STRUCTURE OF SINGULARITIES AND THEIR MOTION IN NONLINEAR ELECTRODYNAMICS

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Submitted to JETP editor May 18, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 45, 1625-1633 (November, 1963)

We study the structure of those singularities in nonlinear electrodynamics whose trajectories are nearly uniform. Equations of motion for the singularities are obtained in this case and it is shown that the uniform and rectilinear motion is stable with respect to perturbations for an arbitrary choice of the Lagrange function for the electromagnetic field. Mass renormalization, generally speaking, upsets the stability of this motion.

INTRODUCTION

THE structure of the singularities and their motion in nonlinear electrodynamics have been considered previously[1] under the assumption that the radius of curvature of the 4-trajectory was large compared with the "singularity radius." It is the purpose of the present work to study the structure of the singularity when its motion deviates little from uniform and rectilinear motion.

Under these conditions the dynamical principle formulated in [1] leads to equations of motion for the singularity that are nonlocal in proper time. These equations of motion satisfy the principle of causality, if one uses the retarded boundary condition in solving the field equations, and permit one to study the question of stability of uniform and rectilinear motion with respect to small perturbations. It turns out that this motion is stable for any choice of the Lagrange function for the nonlinear electromagnetic field. However when mass renormalization is carried out (i.e., when a mass of non-field origin is inserted into the equations of motion) the stability of the uniform and rectilinear motion is, generally speaking, upset.

1. FIELD STRUCTURE IN FIRST APPROXIMATION

As was shown in [1] the variational principle, formulated for the Lagrange function $L(J_1, J_2)$ ($J_1 = -\frac{1}{4}F^2_{\mu\nu}$, $J_2 = -\frac{1}{4}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$, $F_{\mu\nu}$—the electromagnetic field tensor) leads to the following equations for the electromagnetic field:

$$\frac{\partial}{\partial x_\lambda}(\chi_1 F_{\lambda\nu}) + F_{\rho\mu}\varepsilon_{\rho\mu\lambda\nu}\frac{\partial\chi_2}{\partial x_\lambda} = 0, \quad \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + (\mu\nu\lambda) = 0, \tag{1}$$

where $\chi_1 = \partial L/\partial J_1$, $\chi_2 = \partial L/\partial J_2$ and $(\mu\nu\lambda)$ is an abbreviation for the terms obtained from $\partial F_{\mu\nu}/\partial x_\lambda$ by cyclic permutation of the indices.

Denoting by $\xi_\mu(\tau)$ (τ is the proper time, i.e., $\dot{\xi}^2_\mu = -1$) the parametric equation of the world line of the singularity and introducing in place of the variables x_μ, characterizing the point of space-time, the variables $\eta_\mu(x)$, $\tau(x)$, related to x_μ by $\eta_\mu = x_\mu - \xi_\mu(\tau)$, $\eta\dot{\xi}(\tau) = 0$, one can transform the field equations (1) to the form[1]

$$\begin{aligned}
&(\delta_{\nu\lambda} + \dot{\xi}_\nu\dot{\xi}_\lambda)\frac{\partial}{\partial\eta_\lambda}(\chi_1 F_{\mu\nu}) \\
&- \frac{\dot{\xi}_\nu}{1+\eta\ddot{\xi}}\left\{\frac{\partial}{\partial\tau}(\chi_1 F_{\mu\nu}) + (\eta\ddot{\xi})\,\dot{\xi}_\lambda\frac{\partial}{\partial\eta_\lambda}(\chi_1 F_{\mu\nu})\right\} \\
&- F_{\alpha\beta}\varepsilon_{\alpha\beta\lambda\mu}\left\{(\delta_{\lambda\nu} + \dot{\xi}_\lambda\dot{\xi}_\nu)\frac{\partial\chi_2}{\partial\eta_\nu}\right. \\
&\left.- \frac{\dot{\xi}_\lambda}{1+\eta\ddot{\xi}}\left[\frac{\partial\chi_2}{\partial\tau} + (\eta\ddot{\xi})\,\dot{\xi}_\nu\frac{\partial\chi_2}{\partial\eta_\nu}\right]\right\} = 0, \\
&(\delta_{\mu\rho} + \dot{\xi}_\mu\dot{\xi}_\rho)\frac{\partial F_{\lambda\nu}}{\partial\eta_\rho} - \frac{\dot{\xi}_\mu}{1+\eta\ddot{\xi}}\left[\frac{\partial F_{\lambda\nu}}{\partial\tau} + (\eta\ddot{\xi})\,\dot{\xi}_\rho\frac{\partial F_{\lambda\nu}}{\partial\eta_\rho}\right] \\
&+ (\mu\nu\lambda) = 0.
\end{aligned} \tag{2}$$

