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# NONLINEAR EFFECTS IN PLASMA RESONANCE

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The penetration of a longitudinal field having a frequency close to the Langmuir frequency into a plasma is examined by taking into account nonlinear effects. It is demonstrated that hysteresis of the field amplitude in the plasma occurs, that is, two values of the field in the plasma may correspond to a single value of the field strength at the boundary. An equation defining the field distribution in the plasma and resembling the Ginzburg-Landau equation in superconductivity theory is obtained.

## 1. DISTRIBUTION OF PARTICLES IN THE FIELD

THIS paper is devoted to a theoretical investigation of the nonlinear phenomena which occur, in the absence of collisions, in a plasma situated in a high-frequency magnetic field. We consider in greater detail the case when the field frequency is close to the plasma resonance frequency $\omega_0$.

We first calculate the time average of the perturbation of the charged-particle density under the influence of a specified high frequency field 1). The field distribution will be discussed later.

In order to find the perturbation, it is necessary to know the average force acting on the particle in the field. To this end it is simplest to use the results of one of the authors[2], who has shown that a unit volume of the medium situated in an inhomogeneous alternating magnetic field of frequency $\omega$

$$\mathbf{E} = \mathbf{E}(\mathbf{r})\, e^{-i\omega t},$$

is acted upon, in the absence of absorption, by a force

$$\mathbf{f} = \frac{1}{16\pi}\,\mathrm{grad}\left(E_i^* E_k \frac{\partial \varepsilon_{ik}}{\partial n} n\right) - \frac{E_i^* E_k}{16\pi}\,\mathrm{grad}\,\varepsilon_{ik}, \qquad (1)$$

where $\epsilon_{ik}$ is the dielectric constant of the medium 2). The derivative $\partial\epsilon_{ik}/\partial n$ can be transformed by recognizing that the tensor $(\epsilon_{ik} - \delta_{ik})/4\pi$ is the polarizability tensor, which in the case of a plasma is proportional to the electron concentration n.

Thus, in the absence of a magnetic field we have

$$\varepsilon_{ik} = \delta_{ik}\,(1 - 4\pi n e^2/m\omega^2),$$

so that

$$\alpha_{ik} = (\varepsilon_{ik} - \delta_{ik})/4\pi = -e^2\delta_{ik} n/m\omega^2. \qquad (2)$$

In a magnetic field the tensor components $\epsilon_{ik}$ are, as is well known (the x axis is along the magnetic field),

$$\varepsilon_{xx} = 1 - 4\pi e^2 n/m\omega^2, \quad \varepsilon_{yy} = \varepsilon_{yz} = 1 - 4\pi e^2 n/m\,(\omega^2 - \omega_H^2),$$

$$\varepsilon_{yz} = -\varepsilon_{zy} = -i\,(4\pi e^2/m)\, n\omega_H/\omega(\omega^2 - \omega_H^2),$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \varepsilon_{xz} = \varepsilon_{zx} = 0,$$

$$(\omega_H = eH/mc).$$

In this case $\alpha_{ik}$ can be represented in the form

$$\alpha_{ik} = -e^2 n\beta_{ik}/m\omega^2, \qquad (3)$$

$$\beta_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\omega^2}{\omega^2-\omega_H^2} & i\dfrac{\omega_H\omega}{\omega^2-\omega_H^2} \\ 0 & -i\dfrac{\omega_H\omega}{\omega^2-\omega_H^2} & \dfrac{\omega^2}{\omega^2-\omega_H^2} \end{pmatrix}.$$

It is seen from (3) that

$$n\partial\varepsilon_{ik}/\partial n = \varepsilon_{ik} - \delta_{ik} = 4\pi\alpha_{ik}. \qquad (4)$$

Substituting (4) in (1) we get

$$\mathbf{f} = \tfrac{1}{4}\,\alpha_{ik}\,\mathrm{grad}\,E_i^* E_k.$$

In an unmagnetized plasma

$$\mathbf{f} = -(e^2 n/4m\omega^2)\,\mathrm{grad}\,|E|^2.$$

1)Such a problem was considered by Getmantsev and Denisov[1]. The authors were interested in it in connection with the question of perturbations of the ionsphere near antennas mounted on artificial earth satellites. No account was taken in [1] however, of the influence of the external magnetic field.

