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ON THE RADIATIVE ATTENUATION THEORY OF RADIAL OSCILLATIONS OF AN ELECTRON IN A MAGNETIC FIELD

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It is demonstrated that the formulas for the radiative attenuation decrements of an electron in an inhomogensous magnetic field are the same when calculated by quantum or classical theory. The proof is performed for a nonrelativistic electron in a field with weak inhomogeneities with an accuracy to terms of the order of v^2/c^2.

As is well known, a contradiction exists between the classical [1] and quantum [2] theories of the damping of electron oscillations in a magnetic field. According to quantum theory [2] the damping of quantum oscillations, with an account of radiation, is due only to the so-called adiabatic damping, which is connected with the change in the field H and in the gradient $\partial H/\partial R$ on the equilibrium orbit ($r^2 \sim H^{-1}(1-n)^{-1/2}$, $n = -(\partial H/\partial R) \times (R/H)$). According to classical theory, the damping decrements have specific radiation terms. It must be noted that Sokolov and Ternov [3] obtained no radiation damping even in the quasiclassical theory. The question of the existence of radiation damping is of primary significance for the construction of large electron accelerators and storage rings. Therefore the existing contradiction makes it necessary to return each time to this problem [4].

It must be noted that a quantum calculation of the damping of the oscillations is not needed for those real, rather complicated, magnetic field structures encountered in accelerators, provided we show in principle that the results of the classical and quantum theories are in agreement. Inasmuch as the existing contradiction pertains even to the simplest case of an azimuthally-symmetrical field, it is sufficient to consider just this case. Moreover, to attain full clarity, it is best to carry out the proof with a very simple example. In the present paper we show that the results of both theories agree for nonrelativistic particles (accurate to terms $\sim \beta^2 = v^2/c^2$, which corresponds to the account of the dipole and quadrupole radiation) in a field with weak inhomogeneity.

So far we have considered the damping of oscillations of ultrarelativistic particles only. We have therefore carried out a classical calculation of the damping for particles with arbitrary energy, which can be carried out by two essentially different methods.

If we reckon the transverse deviations of the particles from a fixed coordinate curve that does not vary with the particle energy, then the damping decrement can be calculated by merely solving the equations of motion in the linear approximation with respect to small oscillations. In this case, however, the equations for the radial and phase oscillations turn out to be coupled and must be solved simultaneously. On the other hand, if the oscillations are reckoned from the instantaneous equilibrium orbit, the radius of which is in agreement with the instantaneous value of the momentum, then we do not need the equation for the phase oscillations in order to find the damping decrement of the radial oscillations. It is necessary, however, and most important, to take account of the terms that are quadratic in r and dr/dt in the radial oscillations. Failure to take these terms into account is indeed the source of the errors made by Sokolov and Ternov [3]. The same error is also essentially manifest in the quantum calculation [2], since the "parabolization" of the potential energy of the Schrödinger equation which was carried out there corresponds to an incomplete account of the quadratic terms in the classical calculations.

A rigorous classical calculation gives the following damping decrements γ_R and γ_Φ for the radial and phase oscillations [$r^2 \sim \exp(-\int\gamma_R dt)$, $\Phi^2 \sim \exp(-\int\gamma_\Phi dt)$]:

$$\gamma_R = \frac{2e^4H_0^2}{3m^3c^5}\frac{E}{mc^2}\frac{[1-(mc^2/E)^2]\,n + n^2(mc^2/E)^2}{1-n}, \quad (1)$$

$$\gamma_\Phi = \frac{2e^4H_0^2}{3m^3c^5}\frac{E}{mc^2}\frac{3-(mc^2/E)^2 - n\,[4-(mc^2/E)^2]}{1-n}, \quad (2)$$

where H_0 is the field on the equilibrium orbit. In the ultrarelativistic case $E/mc^2 \gg 1$ we obtain from this the known formulas of [1,5].

In the nonrelativistic $E \approx mc^2$, $\beta = v/c \ll 1$, the decrements have a different form

$$\gamma_R = \frac{2e^4H_0^2}{3m^3c^5}\frac{n^2}{1-n}, \qquad \beta \ll 1, \tag{1a}$$

$$\gamma_\Phi = \frac{2e^4H_0^2}{3m^3c^5}\frac{2-3n}{1-n}, \qquad \beta \ll 1. \tag{2a}$$

For comparison with the quantum calculation we give also the formula for γ_R in the case when $n \ll 1$ and $\beta \ll 1$ (accurate to terms n^2 and $n^2\beta^2$):

$$\gamma_R = \frac{2e^4H_0^2}{3m^3c^5}(n\beta^2 + n^2), \qquad n \ll 1, \qquad \beta \ll 1 \tag{1b}$$

Going over to the quantum calculation, we consider the motion of the particle in the $z = 0$ plane and in an inhomogeneous magnetic field

$$H_z = H + gR^2. \tag{3}$$

The vector potential of such a field can be chosen in cylindrical coordinates R, φ, and z in the form

$$A_R = A_z = 0, \qquad A_\varphi = \tfrac{1}{2}HR + \tfrac{1}{4}gR^3. \tag{4}$$

The independence of the potential (4) of the coordinate φ enables us to separate the variables in the wave equation:

$$\psi(R, \varphi) = (2\pi)^{-1/2}e^{il\varphi}u(R), \qquad l = 0, 1, 2 \ldots, \tag{5}$$

where l —orbital quantum number.

