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DRIFT INSTABILITY IN A DENSE PLASMA

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It is shown that the drift instability in an inhomogeneous plasma (with zero temperature gradient) that has been pointed out earlier [1,2] is not excited if the ratio of plasma pressure to magnetic pressure exceeds a critical value $\beta = (8\pi p/B^2) > 0.13$. The boundaries of the drift instability region in the pressure-perturbation wavelength plane are determined.

1. INTRODUCTION

It has been shown earlier [1,2] that an inhomogenous low-pressure plasma ($\beta \equiv 8\pi p/B^2 \ll 1$) is unstable against perturbations with phase velocity ω/k of the order of the particle Larmor drift velocity $v_0 \sim v_T\rho/a$ (v_T is the particle thermal velocity, ρ is the particle Larmor radius and a is the characteristic scale length of the inhomogeneity in density and temperature). This instability has been called the drift instability.[1]

In the present work we investigate the drift instability in a high-pressure plasma $\beta \sim 1$. This question is of interest because it is these values of β that would be required to realize a controlled thermonuclear fusion reaction.

It will be shown below that the drift instability does not occur if β is not small compared with unity, in any case, when $\beta \gtrsim 1$. The mechanism responsible for stabilization of the drift instability in a dense plasma can be understood as follows. This instability is due to the interaction between the wave and resonance electrons, i.e., electrons whose velocity along the lines of force v_z (the z axis is in the direction of the magnetic field B) is approximately the same as the longitudinal phase velocity of the wave ω/k_z. It has been shown in [2] that with a zero temperature gradient ($\nabla T = 0$) drift instabilities are excited at a frequency $\omega \approx k_z c_A/\sqrt{2} \approx k_x v_0$ ($c_A = (B^2/4\pi n_0 m_i)^{1/2}$ is the Alfvén velocity, n_0 is the equilibrium plasma density). It is found that the relative number of resonance electrons is small, going as $c_A/v_{Te} \sim (m_e/m_i\beta)^{1/2}$ (v_{Te} is the thermal velocity of the electrons while m_e and m_i are the electron and ion masses respectively), as does the growth rate $\gamma = \mathrm{Im}\,\omega$ when compared with the frequency.

On the other hand the plasma ions interact with the wave, extracting energy from it and damping the oscillations.

The wave damping arising from the resonant interaction with the ions is due to two mechanisms:

1) The longitudinal phase velocity of the wave ω/k_z can be approximately the same as the velocity along the lines of force of the ions; in this case the wave damping is characterized by an exponential of the form $(-\omega^2/k_z^2 v_{Ti}^2) \approx \exp(-1/\beta)$.

2) The phase velocity of the wave in the drift direction ω/k_x can be approximately the same as the magnetic drift velocity of the ions

$$u_M^i = -(v_\perp^2/2\omega_{Bi})\,\partial \ln B/\partial y, \qquad (\omega_{Bi} = eB/m_i c).$$

Since $\partial \ln B/\partial y = -\frac{1}{2}\beta\partial \ln n_0/\partial y$, the wave damping due to this interaction is characterized by an exponential of the form $\exp(-\omega/k_x \bar{u}_M^i) \approx \exp(-2/\beta)$.

At low values of β both of these effects are exponentially small so that the electron excitation overrides them and the instability is excited. However, if β is not too small (although smaller than unity), the importance of these exponentially small terms increases. Even when $\beta < 1$ the ion damping can be greater than the electron excitation since the electron terms in the dispersion equation are multiplied by the small coefficient $(m_e/m_i)^{1/2}$ indicated above.

It is evident from these qualitative considerations that the drift instability will be stabilized at values of β greater than some critical value $\beta_0 < 1$. It is shown in the present work that $\beta_0 \approx 0.13$.

2. FUNDAMENTAL EQUATIONS

In describing small oscillations of an inhomogeneous plasma we shall find it convenient to introduce the dielectric tensor $\hat{\varepsilon}_{\alpha\beta}$, which characterizes the induced currents j ($\hat{\varepsilon}_{\alpha\beta} = \delta_{\alpha\beta} + 4\pi i\omega^{-1} \times \hat{\sigma}_{\alpha\beta}$, where $j_\alpha = \hat{\sigma}_{\alpha\beta}E_\beta$ while the time dependence of the wave is given by $e^{-i\omega t}$). It is also convenient to introduce the polarizability vector $\hat{\chi}_\alpha$, which relates the charge density ρ and the electric field associated with the wave $\mathbf{E}$, $\rho = \hat{\chi}_\alpha E_\alpha$. It is clear that the vector $\hat{\chi}_\alpha$ is related to the tensor $\hat{\varepsilon}_{\alpha\beta}$ in a definite way by virtue of the connection between the current and charge given by the equation of continuity.

