_j|} - \varkappa\bar{\rho}R^3\int_0^1 \frac{d^3\xi}{|\xi_i - \xi|}\Big)$$

an expression in which the time appears explicitly only in the factor $1/R^2$, since $\bar{\rho}R^3$ is a constant. Multiplying the equation by R^2 and by $m_i d\xi_i/dt$ and summing over i, we get our theorem, in analogy with the derivation of the ordinary integral of energy:

$$R^3\frac{d}{dt}\Big[\sum\frac{1}{2}m_i\Big(\frac{d\xi_i}{dt}\Big)^2\Big] + 4R^2\frac{dR}{dt}\sum\frac{1}{2}m_i\Big(\frac{d\xi_i}{dt}\Big)^2$$
$$= R\frac{d}{dt}\Big[\sum\frac{1}{2}m_iR^2\Big(\frac{d\xi_i}{dt}\Big)^2\Big] + 2\frac{dR}{dt}\sum\frac{1}{2}m_iR^2\Big(\frac{d\xi_i}{dt}\Big)^2$$
$$= R\frac{dT}{dt} + 2\frac{dR}{dt}T$$
$$= \frac{d}{dt}\Big[\sum_{\substack{(i,j)\\ i \neq j}}\frac{\varkappa m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} - \varkappa\bar{\rho}R^3\sum_i m_i\int\frac{d^3\xi}{|\xi_i - \xi|}\Big],$$

where the double sum is taken over all possible pairs (i, j).

When we add to the quantity whose derivative is taken a constant—the potential from the action of the sphere of radius R on itself—this quantity becomes $-F$, calculated in the relative coordinates. In fact, when we go over to the continuous way of writing the expressions, we get

$$\sum_{(i,j)} \frac{\varkappa m_i m_j}{|\xi_i - \xi_j|} = \frac{1}{2}\varkappa \iint \frac{\rho_1 \rho_2}{r_{12}/R} dV_1 dV_2 = \frac{1}{2}\varkappa R \iint \frac{\rho_1 \rho_2}{r_{12}} dV_1 dV_2,$$

$$\varkappa \bar{\rho} R^3 \sum m_i \int \frac{d^3\xi}{|\xi_i - \xi|} = \varkappa R \iint \frac{\rho_1 \bar{\rho}\, dV_2\, dV_1}{r_{12}}.$$

Finally, the constant we have mentioned as added to the quantity whose derivative is taken is

$$\frac{1}{2}\varkappa R \iint \frac{\bar{\rho}^2}{r_{12}} dV_1 dV_2.$$

Collecting, we get on the right hand side

$$\frac{d}{dt}\frac{1}{2}\varkappa R \iint \frac{\rho_1\rho_2 - 2\rho_1\bar{\rho} + \bar{\rho}^2}{r_{12}} dV_1 dV_2.$$

We finally have

$$R\frac{dT}{dt} + 2\frac{dR}{dt}T = -\frac{d}{dt}RF,$$

that is

$$\frac{d}{dt}R(T+F) + \frac{dR}{dt}T = 0,$$

as was to be proved.

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[1] F. Hoyle, La Structure de l'Univers (11th Solvay Congress), Brussels, 1958, page 66.

[2] E. M. Lifshitz, JETP **16**, 587 (1946).

[3] Ya. B. Zel'dovich, JETP **43**, 1982 (1962), Soviet Phys. JETP **16**, 1395 (1963).

[4] W. B. Bonnor, Monthly Notices Roy. Astr. Soc. **117**, 104 (1957).

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