Unlike in [1] we look in the present work for solutions of (2) in the form of an expansion in powers of a parameter that characterizes the deviation of the trajectory $\xi_\mu(\tau)$ from rectilinearity, and confine ourselves here to the first approximation. From the point of view of expansion of the field in the curvature of the 4-trajectory, the first approximation corresponds to the taking into account in the expansion of the field of all terms of the form

$$\ddot{\xi}_\mu, \ldots, \xi^{(n)}_\mu, \ldots, \xi^{(n)}_\mu\dot{\xi}_\nu, \ldots, \xi^{(n)}_\mu\eta_\nu, \ldots,$$

but neglecting all terms of the form

$$\ddot{\xi}_\mu\ddot{\xi}_\nu,\ \ddot{\xi}_\mu\dddot{\xi}_\nu, \ldots,\ \ddot{\xi}_\mu\ddot{\xi}_\nu\dot{\xi}_\lambda, \ldots,\ \dot{\xi}_\mu\dddot{\xi}_\mu = -\ddot{\xi}^2, \ldots$$

This means that we must look for a solution of the Eqs. (2) in the form $F_{\mu\nu} = F^0_{\mu\nu} + F^1_{\mu\nu}$, where

$$F^0_{\mu\nu} = \eta^{-1} z(\eta)(\dot\xi_\mu \eta_\nu - \dot\xi_\nu \eta_\mu), \qquad z\chi(^1/_2 z^2, 0) = C/\eta^2 \tag{3}$$

is the field of the zeroth approximation, determined in [1] ($e = 4\pi C$ is the charge of the singularity) and

$$F^1_{\mu\nu} = \int_{-\infty}^{\infty} d\tau' \{ g_1(\eta, \tau - \tau')[\ddot\xi_\mu(\tau')\dot\xi_\nu(\tau') - \ddot\xi_\nu(\tau')\dot\xi_\mu(\tau')]$$
$$+ g_2(\eta, \tau - \tau')(\eta\ddot\xi(\tau'))[\dot\xi_\mu(\tau')\eta_\nu - \dot\xi_\nu(\tau')\eta_\mu]$$
$$+ g_3(\eta, \tau - \tau')[\ddot\xi_\mu(\tau')\eta_\nu - \ddot\xi_\nu(\tau')\eta_\mu]\} \tag{3'}$$

is the first approximation determined by three unknown functions g_1, g_2, g_3 of two variables η, τ. The various moments of the functions g_1, g_2, g_3 with respect to the variable τ represent certain of the coefficients in the expansion of the field in the curvature of the 4-trajectory.

According to Eq. (2) the equations for the first approximation $F^1_{\mu\nu}$ have the form

$$(\delta_{\nu\lambda} + \dot\xi_\nu\dot\xi_\lambda)\frac{\partial F^1_{\mu\nu}}{\partial \eta_\lambda} - \dot\xi_\nu \frac{\partial F^1_{\mu\nu}}{\partial \tau} + (\delta_{\nu\lambda} + \dot\xi_\nu\dot\xi_\lambda) F^1_{\mu\nu}\frac{\partial \ln \chi^0_1}{\partial \eta_\lambda}$$
$$+ (\delta_{\nu\lambda} + \dot\xi_\nu\dot\xi_\lambda) F^0_{\mu\nu}\frac{\partial}{\partial\eta_\lambda}\ln\frac{\chi_1}{\chi_1^0} - \dot\xi_\nu F^0_{\mu\nu}\frac{\partial}{\partial\tau}\ln\frac{\chi_1}{\chi^0_1} = 0,$$
$$(\delta_{\mu\rho} + \dot\xi_\mu\dot\xi_\rho)\frac{\partial F^1_{\lambda\nu}}{\partial\eta_\rho} - \dot\xi_\mu\frac{\partial F^1_{\lambda\nu}}{\partial\tau} + (\mu\nu\lambda) = 0, \tag{4}$$

where $\chi^0_1 = \chi_1(z^2/2, 0)$.

