PARAMETRIC GENERATION OF ELECTROMAGNETIC WAVES IN A MAGNETOACTIVE PLASMA

L. A. OSTROVSKIĬ and N. S. STEPANOV

Radiophysics Institute, Gor'kiĭ State University

Submitted to JETP editor April 24, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 45, 1473-1478 (November, 1963)

Nonlinear interaction of weak electromagnetic waves with a plasma wave in a magnetoactive plasma at rest is considered. It is shown that under certain conditions a plasma wave propagating along a magnetic field may be unstable and excite electromagnetic waves.

As is well known, the only components of significance in the spectrum of the electron plasma oscillations are those with sufficiently small wave numbers K (large phase velocities), for which Cerenkov damping hardly comes into play.[1,2] However, as will be shown in the present paper, it is precisely the plasma waves with rather small K that can be unstable against electromagnetic perturbations and can generate pairs of electromagnetic waves whose frequencies $\omega_{1,2}$ and wave numbers $k_{1,2}$ satisfy the following relations (which express the energy and momentum conservation laws)

$$\Omega = \omega_1 + \omega_2, \qquad \mathrm{K} = \mathrm{k}_1 + \mathrm{k}_2, \tag{1}$$

where Ω —frequency of the plasma wave.

In order to satisfy condition (1) it is necessary that the waves with frequencies $\omega_{1,2} < \Omega$ be able to propagate at all in the plasma. In the absence of an external magnetic field, this condition cannot be satisfied in a plasma at rest (but can be satisfied if the plasma moves through a dielectric with $\epsilon > 1$[3]). On the other hand, in a magnetoactive plasma the frequency of the propagating electromagnetic waves can be smaller than the frequency of the longitudinal wave, and the effect under consideration does take place[1)].

Let us assume that a plasma wave of frequency Ω propagates along a constant magnetic field $\mathbf{H}_0$ (in the z direction), so that the concentration of the electrons varies in accordance with[2)]

$$N(z, t) = N_0[1 + p\cos(\Omega t - Kz)], \tag{2}$$

where we assume that $0 < p \ll 1$. For a plasma which is on the whole at rest, the longitudinal electron velocity, in accordance with the continuity equation, is (accurate to quantities of order p^2)

$$V(z, t) = p(\Omega/K)\cos(\Omega t - Kz). \tag{3}$$

1. INTERACTION OF WAVES IN LONGITUDINAL PROPAGATION

We shall assume first that weak transverse perturbations also propagate along the z axis. In this case it is convenient to introduce the complex variables

$$A_\pm = A_x \pm iA_y, \qquad v_\pm = v_x \pm iv_y, \tag{4}$$

where A_x, A_y and v_x, v_y are respectively the Cartesian projections of the transverse vector potential $\mathbf{A}_\perp$ and of the vibrational electron velocity in the transverse wave $\mathbf{v}_\perp$ (in this case div $\mathbf{A}_\perp = 0$, so that $\mathbf{E} = -c^{-1}\partial\mathbf{A}_\perp/\partial t$, $\mathbf{H} = \mathrm{curl}\,\mathbf{A}_\perp$), Then, making the usual assumptions of the elementary theory (disregarding ion motion), we obtain from Maxwell's equations and from the electron equations of motion the following independent systems for A_+ and A_-:

$$(\partial^2/\partial z^2 - (\varepsilon/c^2)\,\partial^2/\partial t^2)\,A_\pm = -4\pi eNv_+/c, \tag{5}$$

$$(\partial/\partial t + V\partial/\partial z \pm i\widetilde{\omega}_H)\,v_\pm = (e/mc)\,(\partial/\partial t + V\partial/\partial z)\,A_\pm. \tag{6}$$

Here ϵ is the dielectric constant of the medium (without the plasma), e and m are the charge and mass of the electron, c is the velocity of light,

1)In the solutions obtained below, in particular, assume that ω and k of one of the waves are negative. This case, for which $x = ||\omega_1| - |\omega_2||$, $K = ||k_1| - |k_2||$ corresponds to combination scattering in a plasma[4], but without the usual assumption that the scattered wave is small compared with the incident wave.

2)Since the wave under consideration must have a large phase velocity (on the order of the velocities of the electromagnetic waves), the quasihydrodynamic approach is perfectly suitable for its description. The value of Ω in a plasma at rest is close to the plasma frequency of the electron ω_p (the thermal motion is insignificant).

