THEORY OF QUANTIZATION OF MAGNETIC FLUX IN SUPERCONDUCTORS

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A microscopic theory of quantization of magnetic flux through a superconducting cylinder is developed. The relation between this phenomenon and quantum vortices in a fermion system is indicated. Corrections to the magnitude of the quantum of magnetic flux due to the size of the apparatus and the magnitude of the energy gap are obtained.

1. Quantization of magnetic flux through an opening in a superconducting cylinder has been demonstrated experimentally [1,2]. The "quantum of flux" has turned out to be equal to $2\pi\hbar c/e^* = 2.07 \times 10^{-7}$ gauss cm^2 where $e^* \sim 2e$, and, apparently, depends slightly on the parameters of the apparatus [2]. The phenomenological theory of this phenomenon has been given in a number of papers [3-5]. However, as has been noted in Onsager's paper [5], the detailed kinetic mechanism which leads to the quantization of flux remains unclear on the basis of the microscopic theory of superconductivity.

In a previous paper [6] it has been shown that in a superfluid Fermi gas quantum vortices arise in which the velocity and the energy are quantized similarly to the quantization of these quantities in superfluid He II. For example, a vortex filament has a quantized component of the velocity $v_\varphi = 2\pi\hbar n/m^*r$, where $m^* = 2m$, $n = 1, 2, 3 \ldots$

It is evident that if the fermions are charged then quantization of the velocity leads to the quantization of current density and of the magnetic field. In the case of a superconductor the external magnetic field hardly penetrates inside the metal, and, therefore, the quantized currents flowing in the surface create a field which, on being added to the external field, forms the total magnetic flux in the channel of the superconducting cylinder. As will be shown, such a magnetic flux finally turns out to be quantized in units of $2\pi\hbar c/e^*$ with the magnitude of the quantum also being weakly dependent on the parameters of the apparatus and on the magnitude of the gap in the superconductor.

2. Gor'kov [7] has shown that as $T \to T_c$ the Ginzburg-Landau equation can be obtained from the modern microscopic theory of superconductivity. If in deriving this equation one utilizes iteration discussed in detail previously [6], then one can write the following system of equations connecting the "pair wave function" ψ and the vector potential **A**:

$$\left\{\frac{1}{2m^*}\left(\frac{\partial}{\partial \mathbf{r}} + ie^*\mathbf{A}(\mathbf{r})\right)^2 + \frac{3}{4}\frac{\Delta_T^{02}}{\mu}\left(1 - |\psi(\mathbf{r})|^2\right)\right\}\psi(\mathbf{r}) = 0, \quad (1)$$

$$\text{rot rot } \mathbf{A}(\mathbf{r}) = \mathbf{j}_s,$$

$$\mathbf{j}_s = -\delta_0^{-2}\left\{\frac{i}{2e^*}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \mathbf{A}(\mathbf{r})|\psi|^2\right\} \quad (2)^*$$

($\hbar = c = 1$), where Δ_T^0 is the energy gap in a homogeneous superconductor which is a function of the temperature; δ_0 is the depth of penetration of the field into the superconductor: $\delta_0^{-2} = e^{*2}N_s/m^*$; N_s is the number of "superconducting" electrons, μ is the chemical potential of the system.

A characteristic feature of Eqs. (1) and (2) compared to the Ginzburg-Landau equation [8] is that they contain the doubled electron charge e^*, the doubled mass m^* and the gap Δ_T^0. We note that for a superconductor the nonlinear equation (1) replaces the Schrödinger equation, and it is, generally speaking, unjustified to use the latter equation to describe electrons in superconductors as has been done in the paper of Byers and Yang [3].

We investigate the solution of the system (1), (2) for a circular cylinder of external radius r_2 and internal radius r_1 situated in a weak magnetic field **H** directed along the cylinder axis z.

In this case

$$j_r = j_z = 0, \quad j_\varphi = j(r), \quad A_z = A_r = 0, \quad A_\varphi = A(r),$$

$$\mathbf{H} = H_z = \frac{1}{r}\frac{\partial}{\partial r}(rA(r)). \quad (3)$$

In the absence of a field equation (1) admits a solution in the form $\psi(\mathbf{r}) = R(r)e^{in\varphi}$ which represents a single vortex filament at the center of the

*rot = curl

cylinder. In this case R satisfies the equation

$$\frac{d^2R}{d\xi^2} + \xi^{-1}\frac{dR}{d\xi} - \frac{R}{\xi^2} + R - R^3 = 0, \tag{4}$$

where $\xi = r/l$, $l = (3m^*\Delta_T^{02}/\mu)^{-1/2}$ can be interpreted as the inner radius of the vortex[6]. From the definition of l it can be seen that as $T \to T_c$ the quantity $l \to \infty$ since $\Delta_T^0 \to 0$ and the vortex state is destroyed, as well as the state of superfluidity itself.

We shall be interested here only in the asymptotic behavior of R:

$$R(\rho) = 1 - (l^2/2\delta_0^2)\,\rho^{-2}, \qquad \rho = r/\delta_0, \qquad l < \delta_0. \tag{5}$$

Indeed, in the experiments of Deaver and Fairbank [1] the radii $r_1, r_2 \sim 10^{-3}$ cm, $\delta_0 \sim 10^{-5}$ cm, and therefore $\rho \sim 100$. On substituting (5) into (4) we obtain

$$\frac{d^2A'(\rho)}{d\rho^2} + \rho^{-1}\frac{dA'(\rho)}{d\rho} - \left(1 + \frac{\nu^2}{\rho^2}\right)A'(\rho) = 0, \tag{6}$$

$$A'(\rho) = A(\rho) - \frac{n}{e^*\delta_0\rho}, \quad \nu^2 = 1 - \frac{l^2}{\delta_0^2}. \tag{7}$$

We note that the index ν has the same dependence on the temperature as the gap Δ_T^0.