It is easy to see that the second of these equations gives rise to the following relation between the functions g_1, g_2, g_3:

$$\frac{1}{\eta}g_1' + g_2 + \dot g_3 + \frac{1}{\eta} z\delta(\tau) = 0, \tag{5}$$

where the prime means differentiation with respect to η and the dot—differentiation with respect to τ.

Noting that, according to Eq. (3),

$$\frac{\partial \ln \chi^0_1}{\partial J^0_1} = -\frac{1}{zz'}\left(\frac{2}{\eta} + \frac{z'}{z}\right), \qquad \ln\frac{\chi_1}{\chi^0_1} = (J_1 - J^0_1)\frac{\partial \ln\chi^0_1}{\partial J^0_1},$$
$$\dot\xi_\nu \frac{\partial F^0_{\mu\nu}}{\partial\tau} = 0,$$

we transform the first of the Eqs. (4) to the form

$$(\delta_{\nu\lambda} + \dot\xi_\nu\dot\xi_\lambda)\left\{\frac{\partial F^1_{\mu\nu}}{\partial\eta_\lambda} - \frac{z}{\eta}\dot\xi_\mu\eta_\nu\frac{\partial}{\partial\eta_\lambda}\left[(J_1 - J^0_1)\frac{1}{zz'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)\right]\right.$$
$$\left. - F^1_{\mu\nu}\frac{\eta_\lambda}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right)\right\} - \dot\xi_\nu\frac{\partial F^1_{\mu\nu}}{\partial\tau}$$
$$+ \frac{z}{\eta}\eta_\mu\frac{\partial}{\partial\tau}\left[(J_1 - J^0_1)\frac{1}{zz'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)\right] = 0, \tag{4'}$$

where $J_1 - J^0_1 = -{}^1/_2 F^0_{\mu\nu}F^1_{\mu\nu} = -(z/\eta)\xi_\mu\eta_\nu F^1_{\mu\nu}$.

Substituting into Eq. (4′) the expression (3′) for $F^1_{\mu\nu}$ we obtain

$$\int_{-\infty}^{\infty} d\tau' \{A(\eta, \tau - \tau')\eta_\mu(\eta\ddot\xi(\tau'))$$
$$+ B(\eta, \tau - \tau')\dot\xi_\mu(\tau')(\eta\ddot\xi(\tau'))$$
$$+ C(\eta, \tau - \tau')\ddot\xi_\mu(\tau')\} = 0; \tag{6}$$

$$A = -\frac{1}{\eta}\dot g_3' - \dot g_2 + \frac{1}{\eta}g_3\left(\frac{2}{\eta} + \frac{z'}{z}\right)$$
$$- \frac{z}{\eta^2 z'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)(\dot g_1 - \eta^2\dot g_2),$$
$$B = \frac{2z}{\eta^2 z'}\left\{g_1' - \eta^2 g_2' + \eta^2\left(\frac{z''}{z'} - \frac{1}{\eta}\right)g_2\right.$$
$$\left. + \left(\frac{z'}{z} - \frac{z''}{z'} - \frac{1}{\eta}\right)g_1\right\},$$
$$C = \eta g_3' - \eta\frac{z'}{z}g_3 + \dot g_1. \tag{6'}$$

It follows from Eq. (6) that $A = B = C = 0$.

In this way we obtain finally the following system of equations for the determination of g_1, g_2, g_3:

$$g_1' - \eta^2 g_2' + \eta^2\left(\frac{z''}{z'} - \frac{1}{\eta}\right)g_2 + \left(\frac{z'}{z} - \frac{z''}{z'} - \frac{1}{\eta}\right)g_1 = 0,$$
$$\dot g_3' + \eta\dot g_2 = 0, \qquad g_2 - \frac{1}{\eta^2}\dot g_1 + \frac{z'}{\eta z}\dot g_3 = 0,$$
$$\frac{1}{\eta}g_1' + g_2 + \dot g_3 + \frac{z}{\eta}\delta(\tau) = 0. \tag{7}$$

Since the coefficients of the unknown functions in these equations are independent of τ it is convenient to Fourier-transform in that variable:

$$g_i(\eta, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_i(\eta, \omega)e^{-i\omega\tau}d\omega, \quad i = 1, 2, 3.$$