ν_{eff} is the collision frequency which we assume constant, $\tilde{\omega}_H = \omega_H \pm i\nu_{eff}$; $\omega_H = |e|H_0/mc$ is the electron gyrofrequency.

The quantities N(z,t) and V(z,t) enter into (5) and (6) as variable parameters (it is sufficient to consider from now on only one of the systems, for example for A_+ and v_+). For $\tilde{\omega}_H = 0$ it follows from (6) that the velocity of the forced oscillations v_+ is connected with A_+ by the relation $v_+ = -(e/mc)A_+$, and the parameter V(z,t) is eliminated from the final equation, so that the effect can be connected in this case only with the change in the concentration N(z,t); this does not lead to instability of the longitudinal wave in a plasma at rest [3]. On the other hand, in a magnetic field, as can be readily seen from (6), the presence of an oscillating component leads (in a specified monochromatic field of frequency ω) to the appearance of combination frequencies $\omega \pm \Omega$ for the velocity of transverse motion. These oscillations excite, in turn, fields of the corresponding frequencies and this makes an additional contribution to the wave interaction.

For the analysis of this interaction, we assume that a pair of transverse waves propagates in the plasma:

$$A_+ = P_1 e^{i(\omega_1 t - k_1 z)} + P_2 e^{-i(\omega_2 t - k_2 z)}, \qquad (7)$$

where the coefficient P_1 and P_2 are constant, while $\omega_{1,2}$ and $k_{1,2}$ satisfy conditions (1). Substituting (7) in (5) and (6) and discarding terms of order p^2 and higher (and also those containing the corresponding harmonics $\omega_{1,2} \pm 2\Omega$), we obtain from the condition for the existence of a non-trivial solution a dispersion equation in ω and k. The latter will, generally speaking, be complex quantities. It is natural to seek them in the form

$$k_{1,2} = k_{1,2}^0 \pm ik', \qquad \omega_{1,2} = \omega_{1,2}^0 \mp i\omega', \qquad (8)$$

where $k_{1,2}^0$ and $\omega_{1,2}^0$ satisfy the "unperturbed" dispersion equations

$$(k_{1,2}^0)^2 = (\omega_{1,2}^0)^2 c^{-2} [\varepsilon - \omega_p^2/\omega_{1,2}^0 (\omega_{1,2}^0 \mp \omega_H)] \qquad (9)$$

($\omega_p^2 = 4\pi e^2 N_0/m$), and k' and ω' are small corrections to them; it is assumed here that $\nu_{eff}|\omega_{1,2} \pm \omega_H| \leq p$, so that in the zeroth approximation the losses can also be disregarded.

In particular, for monochromatic waves ($\omega' = 0$), we can easily obtain[3)]

$$k' = -\frac{\nu_{eff}\omega_p^2}{4c}\left[\frac{\omega_1}{k_1(\omega_1-\omega_H)^2} + \frac{\omega_2}{k_2(\omega_2+\omega_H)^2}\right] \pm \frac{1}{2}\left\{\frac{\chi^2}{k_1 k_2} + \left(\frac{\nu_{eff}\omega_p^2}{2c^2}\right)^2\left[\frac{\omega_1}{k_1(\omega_1-\omega_H)^2} - \frac{\omega_2}{k_2(\omega_2+\omega_H)^2}\right]^2\right\}^{1/2}, \qquad (10)$$

$$\chi = \frac{p\omega_p^2}{2c^2}\frac{\omega_1\omega_2 K - \omega_H(\omega_2 k_1 - \omega_1 k_2)}{K(\omega_1-\omega_H)(\omega_2-\omega_H)}, \qquad (11)$$

and for waves that are harmonic in space ($k' = 0$)

$$\omega' = -\frac{\nu_{eff}\omega_p^2}{4c^2}\left[\frac{\omega_1 v_{gr}(\omega_1)}{k_1(\omega_1-\omega_H)^2} + \frac{\omega_2 v_{gr}(\omega_2)}{k_2(\omega_2+\omega_H)^2}\right] \pm \frac{1}{2}\left\{\frac{\chi^2 v_{gr}(\omega_1) v_{gr}(\omega_2)}{k_1 k_2} + \left(\frac{\nu_{eff}\omega_p^2}{2c^2}\right)^2\left[\frac{\omega_1 v_{gr}(\omega_1)}{k_1(\omega_1-\omega_H)} - \frac{\omega_2 v_{gr}(\omega_2)}{k_2(\omega_2+\omega_H)}\right]^2\right\}^{1/2}. \qquad (12)$$