2)We are considering here the general case of a plasma in a constant magnetic field.

Dividing f by the electron concentration, we obtain the force acting on a single electron

$$\mathbf{F} = \frac{\mathbf{f}}{n} = \frac{1}{4}\frac{\alpha_{ik}}{n}\,\mathrm{grad}\,E_i^*E_k = -\frac{e^2}{4m\omega^2}\beta_{ik}\,\mathrm{grad}\,E_i^*E_k. \qquad (5)$$

(This formula was previously obtained by Miller [3].)

In an unmagnetized plasma ($\beta_{ik} = \delta_{ik}$)

$$\mathbf{F} = -(e^2/4m\omega^2)\,\mathrm{grad}\,|E|^2. \qquad (6)$$

The force acting on the ion is M/m times smaller and can be neglected (M is the ion mass). This statement does not apply, however, if $\omega$ is close to the Larmor frequency eH/Mc of the ions. In this case the formulas which we have used for $\epsilon_{ik}$ are likewise incorrect. We are not interested in this case.

We note also that the expression (5) for the average force is valid only under the assumption that the field amplitude changes little over the distance covered by the electron during the oscillation period. This leads to the conditions

$$R \gg v_e/\omega, \qquad R \gg (e\,|\,\mathbf{E}\,|/m\omega)\,\omega^{-1} \qquad (7)$$

($v_e = \sqrt{2T/m}$ —thermal velocity of the electrons, $e\,|\,\mathbf{E}\,|/m\omega$ —ordered velocity of the electrons, R —characteristic distance over which the field amplitude changes noticeably). The condition (7) leads, in particular, to a limitation on the value of the field intensity:

$$|\,\mathbf{E}\,| \ll m\omega^2 R/e.$$

If the plasma contains, in addition to the alternating field, also a constant field with potential $\varphi$, then the total average force acting on the electron will be

$$\mathbf{F} = \mathrm{grad}\,(e\varphi - e^2\beta_{ik}E_i^*E_k/4m\omega^2). \qquad (8)$$

From this formula we see that the expression in the bracket is equal to $-U$ (U is the average potential energy of the electron).

To obtain the time-averaged perturbation in the electron concentration, we must substitute the force (8) into the kinetic equation for the electron distribution function and solve this equation. On the other hand, if, as is customary, the rate of change of the field amplitude is sufficiently small, so that the following inequality is satisfied

$$v_e = \sqrt{2T/m} \gg R/t_0, \qquad (9)$$

where $t_0$ is the characteristic variation time of the field amplitude, then it can be stated beforehand that the electrons will have a Boltzmann distribution with a potential energy given by (8), so that

$$n\,(\mathbf{r}) = n_0 \exp\,[(e\varphi - (e^2/4m\omega^2)\,\beta_{ik}E_i^*E_k)/T], \qquad (10)$$

where $n_0$ is the unperturbed electron concentration and T is the plasma temperature in energy units.