Introducing $\hat{\varepsilon}_{\alpha\beta}$ and $\hat{\chi}_\alpha$ we write Poisson's equation and Maxwell's equations in the form

$$\operatorname{div} \mathbf{E} = 4\pi\hat{\chi}_\alpha E_\alpha, \qquad (2.1)$$

$$(\operatorname{rot}\operatorname{rot}\mathbf{E})_\alpha = \omega^2 c^{-2}\hat{\varepsilon}_{\alpha\beta}E_\beta. \qquad (2.2)^*$$

The field $\mathbf{E}$ is expressed in terms of the scalar and vector potentials φ and $\mathbf{A}$:

$$\mathbf{E} = -\nabla\varphi + i\omega c^{-1}\mathbf{A}, \qquad (2.3)$$

where $\mathbf{A}$ satisfies the condition

$$\operatorname{div}\mathbf{A} = 0. \qquad (2.4)$$

It then follows from (2.1) and (2.2) that

$$\Delta\varphi = 4\pi\hat{\chi}_\alpha\nabla_\alpha\varphi - 4\pi i\omega c^{-1}\hat{\chi}_\alpha A_\alpha, \qquad (2.5)$$

$$-(\Delta A)_\alpha = \omega^2 c^{-2}\hat{\varepsilon}_{\alpha\beta}A_\beta + i\omega c^{-1}\hat{\varepsilon}_{\alpha\beta}\nabla_\beta\varphi. \qquad (2.6)$$

If it is assumed that the magnetic force lines are along the z axis and that the plasma inhomogeneity is along the y axis the spatial dependence of the field on the "homogeneous" coordinates x and z can be written in the form $(ik_x x + ik_z z)$. If the wave length is small compared with the characteristic scale size of the plasma inhomogeneity a, $\lambda_y \ll a$, the dependence of the field on the "inhomogeneous" coordinate y can be written in the semi-classical (WKB) form $\exp[i\int k_y(y)\,dy]$. We then have from (2.5) and (2.6)

$$\begin{aligned} k^2\varphi &= -4\pi i\chi\mathbf{k}\varphi + 4\pi i\omega c^{-1}\chi\mathbf{A},\\ k^2A_y &= \omega^2c^{-2}\varepsilon_{y\beta}A_\beta - \omega c^{-1}\varepsilon_{y\beta}k_\beta\varphi,\\ k^2A_z &= \omega^2c^{-2}\varepsilon_{z\beta}A_\beta - \omega c^{-1}\varepsilon_{z\beta}k_\beta\varphi. \end{aligned} \qquad (2.7)$$

Here, χ_α and $\varepsilon_{\alpha\beta}$ are functions of $\mathbf{k}$, ω, and y, obtained by applying the operators $\hat{\chi}_\alpha$ and $\hat{\varepsilon}_{\alpha\beta}$ to expressions of the form $\{ik_x y + i\int k_y\,dy + ik_z z\}$. These functions have been given earlier.[3]

In the WKB approximation with $k_z \ll k_\perp$ the condition in (2.4) becomes

$$A_x k_x + A_y k_y = 0. \qquad (2.8)$$

Expressing A_y in terms of A_x in Eq. (2.7) and setting the determinant of this system equal to zero we obtain the eikonal equation[4] which, for low-frequency oscillations $\omega \ll \omega_{Bi}$ ($\omega_{Bi} = eB/m_i c$ is the ion cyclotron frequency), can be written in the following symmetric form:

$$\begin{pmatrix} -k^2\varepsilon_0, & i\omega c^{-1}\alpha_2, & \omega c^{-1}\alpha_3,\\ -i\omega c^{-1}\alpha_2, & k^2-\omega^2c^{-2}\varepsilon_2, & i\omega^2c^{-2}\alpha_{23},\\ \omega c^{-1}\alpha_3, & -i\omega^2c^{-2}\alpha_{23}, & k^2-\omega^2c^{-2}\varepsilon_3 \end{pmatrix} = 0. \qquad (2.9)$$