Assume for concreteness that $v_{gr}(\omega_{1,2}) > 0$ and $V_{ph}(\omega_{1,2}) = \omega_{1,2}/k_{1,2} > 0$. It is then clear from (11) and (12) that for sufficiently low losses one of the roots (for both k' and ω') is positive; a weak transverse perturbation will then grow exponentially in space or in time. Thus, the plasma wave turns out to be unstable, and its energies transform into transverse electromagnetic waves satisfying the conditions (1). Since we should have here $k_1k_2 > 0$ and $\omega_1\omega_2 > 0$, it follows from (1) that $\Omega > |\omega_{1,2}|$ and $K > |k_{1,2}|$.

We note also that when $\omega_{1,2} > 0$ the first term in (17) corresponds to the extraordinary wave in a magnetoactive plasma, while the second corresponds to the ordinary wave, i.e., exponential intensification occurs when two waves of different types interact (with opposite directions of rotation of the vector E, and consequently of the electrons)[4)]. On the other hand, interaction of waves of the same type, corresponding, as is clear from (11) and (12), to the conditions $\omega_1\omega_2 < 0$ and $k_1k_2 < 0$ leads only to periodic transfer of energy from one frequency to the other (when $\nu_{eff} = 0$ the energy varies as a result of such a transformation in proportion to the frequency) and represents, as indicated, one of the modes of combination scattering.

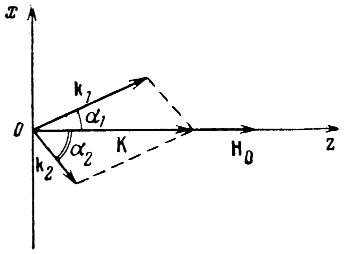
If the group velocities of the transverse waves have opposite signs, but $\omega_1\omega_2 > 0$, the regenerative effect of resonant reflection occurs (see [3,5]). Equation (12) then remains in force for a plasma that is unbounded in the z direction.

2. INTERACTION BETWEEN WAVES PROPAGATING AT AN ARBITRARY ANGLE TO THE MAGNETIC FIELD

Let us consider now a general case, when the electromagnetic waves propagate at angles α_1 and α_2 respectively to the plasma (for which, as be-

3)For the sake of brevity we omit the zero superscripts for ω and k.

4)The impossibility of simultaneous growth of two waves of the same type (in particular, of 'degenerate' generation of a wave with $\omega = \Omega/2$, $k = k/2$) follows from the conservation of the angular momentum of the electrons.



fore, $\mathbf{K} \parallel \mathbf{H}_0$, see the figure). Then the electron velocity in the electromagnetic wave also has a component along the z axis, and the equation of motion of the electrons without account of collisions, is of the form

$$\eta^{-1}(\partial/\partial t + V\partial/\partial t)\,\mathbf{v} = \mathbf{E} + c^{-1}\{[\mathbf{v}\mathbf{H}_0] + [\mathbf{V}\mathbf{H}]\} - \eta^{-1}v_z\mathbf{z}_0\partial V/\partial z, \qquad (13)^*$$

where $\mathbf{z}_0$ is a unit vector in the z direction (we neglect the nonlinear terms).

We seek the solution in the form

$$\mathbf{E} = \mathbf{E}_1 e^{i(\omega_1 t - \mathbf{k}_1\mathbf{r})} + \mathbf{E}_2 e^{-i(\omega_2 t - \mathbf{k}_2\mathbf{r})} \qquad (14)$$