The Fourier coefficients $g_i(\eta, \omega)$ satisfy, according to Eq. (7), the following system of ordinary differential equations:

$$g_1' - \eta^2 g_2' + \eta^2\left(\frac{z''}{z'} - \frac{1}{\eta}\right)g_2 + \left(\frac{z'}{z} - \frac{z''}{z'} - \frac{1}{\eta}\right)g_1 = 0,$$
$$g_3' - i\omega\eta g_2 = 0, \qquad g_3 - \frac{i\omega z}{\eta z'}(\eta^2 g_2 - g_1) = 0,$$
$$g_1' + \eta g_2 - i\omega\eta g_3 + z = 0. \tag{7'}$$

It is easy to see that the first of these equations is a consequence of the second and third. Expressing therefore g_1 and g_2 in terms of g_3 from the second and third equations:

$$g_1 = \frac{\eta}{i\omega}\left(g_3' - \frac{z'}{z}g_3\right), \qquad g_2 = \frac{1}{i\omega\eta}g_3' \tag{8}$$

and substituting these expressions into the fourth of Eqs. (7), we find that $g_3(\eta, \omega)$ satisfies the following second order differential equation:

$$g_3'' + \left(\frac{2}{\eta} - \frac{z'}{z}\right)g_3' + \left[\omega^2 - \frac{z'}{z}\left(\frac{1}{\eta} + \frac{z''}{z'} - \frac{z'}{z}\right)\right]g_3$$
$$+ i\omega\frac{z}{\eta} = 0. \tag{9}$$

Equations (8) and (9), combined with formula (3′), solve the problem of determining the field of one singularity in the first approximation.

2. EQUATIONS OF MOTION OF THE SINGULARITY IN THE FIRST APPROXIMATION

The equations of motion of the singularity may be obtained from the dynamical principle:

$$\lim_{S\to 0} \dot{\xi}_\lambda \oint_S T_{\mu\nu}\, df^{\lambda\nu} = 0, \qquad (10)$$

where $df^{\lambda\nu}$ is an element of the surface S that surrounds the singularity and lies in the hyperplane orthogonal to $\dot{\xi}$; $T_{\mu\nu}$ is the field energy-momentum tensor:

$$T_{\mu\nu} = (L - J_2\chi_2)\,\delta_{\mu\nu} + \chi_1 F_{\mu\lambda}F_{\nu\lambda}. \qquad (11)$$

Choosing a sphere as the surface S we express the dynamical principle (10) in the form

$$\lim_{\eta\to 0} \eta \overline{T_{\mu\nu}\eta_\nu} = 0, \qquad (10')$$

where the bar denotes averaging over angles:

$$\bar{f} = \frac{1}{4\pi}\oint f\, do.$$

In the approximation under consideration $F_{\mu\nu} = F^0_{\mu\nu} + F^1_{\mu\nu} + f_{\mu\nu}$, where $f_{\mu\nu}$ was defined in [1] and represents the effect on the field near the singularity being discussed of other sufficiently distant singularities. The contribution to the equations of motion of the field $f_{\mu\nu}$ is equal to [1]

$$\lim_{\eta\to 0} \eta\, \overline{(T_{\mu\nu}\eta_\nu)}_f = -Cf^0_{\mu\nu}\dot{\xi}_\nu, \qquad (12)$$

where $f^0_{\mu\nu}$ is the field of all remaining singularities under the assumption that the singularity being discussed is absent ($C = 0$).

The contribution to $\lim_{\eta\to 0} \eta \overline{T_{\mu\nu}\eta_\nu}$ from the self field $F^1_{\mu\nu}$ is calculated quite simply if it is noted that in the approximation in question

$$L = L(^1\!/_2 z^2, 0) + (J_1 - J_1^0)\,\chi_1(^1\!/_2 z^2, 0),$$

$$\chi_1 = \chi_1(^1\!/_2 z^2, 0) - \frac{1}{zz'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)(J_1 - J_1^0)\,\chi_1(^1\!/_2 z^2, 0),$$

$$J_1 - J_1^0 = -\frac{z}{\eta}\dot{\xi}_\mu\eta_\nu F^1_{\mu\nu}, \qquad \bar{\eta}_\mu = 0,$$

$$\overline{\eta_\mu\eta_\nu} = \frac{\eta^2}{3}(\delta_{\mu\nu} + \dot{\xi}_\mu\dot{\xi}_\nu).$$

As a result of the calculations we obtain

$$\lim_{\eta\to 0} \eta\, \overline{(T_{\mu\nu}\eta_\nu)}_{F^1} = \frac{2}{3}C^2 \int_{-\infty}^{\infty} g(0, \tau - \tau')\, \ddot{\xi}_\mu(\tau')\, d\tau',$$

$$Cg(\eta, \tau) = g_1(\eta, \tau) + \frac{z}{\eta z'}[\eta^2 g_2(\eta, \tau) - g_1(\eta, \tau)]. \qquad (13)$$