The Klein-Gordon equation for the radial part of the wave function of a state with energy E has in our case the form

$$\frac{d^2u}{dR^2} + \frac{1}{R}\frac{du}{dR} + u\left[\frac{E^2 - m^2c^4 - ec\hbar l(H + gR^2/2) - e^2(HR/2 + gR^3/4)^2}{c^2\hbar^2} - \frac{l^2}{R^2}\right] = 0. \tag{6}$$

The terms proportional to g in this equation do not make it possible to solve it accurately. We shall assume henceforth that the inhomogeneity is weak, that is, it satisfies the condition

$$gR^2 \ll H. \tag{7}$$

For a magnetic field with weak inhomogeneity, the wave function of the particle and the eigenvalues of its energy can be sought in the form of series in powers of a small parameter $g_0 \equiv g\hbar c/eH^2$. For our problem [quantum derivation of (1b)] it is sufficient to retain the terms linear in g_0 (see below). The wave function has in this perturbation-theory approximation the form

$$\begin{aligned}\psi_{l,s}(\varphi, y) &= \psi_{l,s}^{(0)}(\varphi, y) \\ &+ \tfrac{1}{2}g_0\Big\{(3l + 4s + 4)\sqrt{(s+1)(l+s+1)}\,\psi_{l,s+1}^{(0)} \\ &- (3l + 4s)\sqrt{s(l+s)}\,\psi_{l,s-1}^{(0)} \\ &+ \tfrac{1}{2}\sqrt{s(s-1)(l+s)(l+s-1)}\,\psi_{l,s-2}^{(0)} \\ &- \tfrac{1}{2}\sqrt{(s+1)(s+2)(l+s+1)(l+s+2)}\,\psi_{l,s+2}^{(0)}\Big\},\end{aligned} \tag{8}$$

where

$$\psi_{l,s}^{(0)} = \frac{1}{\sqrt{2\pi}}e^{il\varphi}\left(\frac{(l+s)!}{s!\,l!^2}\right)^{1/2}e^{-y/2}y^{l/2}F(-s, l+1, y), \tag{9}$$

$$y = R^2eH/2c\hbar, \tag{10}$$

s —radial quantum number, $F(-s, l+1, y)$ —confluent hypergeometric function.

The eigenvalues of the energy are

$$E_{l,s} = \Big\{m^2c^4 + 2c\hbar eH\Big[l + s + \tfrac{1}{2} + g_0(l^2 + 4ls + 3s^2 + 2l + 3s + 1)\Big]\Big\}^{1/2}. \tag{11}$$

The zeroth-approximation function (9) is so normalized that

$$\int_0^{2\pi}\int_0^\infty \psi_{L\Sigma}^*(\varphi, y)\,\psi_{l,s}(\varphi, y)\,dy\,d\varphi = \delta_{Ll}\delta_{\Sigma s}. \tag{12}$$

The square of the amplitude of the radial oscillations in the state l, s, obtained with the aid of the functions (8), is equal to

$$\begin{aligned}A_{l,s}^2 &= 2(\overline{R^2} - \bar{R}^2) = (c\hbar/eH)\,[1 - 4g_0(l+s)]\,s \\ &= (c\hbar/eH)(1 + n)\,s.\end{aligned} \tag{13}$$

In the last equation we have made use of the fact that

$$n = -(R/H)\,\partial H/\partial R = -2g\bar{R}^2/H = -4g_0(l+s). \tag{14}$$

Formula (13) coincides with the analogous expression of Sokolov and Ternov [2] and corresponds to classical adiabatic damping, which arises in the case of slow variation of H and n; the quantum number s remains constant in this case:

$$A^2 \sim 1/H_z(R)\sqrt{1-n} \approx 1/H(1 - n/2)\sqrt{1-n} \approx 1 + n. \tag{15}$$

Radiation causes changes in l and s; however, the change in l due to radiation is fully offset by the adiabatic damping connected with the accelerating electric field. This is seen from the fact that the energy and the radius of the equilibrium particle, which depend precisely on the value of l, remain in the mean unchanged when $s = 0$. It is precisely for this reason that only the terms linear in g_0 play any role in the expansion of $\psi_{l,s}$ in powers of g_0.