Here we have introduced the notation

$$\begin{aligned} \varepsilon_0 &= 1 + 4\pi i\chi\mathbf{k}/k^2, \quad \varepsilon_2 = \varepsilon_{yy} - \varepsilon_{yx}k_y/k_x, \quad \varepsilon_3 = \varepsilon_{zz},\\ \alpha_2 &= i\varepsilon_{y\beta}k_\beta/\cos\psi = 4\pi(\chi_y\cos\psi - \chi_x\sin\psi),\\ \alpha_{23} &= i\varepsilon_{yz}/\cos\psi = -i\cos\psi\,(\varepsilon_{zy} - \varepsilon_{zx}k_y/k_x),\\ \alpha_3 &= \varepsilon_{z\beta}k_\beta = 4\pi i\chi_z, \quad \psi = \operatorname{arc\,tg}(k_y/k_x). \end{aligned} \qquad (2.10)^*$$

The quantities ε_0, ε_2 etc. are of the form

$$\varepsilon_0 = 1 + \sum_{i,e}\frac{4\pi e^2}{Tk^2}\left(1 + \frac{k_xT}{m\omega\omega_B}\frac{\partial^0}{\partial y}\right)n_0\left(1 - \omega\int\frac{J_0^2 f_0\,d\mathbf{v}}{\omega - k_xu_M - k_zv_z}\right),$$

$$\varepsilon_2 = 1 + \left(\frac{\omega_{0i}}{\omega_{Bi}}\right)^2 - \sum_{i,e}\frac{4\pi e^2}{\omega T}\left(1 + \frac{k_xT}{m\omega\omega_B}\frac{\partial^0}{\partial y}\right)n_0\int\frac{J_1^2 f_0 v_\perp^2\,d\mathbf{v}}{\omega - k_xu_M - k_zv_z},$$

$$\varepsilon_3 = 1 - \sum_{i,e}\frac{4\pi e^2}{\omega T}\left(1 + \frac{k_xT}{m\omega\omega_B}\frac{\partial^0}{\partial y}\right)n_0\int\frac{J_0^2 v_z^2 f_0\,d\mathbf{v}}{\omega - k_xu_M - k_zv_z},$$

$$\alpha_2 = -\sum_{i,e}\frac{4\pi e^2}{T}\left(1 + \frac{k_xT}{m\omega\omega_B}\frac{\partial^0}{\partial y}\right)n_0\int\frac{v_\perp J_0J_0' f_0\,d\mathbf{v}}{\omega - k_xu_M - k_zv_z},$$

$$\alpha_3 = -\sum_{i,e}\frac{4\pi e^2}{T}\left(1 + \frac{k_xT}{m\omega\omega_B}\frac{\partial^0}{\partial y}\right)n_0\int\frac{v_zJ_0^2 f_0\,d\mathbf{v}}{\omega - k_xu_M - k_zv_z},$$

$$\alpha_{23} = -\sum_{i,e}\frac{4\pi e^2}{T\omega}\left(1 + \frac{k_xT}{m\omega\omega_B}\frac{\partial^0}{\partial y}\right)n_0\int\frac{v_zv_\perp J_0J_0' f_0\,d\mathbf{v}}{\omega - k_xu_M - k_zv_z}. \qquad (2.11)$$

Here

$$f_0 = \left(\frac{m}{2\pi T}\right)^{1/2}\frac{m}{T}\exp\left(-\frac{mv^2}{2T}\right), \qquad \frac{\partial^0}{\partial y} = \frac{\partial}{\partial T}\frac{dT}{dy} + \frac{\partial}{\partial n_0}\frac{dn_0}{dy},$$

$$d\mathbf{v} = v_\perp\,dv_\perp\,dv_z;$$

J_0, J_1 are Bessel functions, $J_n = J_n(k_\perp v_\perp/\omega_B)$, $u_M = -(v_\perp^2/2\omega_B)\partial\ln B/\partial y$ is the magnetic drift velocity, $\omega_B = eB/mc$, $\omega_{0i}^2 = 4\pi n_0 e^2/m_i$, $k_\perp = \sqrt{k_x^2 + k_y^2}$; the summation in Eq. (2.11) is taken over the ions and electrons.

The relation in (2.9) is the basic equation for describing low-frequency oscillations of an inhomogeneous plasma with an arbitrary ratio of gas pressure to magnetic pressure.

To obtain the required information concerning the plasma oscillations and stability it is sufficient to study Eq. (2.9) in the vicinity of an arbitrary fixed point $y = y^*$ with an arbitrary real value of the wave number $k_y = k_y(y^*)$.[5,6] This is the "local" approach (in contrast with the "integral" method[7] which is equivalent to it in many ways) and will be used below.

3. BOUNDARIES OF THE DRIFT INSTABILITY REGION IN A DENSE PLASMA

We now apply the results of the preceding sec-

*rot = curl.