(we seek the quantities $\mathbf{H}$ and $\mathbf{v}$ in analogous form), where $\omega_{1,2}$ and $\mathbf{k}_{1,2}$ satisfy equations (1). We substitute the last expressions in Maxwell's equations and in formula (13) and then take relations (2) and (3) into account; this yields, after elimination of the variables $\mathbf{H}$ and $\mathbf{v}$, two equations for the complex vectors $\mathbf{E}_i$:

$$\hat{\mathbf{L}}_i(\mathbf{E}_i, \omega_i) = \tfrac{1}{2}p\mathbf{f}_j(\mathbf{E}_j, \omega_j, \omega_i) \quad (i, j = 1, 2, \quad i \neq j), \qquad (15)$$

where

$$\hat{\mathbf{L}}_i = c^2\omega_p^{-2}(\mathbf{k}_i\mathbf{E}_i)\,\mathbf{k}_i - u_i\mathbf{E}_i + ih_i\{c^2\omega_i^{-2}(\mathbf{k}_i\mathbf{E}_i)\,[\mathbf{k}_i\mathbf{z}_0] - q_i\,[\mathbf{E}_i\mathbf{z}_0]\},$$

$$\mathbf{f}_j = (\mathbf{k}_j/\omega_j)\,[\Omega K^{-1}E_{jz} + (\mathbf{k}_j\mathbf{E}_j)\,c^2\omega_p^{-2}(\Omega k_{jz}/K - \omega_i)] + \mathbf{E}_j\,(q_j\omega_i\omega_j/\omega_p^2 - \Omega k_{jz}u_j/K) + i\,(-1)^{j+1}h_j\,\{q_j\,[\mathbf{E}_j\mathbf{z}_0]\,c^2\omega_j^{-2}(\mathbf{k}_j\mathbf{E}_j)\,[\mathbf{k}_j\mathbf{z}_0]\} + c\Omega\mathbf{z}_0\omega_p^{-2}\,[\omega_j q_j E_{jz}/c - c\,(\mathbf{k}_j\mathbf{E}_j)\,k_{jz}/\omega_j],$$

$$q_i = n_i^2 - 1, \qquad u_i = 1 + \omega_i^2(n_i^2 - 1)/\omega_p^2, \qquad h_i = \omega_i\omega_H/\omega_p^2, \qquad n_i = ck_i/\omega_i$$

As before, we seek the small corrections to the values of the corresponding quantities which were "unperturbed" by the plasma wave

$$\mathbf{E}_i = \mathbf{E}_i^0 + \mathbf{E}_i', \quad \omega_i = \omega_i^0 \mp i\omega' \qquad (16)$$

(for concreteness we assume the wave to be harmonic in space). For the unperturbed quantities we then obtain the equation

$$\hat{\mathbf{L}}_i(\mathbf{E}_i^0, \omega_i^0) = 0. \qquad (17)$$

The condition for the existence of a nontrivial solution of the last equations leads, as expected, to the well known dispersion equation describing the oblique propagation of waves in a gyrotropic plasma[2].

In the next approximation we have from (15) and (16)

$$\hat{\mathbf{L}}_i(\mathbf{E}_i', \omega_i^0) = \tfrac{1}{2}p\mathbf{f}_j(\mathbf{E}_j^0, \omega_j^0, \omega_i^0) \pm j\partial\hat{\mathbf{L}}_i(\mathbf{E}_i^0, \omega_i^0)/\partial\omega_i. \qquad (18)$$

According to (17), the determinant of this system of equations (with respect to $\mathbf{E}_i'$) is equal to zero. We can easily obtain the frequency correction ω' from the condition of orthogonality of the right halves of (18) to the solution of the transposed system (17)

$$(\omega')^2 = \tfrac{1}{4}p^2\omega_1\omega_2 s_1 s_2, \qquad (19)$$

where

$$s_1 = \left[\frac{2 + q_1u_1}{q_1} + \frac{h_1^2(q_1u_1 - r_1)}{q_1u_1^2}\sin^2\alpha_1\right]^{-1}\left\{\frac{\omega_2 q_2}{\omega_1 q_1 u_2}(u_1u_2 - h_1h_2q_1q_2) + \left(\frac{\Omega k_{2z}}{\omega_2 K} - \frac{\omega_1\omega_2 q_2}{\omega_{p0}^2 u_2}\right)\frac{\omega_1 q_2 u_2 - \omega_2 q_2 u_1}{\omega_1 q} + n_2^2\sin^2\alpha_2\left[1 + \frac{h_2^2q_2^2\omega_2^2}{u_2r_2\omega_p^2} + \frac{u_1q_2\omega_2^2}{\omega_1q_1u_2r_2\omega_{p0}^2}\left(\omega_1 - \frac{\Omega k_{2z}}{K}\right)\right] - \frac{h_1h_2q_2n_1^2\omega_2}{4u_1r_1r_2\omega_{p0}^2}\left[\Omega + \frac{n_2^2}{u_2}\left(\omega_2 - \frac{\Omega k_{2z}}{K}\right)\right]\sin 2\alpha_1\sin 2\alpha_2\right\}. \qquad (20)$$

The expression for s_2 is obtained from this by making the substitutions $\alpha_1 \to \alpha_2$, $\omega_1 \to -\omega_2$ and $k_1 \to -k_2$.