Equation (6) has a solution in the form of combinations of cylindrical functions of complex argument and index ν:

$$A'(\rho) = \delta_0\,[aI_\nu(\rho) + bK_\nu(\rho)]. \tag{8}$$

Utilizing (3) we obtain the intensity of the magnetic field:

$$H(\rho) = a\,[I_{\nu-1}(\rho) - (1-\nu)\,I_\nu(\rho)/\rho] - b\,[K_\nu(\rho) - (1-\nu)\,K_\nu(\rho)/\rho]. \tag{9}$$

On determining a and b from the boundary conditions we obtain

$$a = f_\nu^{-1}(\rho_1\rho_2)\,[H_2K_{\nu-1}(\rho_1) - H_1K_{\nu-1}(\rho_2) - (1-\nu)\,H_2K_\nu(\rho_1)/\rho_1 + (1-\nu)\,H_1K_\nu(\rho_2)/\rho_2],$$

$$b = f_\nu^{-1}(\rho_1\rho_2)\,[H_2I_{\nu-1}(\rho_1) - H_1I_{\nu-1}(\rho_2) + (1-\nu)\,H_2I_\nu(\rho_1)/\rho_1 - (1-\nu)\,H_1I_\nu(\rho_2)/\rho_2], \tag{10}$$

where

$$f_\nu(\rho_1\rho_2) = \varphi_{\nu-1}(\rho_1\rho_2) + (1-\nu)\,\chi_\nu(\rho_2\rho_1)/\rho_2 - (1-\nu)\,\chi_\nu(\rho_1\rho_2)/\rho_1; \tag{11}$$

$$\varphi_\nu(\rho\rho_1) = K_\nu(\rho)\,I_\nu(\rho_1) - I_\nu(\rho)\,K_\nu(\rho_1), \tag{12}$$

$$\chi_\nu(\rho\rho_1) = I_\nu(\rho)\,K_{\nu-1}(\rho_1) + K_\nu(\rho)\,I_{\nu-1}(\rho_1); \tag{13}$$

H_1 and H_2 are the values of the field inside and outside the cylinder respectively.

From the known $A'(\rho)$ one can easily obtain j_s, $H(\rho)$ and $\Phi(\rho)$:

$$H_1 = \left\{\frac{2n}{e^*\delta_0^2\rho_1^2} + \frac{2}{\rho_1}\frac{H_2\chi_\nu(\rho_1\rho_1)}{f_\nu(\rho_1\rho_2)}\right\} \times \left\{1 + \frac{2}{\rho_1}\frac{\chi_\nu(\rho_2\rho_1) - (1-\nu)\,\varphi_\nu(\rho_1\rho_2)/\rho_2}{f_\nu(\rho_1\rho_2)}\right\}^{-1}, \tag{14}$$

$$\Phi(\rho) = \frac{2\pi n}{e^*} + \frac{2\pi\rho\delta_0^2}{f_\nu(\rho_1\rho_2)}\{H_2[\chi_\nu(\rho\rho_1) + (1-\nu)\,\varphi_\nu(\rho\rho_1)/\rho_1] - H_1[\chi_\nu(\rho_2\rho) - (1-\nu)\,\varphi_\nu(\rho\rho_2)/\rho_2]\}. \tag{15}$$

Formulas (14) and (15) give values of H_1 and $\Phi(\rho)$ for cylinders of arbitrary thickness at a temperature $T < T_c$ (for $\nu = 1$ all these expressions reduce to the corresponding formulas obtained in Ginzburg's paper[4]).

Approximate values for Φ and H can be obtained by utilizing the asymptotic expressions for I_ν and K_ν:

$$I_\nu(\rho) = \sqrt{1/2\pi\rho}\,e^\rho, \qquad K_\nu(\rho) = \sqrt{\pi/2\rho}\,e^{-\rho}. \tag{16}$$

In this case we have

$$\Phi(\rho_1) \approx (2\pi n/e^*)\{1 - 2\rho_1^{-1}(1 + dl^2/2\rho_1\rho_2\delta_0^2)\} + 4\pi\delta_0^2\sqrt{\rho_1\rho_2}\,e^{-d}H_2,$$

$$H(\rho_1) \approx 2n/e^*\delta_0^2\rho_1^2 + 4\sqrt{\rho_1\rho_2}\,e^{-d}H_2/\rho_1^2, \tag{17}$$

where $d = \rho_2 - \rho_1$ is the thickness of the cylinder.

For the samples utilized in the experiments[1,2] $e^{-d} \sim 10^{-9}$ and, therefore, the term containing H_2 in (17) can be neglected.

The expression obtained for $\Phi(\rho_1)$ shows that the magnetic flux through the opening in the cylinder is indeed quantized. The dependence of this quantum on the dimensions of the apparatus which is contained in the term $2/\rho$ has been already obtained by Ginzburg[4]. The last term also contains the dependence on the gap Δ_T^0. However, for the dimensions of the cylinders used in experiments [1,2] this term is of the order of $10^{-3} \sim 10^{-4}$. For thinner samples for which one would have to utilize formulas (14), (15), and not (17) the contribution of this term can increase considerably.

From the preceding discussion it follows that the discovery of the quantization of the magnetic flux is physically equivalent to a proof of the presence of quantum vortices in superconductors.

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