The potential $\varphi$ is given by the Poisson equation

$$\begin{aligned}\Delta\varphi &= -4\pi e\,(n_i - n_e)\\ &= -4\pi e\,\{n_i - n_0 \exp\,[(e\varphi - (e^2/4m\omega^2)\,\beta_{ik}E_i^*E_k)/T]\}.\end{aligned} \qquad (11)$$

In the case when the characteristic distance R over which the fields vary noticeably is much larger than the Debye radius in the plasma a,

$$R \gg a = \sqrt{T/4\pi n_0 e^2}, \qquad (12)$$

the term with $\Delta\varphi$ in the left half of (11) is small and can be neglected. In this case (11) reduces to the condition for the quasi-neutrality of the plasma:

$$n_i = n_e.$$

Solving this equation with respect to $\varphi$, we obtain

$$e\varphi = (e^2/4m\omega^2)\,\beta_{ik}E_i^*E_k + T\,\ln\,(n_i/n_0). \qquad (13)$$

To obtain $n_i$ it is necessary to solve (13) and the kinetic equation for the ion distribution function simultaneously. The force acting on each ion is simply

$$\mathbf{F}_i = -e\,\mathrm{grad}\,\varphi. \qquad (14)$$

If, in addition to the inequality (9), there is satisfied also the stronger inequality

$$v_i = \sqrt{2T/M} \gg R/t_0,$$

then the ions have a Boltzmann distribution so that

$$n_i = n_0 \exp\,(-e\varphi/T). \qquad (15)$$

Solving (13) and (15) simultaneously we obtain for this case

$$e\varphi = (e^2/8m\omega^2)\,\beta_{ik}E_i^*E_k, \qquad (16)$$

$$n = n_i = n_0 \exp\,(-e^2\beta_{ik}E_i^*E_k/8m\omega^2 T). \qquad (17)$$

In the absence of a magnetic field we obtain the formula derived in [1]

$$n = n_0 \exp\,(-e^2\,|\,\mathbf{E}\,|^2/8m\omega^2 T). \qquad (18)$$

We see therefore that at sufficiently large values of E, i.e., when

$$|\,\mathbf{E}\,| \gtrsim \sqrt{8Tm}\,\omega/e \approx 2\cdot 10^{-8}\omega\sqrt{T^\circ/1000}\ \mathrm{V/cm}, \qquad (19)$$

the perturbations produced by the field in the plasma are large.

Using the explicit expression for the tensor $\beta_{ik}$, we can rewrite (17) in the form

$$n = n_0 \exp\left\{-\frac{e^2}{8mT}\left(\frac{|\,\mathbf{E}_\parallel|^2}{\omega^2} + \frac{|\,\mathbf{E}_\perp|^2}{\omega^2 - \omega_H^2} + i\frac{\omega_H}{\omega\,(\omega^2 - \omega_H^2)}\,[\mathbf{E}^*\mathbf{E}]\,\mathbf{H}\right)\right\}, \qquad (20)^*$$

*$[\mathbf{E}^*\mathbf{E}] = \mathbf{E}^* \times \mathbf{E}'$

where $E_{||}$ is the projection of $\mathbf{E}$ on the $\mathbf{H}$ direction, and $\mathbf{E}_\perp$ is the projection of $\mathbf{E}$ on the plane perpendicular to $\mathbf{H}$.

So far we have dealt with a determination of $n(\mathbf{r})$ for a specified field distribution $\mathbf{E}(\mathbf{r})$. Let us now proceed to determine $\mathbf{E}(\mathbf{r})$. This problem can be solved independently of the determination of $n(\mathbf{r})$ only if the field frequency is much larger than the plasma Langmuir frequency $\omega_0$:

$$\omega \gg \omega_0 = 4\pi n_0 e^2/m. \qquad (21)$$

In this case we can assume in the determination of $\mathbf{E}(\mathbf{r})$, that the dielectric constant of the plasma is equal to unity, i.e., we can calculate $\mathbf{E}(\mathbf{r})$ using the same formulas as for the vacuum.

On the other hand, if condition (21) is not satisfied, then the perturbation $n(\mathbf{r})$ changes the dielectric constant of the plasma. Since $n(\mathbf{r})$ itself depends on $\mathbf{E}$, the problem determining $\mathbf{E}$ becomes nonlinear. As is well known, in any nonlinear problem it is possible for harmonics of the fundamental frequency to appear, i.e., oscillations with frequencies $2\omega$, $3\omega$, etc. An account of these harmonics would complicate the solution of the problem. We shall show, however, that under condition (12) the amplitude of the harmonics is small and they can be disregarded.