*arc tg = $\tan^{-1}$.

tion to the analysis of the drift instability of a dense plasma with zero temperature gradient ($\nabla T = 0$). Below we shall find the boundaries of the instability region in the pressure-perturbation wavelength-plane.

It is assumed that the boundary of the drift instability lies in the region $\beta \ll 1$. Then, keeping terms which are exponentially small [as $\exp(-1/\beta)$], taking $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$ and $\omega \gg k_x \bar{u}_M$, and using Eqs. (2.9) and (2.11), we obtain

$$\varepsilon_0\left(1-\frac{\omega^2}{c^2k^2}\varepsilon_3\right)+\frac{\omega^2}{c^2k^4}\alpha_3^2=0 \tag{3.1}$$

or

$$2-I_0e^{-z}\left(1+\frac{k_xv_0}{\omega}\right)+\frac{(k_xv_0)^2-\omega^2}{(k_zc_A)^2}\frac{1-I_0e^{-z}}{z}$$
$$+\,i\sqrt{\pi}S\left\{\frac{\omega}{k_zv_{Te}}\left(1-\frac{k_xv_0}{\omega}\right)+\frac{\omega}{k_zv_{Ti}}\left(1+\frac{k_xv_0}{\omega}\right)Q\right\}=0,$$
$$S=1-\left(\frac{\omega}{k_zc_A}\right)^2\frac{1-I_0e^{-z}}{z}\left(1+\frac{k_xv_0}{\omega}\right); \tag{3.2}$$

$$Q=\int_0^\infty J_0^2(\sqrt{2z\varepsilon})\exp\left\{-\varepsilon-\frac{m_i}{2Tk_z^2}(\omega-k_x\bar{u}_M^i\varepsilon)^2\right\}d\varepsilon,$$
$$T_i=T_e=T,\qquad z=\frac{k_\perp^2T}{m_i\omega_{Bi}^2},\qquad \bar{u}_M^i=-\frac{T}{m_i\omega_{Bi}}\frac{\partial\ln B}{\partial y},$$
$$v_0=\frac{T}{m_i\omega_{Bi}}\frac{\partial\ln n_0}{\partial y}, \tag{3.3}$$

v_0 is the electron Larmor drift velocity, and I_0 is the Bessel function of imaginary argument $I_0 = I_0(z)$.

The imaginary terms in this equation describe the effect of the resonant interactions between the wave and the electrons (given by the $\omega/k_z v_{Te}$ term) and the ions (given by the term with the integral Q).

At the stability boundary, i.e., where $\mathrm{Im}\,\omega = 0$, the real and imaginary parts of Eq. (3.2) yield

$$2-I_0e^{-z}\left(1+\frac{k_xv_0}{\omega}\right)+\frac{(k_xv_0)^2-\omega^2}{(k_zc_A)^2}\frac{1-I_0e^{-z}}{z}=0, \tag{3.4}$$
$$1-\frac{k_xv_0}{\omega}+\left(\frac{m_i}{m_e}\right)^{1/2}\left(1+\frac{k_xv_0}{\omega}\right)Q=0. \tag{3.5}$$

It is evident from Eq. (3.4) that $\omega = \omega(\mathbf{k})$ has three branches. Of these, it is shown in [6] that only one branch satisfies the condition $1 - k_x v_0/\omega < 0$; if one neglects ion damping (the terms in Eq. (3.2) containing the integral) this mode is unstable. However, the ion damping can compensate the electron excitation and the instability can be stabilized. It follows from Eq. (3.5) that the ion damping is a minimum at the maximum value of ω/k_z; hence the hardest perturbations to stabilize are those for which k_z^0 satisfies the relation

$$\left.\frac{\partial(\omega/k_z)}{\partial k_z}\right|_{k_z=k_z^0}=0. \tag{3.6}$$

As indicated by Eq. (3.4), the dependence of k_z^0 on z and β is

$$(k_z^0)^2=\tfrac{1}{2}\lambda\beta\varkappa^2(1-I_0e^{-z})(k_x/k_\perp)^2,\qquad \varkappa=\partial\ln n_0/\partial y, \tag{3.7}$$

while the corresponding value of the frequency $\omega = \omega(k_z^0)$ is

$$\omega=k_xv_0\lambda I_0e^{-z}, \tag{3.8}$$

where

$$\lambda=\lambda(z)\equiv\left(\frac{1-\sqrt{1-I_0e^{-z}}}{I_0e^{-z}}\right)^2. \tag{3.9}$$