Indeed, the expression of A^2_{ls}, calculated with account of g_0^2, has again a form analogous to (13):

$$A^2_{l,s} = (c\hbar/eH)\,(1 - 4g_0 l)\, s\left(1 + O_l\,(g_0^2) + O\left(\frac{s}{l}\right)\right), \qquad \text{(13a)}$$

where the correction $O_l(g_0^2)$ depends only on l. The additional radiative damping is connected only with the change in s during radiation; it therefore follows from (13a) that

$$\gamma_R \equiv -\frac{1}{A^2}\left(\frac{dA^2}{dt}\right)_{\text{рад}} = -\frac{1}{s}\frac{\overline{ds}}{dt} = \frac{1}{s}\sum_{\Sigma}(s - \Sigma)W_{s\to\Sigma}\,. \qquad \text{(16)}$$

Here $W_{s\to\Sigma}$ is the probability per unit time of radiation with transition from the state s to the state Σ. This probability can be obtained from the formula[6]:

$$W_{l\to L,\, s\to\Sigma} = \frac{e^2 c\Delta E}{2\pi E^2} \times \sum_{\lambda=1}^{2}\int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\theta\, d\Phi \left|\int d\mathbf{r}\psi^*_{L\Sigma} e^{-i\mathbf{kr}}\boldsymbol{\eta}_\lambda \left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right)\psi_{ls}\right|^2, \qquad \text{(17)}$$

where $\Delta E = E_{ls} - E_{L\Sigma}$, $\mathbf{k}$ and η_λ —wave vector and unit polarization vector of the quantum, $\mathbf{k}\cdot\eta_\lambda = 0$; the spherical components of the vector $\mathbf{k}$ are $\Delta E/\hbar c$, θ, and Φ.

For a nonrelativistic particle, the distances essential for the integrals in (17) satisfy the inequality

$$kr \approx \beta \ll 1. \qquad \text{(18)}$$

In order to obtain the results with accuracy to β^2 inclusive, we must retain the first three terms in the expansion of $\exp(-i\mathbf{k}\cdot\mathbf{r})$

$$e^{-i\mathbf{kr}} = 1 - i\mathbf{kr} - \tfrac{1}{2}(\mathbf{kr})^2. \qquad \text{(19)}$$

We shall retain henceforth only terms of order g_0^2 and $g_0\beta^2$ in the expressions for the transition probability. In this approximation the only nonvanishing terms of the sum determining the damping decrement (16) are

$$\gamma_R = s^{-1}\,(W_{l\to l,\, s\to s-1} + W_{l\to l-1,\, s\to s-1} - W_{l\to l-2,\, s\to s+1}). \qquad \text{(20)}$$

After integration over the angles of the quantum and summing over the polarizations, we obtain the following expression for the damping decrement:

$$\begin{aligned}\gamma_R = {} & \frac{2}{3}\frac{e^3 H\Delta E}{sE^2\hbar}\left|\left(2\sqrt{y}\frac{\partial}{\partial y} + \frac{l}{\sqrt{y}} + \sqrt{y} + g_0 y^{3/2}\right)^{l-1,\, s-1}_{ls}\right|^2 \\ & + \frac{e^2(\Delta E)^3}{sE^2\hbar^2 c}\left\{\frac{1}{15}\left|\left(2y\frac{\partial}{\partial y}\right)^{ls-1}_{ls}\right|^2 + \frac{1}{3}\left|(l + y + g_0 y^2)^{ls-1}_{ls}\right|^2\right. \\ & - \frac{1}{10}\left|\left(2y\frac{\partial}{\partial y} + l + y + g_0 y^2\right)^{l-2,\, s+1}_{ls}\right|^2 \\ & \cdot -\frac{32}{15}\left(2\sqrt{y}\frac{\partial}{\partial y} + \frac{l}{\sqrt{y}} + \sqrt{y} + g_0 y^{3/2}\right)^{l-1,\, s-1}_{ls} \\ & \left.\times\left(y^{3/2}\frac{\partial}{\partial y} + l\sqrt{y} + y^{3/2} + g_0 y^{5/2}\right)^{l-1,s-1}_{ls}\right\} \end{aligned} \qquad \text{(21)}$$

After rather cumbersome algebraic calculations we obtain ultimately

$$\gamma_R = \tfrac{2}{3}\,(e^4 H^2/m^3 c^5)\,(n^2 + n\beta^2). \qquad \text{(22)}$$

The formula obtained coincides with the result of the expansion of the classical expression (1) in powers of $n \ll 1$. The first term of (22) (dipole approximation) was obtained in a paper by the authors[7].

We can therefore assume that the classical and quantum theories of damping lead to the same radiative-damping decrement, and the previously existing discrepancy between them was the result of an insufficient account of all the required terms.

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