The general form of (20) is quite elaborate, but the character of the solution is made sufficiently clear by examination of some particular cases. When $\alpha_i = 0$ we obtain, of course, formula (12). Let us consider also another case—the interaction of two symmetrical waves. Putting $\alpha_1 = \alpha_2 = \alpha$, $\omega_1 = \omega_2 = \omega$, and $k_1 = k_2 = k$, we have

$$\frac{\omega'}{\omega} = \pm\frac{pn^2}{2}\sin^2\alpha\left[1 + \frac{qh^2\omega^2}{ur\omega_p^2}\left(\varepsilon - \frac{2\cos^2\alpha}{r}\right)\right] \times \left[\frac{2 + uq}{q} + \frac{h^2(uq - 2)}{ur^2}\sin^2\alpha\right]. \qquad (21)$$

In this formula ω' is always real, i.e., the plasma wave is also unstable against "oblique" perturbations propagating at an angle $\alpha < \pi/2$ to the plasma. Only when $\alpha = 0$ do we get here $\omega' = 0$, i.e., in accordance with the statements made in Sec. 1, only different modes can be excited here. This result follows also directly from (19) and (20) (including the case when $k_1 = k_2$).

*$[\mathbf{VH}] = \mathbf{V} \times \mathbf{H}$.

Thus, a plasma wave can generate a whole spectrum of electromagnetic waves (both "ordinary" and "extraordinary") propagating at different angles to the direction of the magnetic field $\mathbf{H}_0 \parallel \mathbf{K}$. The mechanism of this instability is analogous to parametric resonance in coupled oscillating systems—variation of N and V in accordance with (2) and (3) is equivalent to the presence of coupling between the waves with frequencies ω_1 and ω_2, satisfying conditions (11). In particular, a similar variation in the electron concentration causes a periodic disturbance of the effective dielectric constant of the plasma; it is known that in a medium with variable ϵ or μ, amplification and generation of electromagnetic waves at the expense of the source energy is possible under conditions (1), and these waves change the parameters [5-7]. We note that instabilities of this type were discussed also for low frequency waves in a plasma [8,9] with analysis, for example, of the decay of electron waves in an isotropic plasma into electron and ion waves. The transitions corresponding to (1) are also characteristic of quantum generators.

Of course, the formulas presented here are valid only so long as the excited electromagnetic waves are sufficiently weak compared with the plasma wave. When this condition is violated, the problem becomes essentially nonlinear.

It appears that the parametric excitation of electromagnetic waves in a plasma, which we have considered here, can play a definite role in cosmic conditions (for example, in the mechanism of radio emission from the sun), and can also be employed in principle to produce plasma parametric amplifiers.

In conclusion the authors are grateful to A. V. Gaponov and V. V. Zheleznyakov for a useful discussion of the results of the work.

[1] L. D. Landau, JETP **16**, 574 (1946).

[2] V. L. Ginzburg, Rasprostranenie elektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in a Plasma), Fizmatgiz, 1960.

[3] N. S. Stepanov, Izv. vyssh. uch. zav., Radiofizika **6**, 461 (1963).

[4] Akhiezer, Prokhoda, and Sitenko, JETP **33**, 750 (1957), Soviet Phys. JETP **6**, 576 (1958).

[5] N. S. Stepanov, Izv. vyssh. uch. zav., Radiofizika **4**, 656 (1961).

[6] P. Tien and H. Suhl, Proc. IRE **46**, 700 (1958).

[7] S. A. Akhmanov and R. V. Khokhlov, JETP **43**, 351 (1962), Soviet Phys. JETP **16**, 252 (1963).

[8] V. N. Oraevskiĭ and R. Z. Sagdeev, ZhTF **32**, 1291 (1962), Soviet Phys. Tech. Phys. **7**, 955 (1963).

[9] V. N. Oraevskiĭ, Prikl. mekh. i tekhn. fiz. (Appl. Mech. and Tech. Physics) No. 5, 38 (1962).

Translated by J. G. Adashko
242