To prove this, we write out the equation for the alternating component of the electron distribution function. This is precisely the equation which becomes nonlinear in a strong field. We take account of the fact that under condition (12) it is possible to neglect the term with derivatives with respect to the coordinates in this equation. The equation then assumes the form

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{v}} \frac{e\mathbf{E}}{m} = 0\,. \qquad (22)$$

Multiplying this equation by $e\mathbf{v}$ and integrating with respect to $d^3v$, we reduce it to the form[3)]

$$\partial \mathbf{j}/\partial t - e^2 n(\mathbf{r})\,\mathbf{E}/m = 0, \qquad (23)$$

where $\mathbf{j} = e\int \mathbf{v} f\, d^3\mathbf{v}$ is the current density.

Under condition (12) we have $n(\mathbf{r}) = n_i(\mathbf{r})$. However, owing to the large mass of the ions, the alternating field hardly influences their motion. Therefore $n_i(\mathbf{r})$ is equal to its time-averaged value and does not contain a high-frequency component. Thus, the equation relating $\mathbf{j}$ with $\mathbf{E}$ has a time-independent coefficient (which does depend, however, on the amplitude $\mathbf{E}$), so that no higher harmonics arise. It also follows from (23) that under condition (12) $\mathbf{j}$ is connected with $\mathbf{E}$ by the usual formula, in which $\mathbf{n}$ is taken to mean the average value of $n(\mathbf{r})$. This means that we can use for $\mathbf{E}$ the usual macroscopic material field equations in which the dielectric constant is expressed in terms of the average concentration $n(\mathbf{r})$, defined in turn by (10) or (17).

In the latter case we have

$$\varepsilon_{ik}(\mathbf{r}) = \delta_{ik} - (\omega_0/\omega)^2 \exp\left(-e^2\beta_{lm}E_l^*E_m/8m\omega^2T\right)\beta_{ik}. \qquad (24)$$

If there is no magnetic field, then

$$\varepsilon(\mathbf{r}) = 1 - (\omega_0/\omega)^2 \exp\left(-e^2|\mathbf{E}|^2/8m\omega^2T\right). \qquad (25)$$

## 2. NONLINEAR EFFECTS NEAR THE PLASMA RESONANCE FREQUENCY

An interesting example of the application of the formulas obtained in the preceding section is the question of the occurrence in a plasma of a weak longitudinal field, with a frequency close to the plasma frequency $\omega_0$. Specifically, we may deal with a plasma layer between plates of a capacitor or with a plasma near the surface of a metallic antenna, etc.

Assume that an electric field $E_0$ is situated outside a semi-infinite layer of plasma and is normal to the latter. In this case the electric induction $D$ inside the plasma will be equal to $E_0$, in accordance to the condition $\operatorname{div}\mathbf{D} = 0$,

$$D = E_0. \qquad (26)$$

If we neglect the influence of $E$ on $\epsilon$, i.e., if we solve the problem in the linear approximation, then the field intensity in the plasma will be

$$E = D/\varepsilon_0 = E_0/\varepsilon_0, \qquad (27)$$

where

$$\varepsilon_0 = 1 - \omega_0^2/\omega^2$$

is the dielectric constant of the unperturbed plasma.

If $\omega \to \omega_0$, then

$$\varepsilon_0 \approx 2(\omega - \omega_0)/\omega_0 \to 0$$

and $E$ increases without limit for constant $E_0$. Of course, the true value of $E$ will be finite. To determine this finite value we must take into account the nonlinear corrections, i.e., the dependence of $\epsilon$ on $E$.