In particular, as $z \to 0$

$$\lambda=1,\qquad (k_z^0)^2=\tfrac{1}{2}\beta\varkappa^2z\,(k_x/k_\perp)^2,\qquad \omega=k_xv_0, \tag{3.10}$$

whereas when $z \gg 1$

$$\lambda=\tfrac{1}{4},\quad (k_z^0)^2=\tfrac{1}{8}\varkappa^2\beta\,(k_x/k_\perp)^2,\quad \omega=k_xv_0/4\sqrt{2\pi z}. \tag{3.11}$$

Eliminating the oscillation frequency from Eq. (3.5), using Eqs. (3.7) and (3.8), we obtain the critical value of β as a function of the transverse wave number:

$$\frac{1-\lambda I_0e^{-z}}{1+\lambda I_0e^{-z}}=\left(\frac{m_i}{m_e}\right)^{1/2}\int_0^\infty J_0^2(\sqrt{2z\varepsilon})\exp\left\{-\varepsilon-\frac{z(\lambda I_0e^{-z}-\beta\varepsilon/2)^2}{\lambda\beta(1-I_0e^{-z})}\right\}d\varepsilon. \tag{3.12}$$

It is evident from Eq. (3.11) that the longitudinal damping exceeds the transverse damping (due to magnetic drift) when $z < z_0 = 8\pi$. When $z \gg 1$

$$\exp(-m_i\omega^2/2Tk_z^2)=\exp(-1/8\pi\beta),$$
$$\exp(-\omega/k_x\bar{u}_M^i)=\exp(-1/2\sqrt{2\pi z}\beta), \tag{3.13}$$

whence follows the expression written above for z_0. Hence, up to these values of z we use the following simplified equation in place of Eq. (3.12):

$$\frac{1-\lambda I_0e^{-z}}{1+\lambda I_0e^{-z}}=\left(\frac{m_i}{m_e}\right)^{1/2}I_0e^{-z}e^{-\mu(z)/\beta},\quad \mu(z)=\frac{z\lambda(I_0e^{-z})^2}{1-I_0e^{-z}}. \tag{3.14}$$

When $z > z_0$, omitting the longitudinal damping and taking account only of the transverse damping, we have approximately

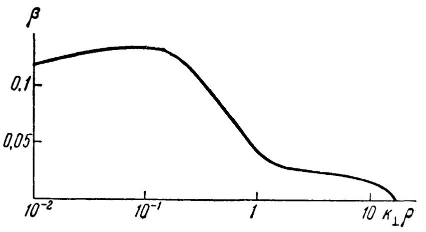
$$1=(m_i/m_e)(2\pi/z^3)^{1/2}\exp[-1/\beta\sqrt{2\pi z}]. \tag{3.15}$$

Here we have used the approximate expression for $J_0^2(x)$ at large values of x: $J_0^2(x) = 1/\pi x$.

The expression for the stability boundary $\beta = \beta(z)$ obtained from Eqs. (3.14) and (3.15), is

$$\beta=\mu(z)/\ln\left\{\left(\frac{m_i}{m_e}\right)^{1/2}I_0e^{-z}\frac{1+\lambda I_0e^{-z}}{1-\lambda I_0e^{-z}}\right\}\quad\text{for}\quad z<8\pi,$$
$$\beta=\sqrt{\frac{2}{\pi z}}\Big/\ln\left\{\left(\frac{m_i}{m_e}\right)^2\frac{2\pi}{z^3}\right\}\quad\text{for}\quad z>8\pi.$$

The function $\beta = \beta(z)$ is shown in the figure.



The region above the curve $\beta = \beta(k_\perp \rho)$ corresponds to stability.

When $z \approx (m_i/m_e)^{2/3}$ the imaginary terms in the eikonal equation are of the same order as the real terms and the stability boundary can only be determined qualitatively. In this case $\omega \sim k_x \bar{u}^i_M$; on the other hand, $\omega \approx k_x v_0/4\sqrt{2\pi z}$, whence we find that when $z \approx (m_i/m_e)^{2/3}$ the function $\beta(z)$ decreases as $z^{-1/2}$. This functional relation $\beta = \beta(z)$ will hold as long as $\omega \sim k_z v_{Ti}$ i.e., (see [8]) up to $z \approx m_i/2\pi m_e$; beyond this value the instability disappears for all values of β.

Thus, in the present work we have shown that if the ratio of plasma pressure to magnetic pressure $\beta = 8\pi p/B^2 > 0.13$ is an isothermal plasma ($\nabla T = 0$) the plasma is stable against drift perturbations.

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