If the field $E$ is weak, then the exponential in (25) can be expanded and we obtain (for $\omega \to \omega_0$)

$$\varepsilon \approx \varepsilon_0 + e^2|E|^2/8m\omega_0^2T = \varepsilon_0 + \varkappa|E|^2, \qquad (28)$$

$$\varkappa = e^2/8m\omega_0^2T. \qquad (29)$$

[3)]For simplicity we present the proof without the magnetic field.

Substituting (29) in (26) we obtain the equation for E:

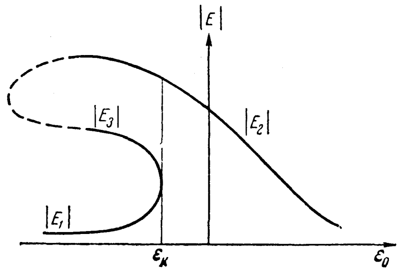
$$\varepsilon_0 E + \varkappa |E|^2 E = E_0. \quad (30)$$

The equation for the amplitude $|E|$ has the form

$$(\varepsilon_0 + \varkappa |E|^2)^2 |E|^2 = |E_0|^2. \quad (31)$$

It is interesting to note that this equation has the same form as the equation for the amplitude of the forced oscillations of an anharmonic oscillator without friction near resonance [4].

Equation (31) is cubic in $|E|^2$ and has, in the general case, three roots: $|E_1|^2$, $|E_2|^2$, $|E_3|^2$. The dependence of these roots on $\epsilon_0$ is shown schematically in the figure. We see that if $\epsilon_0$ exceeds a certain value $\epsilon_k$, then one value of $|E_0|^2$ corresponds to one value of $|E|^2$. On the other hand, if $\epsilon < \epsilon_k$, then to each value of the field amplitude outside the plasma $|E_0|^2$ there correspond three values of the field inside the plasma, i.e., hysteresis sets in. It can be shown, however, that the states corresponding to the middle root $|E_3|^2$ are unstable and cannot be realized.




To determine the critical value $\epsilon_k$ we note that when $\epsilon = \epsilon_k$, we have

$$d\varepsilon_0 / d|E|^2 = 0.$$

Differentiating (31) with respect to $|E|^2$ we obtain

$$(\varepsilon_0 + \varkappa |E|^2) + 2|E|^2 (d\varepsilon_0 / d|E|^2 + \varkappa) = 0. \quad (32)$$

Putting $d\epsilon_0/d|E|^2 = 0$ and simultaneously solving (31) and (32), we get

$$\varepsilon_k = -3 \cdot 2^{-2/3} (\varkappa |E_0|^2)^{1/3}. \quad (33)$$

At this value of $\epsilon_0$, $|E|^2$ can assume two values: $|E_{1k}|^2$ and $|E_{2k}|^2$:

$$|E_{1k}|^2 = |E_0|^{2/3} / 2^{2/3}\varkappa^{2/3}, \qquad |E_{2k}|^2 = 4|E_{1k}|^2 = 2^{4/3}|E_0|^{2/3} / \varkappa^{2/3}.$$

It is easy to show that the root $|E_2|^2$ increases without limit as $\epsilon_0$ decreases. This, however, is connected with the neglect of damping, i.e., energy dissipation. If dissipation is taken into account, the third and second branches merge together for some negative value of $\epsilon_0$, as shown in the figure by the dashed line.

So far we have considered the corrections to $\epsilon$ due to nonlinearity. There exist, however, corrections of a different kind, connected with the dependence of **E** on the coordinates, i.e., with spatial dispersion. The spatial dispersion is also expressed in the dependence of $\epsilon$ not only on the frequency $\omega$, but also on the wave vector **k**. For small k, when

$$ka \ll 1, \quad (34)$$

we can expand $\epsilon$ in powers of k. This expansion has the form

$$\varepsilon(\omega, \mathbf{k}) \approx \varepsilon(\omega) - 3(ka)^2 (\omega_0/\omega)^4. \quad (35)$$

So long as the nonlinear corrections and the corrections in **k** are small, they can be considered separately. The final expression for $\epsilon$ has therefore, in the case when $\omega \approx \omega_0$, the form

$$\varepsilon(\omega, \mathbf{k}) \approx \varepsilon_0 + \varkappa |E|^2 - 3(ka)^2. \quad (36)$$

Formula (36) means that $\mathbf{D}(\mathbf{r})$ is connected with $\mathbf{E}(\mathbf{r})$ by the relation

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varkappa |\mathbf{E}|^2 \mathbf{E} + 3a^2 \Delta \mathbf{E}, \quad (37)$$

or, in the one-dimensional case [4)]

$$D = \varepsilon_0 E + \varkappa |E|^2 E + 3a^2 d^2E / dx^2$$

(the x axis is in the E direction, i.e., normal to the surface of the plasma layer). Substituting the last expression for D in (26), we obtain an equation for the field distribution in the plasma:

$$3a^2 d^2E / dx^2 + \varepsilon_0 E + \varkappa |E|^2 E = E_0. \quad (38)$$

Equation (38) must be solved with the boundary condition $E = E_0$ for $x = 0$ (i.e., on the plasma boundary).

However, since the value of E inside the plasma is much larger than $E_0$ when $\epsilon_0 \ll 1$, we can assume with sufficient accuracy that the boundary value of E is equal to 0:

$$E = 0, \qquad x = 0. \quad (39)$$

If we neglect the nonlinear term in (38), the solution under condition (39) assumes the form

$$E = \frac{E_0}{\varepsilon}\left[1 - \exp\left(-\frac{x}{a}\sqrt{\frac{|\varepsilon_0|}{3}}\right)\right], \qquad \varepsilon_0 < 0,$$
$$E = \frac{E_0}{\varepsilon}\left[1 - \exp\left(\frac{ix}{a}\sqrt{\frac{\varepsilon_0}{3}}\right)\right], \qquad \varepsilon_0 > 0. \quad (40)$$

[4)]Strictly speaking, in order to be able to assume that the plasma particles have a Boltzmann distribution, it is necessary that the electric field approach zero at infinity. Therefore, the one-dimensional case considered here must be

Expressions (40) coincide, as they should, with those obtained by Landau for this case directly with the aid of the kinetic equation [6].

We note, however, that the small Landau-damping term is missing from the second formula of (40). Equation (38) does not describe this damping. It is negligibly small for small $\epsilon_0$. It is seen from (40) that when $\epsilon_0 \ll 1$ the values of importance in (38) are $k \sim \sqrt{\epsilon_0}/a \ll 1/a$. This justifies the use of the expansion (35) and consequently of Eq. (38).

In order to ascertain when the nonlinear term can be neglected in (38), i.e., when solutions of the type (40) are correct, we rewrite (38) in dimensionless variables. To this end we put

$$E = E_0 y / (\varkappa E_0^2)^{1/3}, \qquad x = a\sqrt{3}\, t/(\varkappa E_0^2)^{1/6}, \qquad (41)$$

after which (38) assumes the form

$$d^2y/dt^2 + \mu y + y^3 = 1, \qquad \mu = \varepsilon_0/(\varkappa E_0^2)^{1/3}. \qquad (42)$$

It is clear from (42) that the term with $y^3$ can be neglected if $\mu \gg 1$, i.e., if

$$\varepsilon_0 \gg (\varkappa E_0^2)^{1/3} = (e^2 E_0^2 / 8m\omega_0^2 T)^{1/3}.$$

In the opposite limiting case, when $\mu \ll 1$, i.e., in the direct vicinity of the plasma frequency, we can put in (42) $\mu = 0$. In this case the distribution of the field in the plasma is determined by the universal equation

$$d^2y/dt^2 + y^3 = 1. \qquad (43)$$

It is seen from (43) that the field intensity in the plasma has the order of magnitude

$$E \sim E_0/(\varkappa E_0^2)^{1/3}, \qquad (44)$$

and the characteristic dimension $x_0$, over which the field varies, is

$$x_0 \sim a\sqrt{3}/(\varkappa E_0^2)^{1/6}. \qquad (45)$$

From the statements made earlier it is clear that (38) is valid under the following conditions:

$$\varkappa E^2 \ll 1 \text{ and } x_0 \gg a. \qquad (46)$$

[The first of these equations insures the possibility of expanding the exponential in (25), while the second ensures the possibility of confining oneself to the term with $(ka)^2$ in (35)]. It is seen from (44) and (45) that both conditions are satisfied if

$$\varkappa E_0^2 \ll 1. \qquad (47)$$

The solution of the nonlinear equation (42) with boundary condition (39) can be obtained for the case for real y. We shall not deal here with these calculations [5)].

It is curious that the left half of (38) has the same form as the Ginzburg-Landau equation in the theory of superconductivity near the transition point [7], that of the equation for the theory of superfluidity near the λ point [8], and that of the equation describing vortex filaments in a non-ideal Bose gas [9]. This similarity is not an accident. In all cases the need for taking into account the nonlinear terms and the possibility of retaining the second derivatives is connected with the smallness of the coefficient of the linear term. In [7,8] this smallness is ensured by closeness to the transition point, while in [9] by the weakness of the interaction, and in the present work, it is ensured by closeness to the resonant frequency. Analogous equations are also encountered in nonlinear theory of the propagation of radio waves in a plasma. [10]

So far we have considered only the one-dimensional problem of field penetration in a plasma. Let us consider now the axially-symmetrical case, for example, a field at a short distance from a cylindrical antenna of radius $\rho_0$ situated in a plasma. In this case the field is directed from the antenna along the radius and depends only on the distance to the antenna axis $\rho$. The equation $\operatorname{div} \mathbf{D} = 0$ has in this case the form

$$\rho^{-1} d(\rho D)/d\rho = 0.$$

Integrating, we obtain an equation which replaces (26) in the cylindrical case:

$$D = \rho_0 E_0/\rho, \qquad (48)$$

where $E_0$ is the field on the antenna surface itself.

Substituting in (48) the previously obtained relation (37) between **D** and **E**, we obtain for E, the equation

$$3a^2\rho^{-1}\frac{d}{d\rho}\rho\frac{dE}{d\rho} + \varepsilon_0 E + \varkappa |E|^2 E = \frac{\rho_0}{\rho} E_0. \qquad (49)$$

At short distances from the antenna surface, i.e., when $\rho - \rho_0 \gg \rho_0$, this equation, as it should, goes over into the one-dimensional equation (38).

If we neglect the spatial dispersion, i.e., the first term in (48), we obtain an algebraic equation for E:

$$\varepsilon_0 E + \varkappa |E|^2 E = \rho_0 E_0/\rho, \qquad (50)$$

which differs from (30) only in that $E_0$ is replaced by $\rho_0 E_0/\rho$. The entire investigation of (30) given above therefore applies fully to (50). It is seen from (50) that if $\omega = \omega_0$, i.e., $\epsilon_0 = 0$, then E de-

regarded as the limiting case of a cylindrical one (see below). The cylindrical case goes over into the one dimensional case if $x \ll \rho_0$ ($\rho_0$—radius of the antenna).

[5)]Equation (38) has as its first integral

$$3a^2|dE/dx|^2 + \varepsilon_0|E|^2 + \varkappa|E|^4/2 - E_0E^* - E_0^*E = \text{const}.$$

creases at great distances in accordance with

$$|E| = (\rho_0 |E_0| / \varkappa)^{1/3} \rho^{-1/3}, \qquad (51)$$

and not like $1/\rho$ as in the case of $\epsilon_0 \neq 0$.

It must be borne in mind, however, that owing to the influence of the spatial dispersion, formula (51) can apply only at a rather large $\rho/a$.

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