The Fourier component of this function can be, according to Eq. (8), expressed in the form

$$g(\eta, \omega) = \frac{z}{i\omega C}\left(\frac{\eta g_3(\eta, \omega)}{z}\right)'. \qquad (14)$$

According to the dynamical principle (10′) and Eqs. (12) and (13), the equations of motion of the singularity take the form

$$\frac{2}{3}C^2 \int_{-\infty}^{\infty} g(0, \tau - \tau')\, \ddot{\xi}_\mu(\tau')\, d\tau' = Cf^0_{\mu\nu}\dot{\xi}_\nu. \qquad (15)$$

It is easy to show that if $g_3(\eta, \omega)$ satisfies Eq. (9) then the function $g(\eta, \omega)$, defined by Eq. (14), satisfies the equation

$$z\frac{d}{d\eta}\frac{1}{z}\frac{dg}{d\eta} + \omega^2 g = 0. \qquad (16)$$

To find the boundary conditions to this equation we note that for $\eta \to \infty$ the solutions of the nonlinear equations should go over into solutions of linear equations corresponding to the motion of a charge e along the trajectory $\xi_\mu(\tau)$.

It is not difficult to find that in the linear theory

$$g_3(\eta, \omega) = -\frac{iC}{\omega\eta^3}\{1 + (\pm i\omega\eta - 1)\, e^{\pm i\omega\eta}\}$$

and, consequently, $g(\eta, \omega) = \eta^{-1}e^{\pm i\omega\eta}$, where the + sign corresponds to advanced, and the − sign to retarded, solutions. Consequently the function $g(\eta, \omega)$ should satisfy the following boundary condition:

$$g(\eta, \omega) \to \frac{1}{\eta}e^{\pm i\omega\eta}, \quad \eta \to \infty. \qquad (17)$$

It is easy to verify that the asymptotic behavior (17) is not in contradiction with Eq. (16), since $z \to C/\eta^2$ when $\eta \to \infty$. Together with the boundary condition (17) Eq. (16) determines uniquely the function $g(\eta, \omega)$.

In the following we confine ourselves to the consideration of boundary conditions corresponding to retarded solutions. It can be shown that the function $g(\eta, \omega)$ [and, consequently, also the function $g(0, \omega)$] analytically continued into the upper half of the ω plane, has no poles in the upper half plane. In addition, it follows from Eqs. (16), (17) that $g(\eta, \omega)$ has no other singular points and falls off exponentially for $\omega \to \infty$, Im $\omega > 0$. Therefore the function

$$g(0, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(0, \omega)\, e^{-i\omega\tau} d\omega$$

vanishes for $\tau < 0$:

$$g(0, \tau) = 0, \quad \tau < 0. \qquad (18)$$

With Eq. (18) taken into account Eq. (15) takes on the form

$$-\frac{2}{3}C^2\int_{-\infty}^{\tau} g(0, \tau-\tau')\,\ddot{\xi}_\mu(\tau')\,d\tau' = Cf^0_{\mu\nu}\dot{\xi}_\nu. \qquad (15')$$

This equation shows that a change in the field after the instant of time τ can have no effect on the motion of the singularity up to the instant of time τ. In this fashion the principle of causality is connected with the absence of poles in the upper half plane for the function $g(0, \omega)$, and with the use of the retarded boundary conditions.

3. STABILITY OF UNIFORM AND RECTILINEAR MOTION

The question of stability of uniform and rectilinear motion (the question of self-acceleration of a charged particle) has been discussed by Fradkin[2] and Natanzon[3] under the assumption that the charge is spread out in some fashion in space (nonlocal theory). In nonlinear electrodynamics with a point singularity this question is solved rather simply and is connected with certain analytic properties of the function $g(0, \omega)$.

Let $f^0_\mu(\tau) = Cf^0_{\mu\nu}\xi_\nu = 0$ for $\tau < 0$. Then the equation

$$-\frac{2}{3}C^2\int_{-\infty}^{\tau} g(0, \tau-\tau')\,\ddot{\xi}_\mu(\tau')\,d\tau' = 0, \quad \tau < 0$$

is satisfied if

$$\ddot{\xi}_\mu(\tau) = 0, \quad \tau < 0. \qquad (19)$$

For $\tau > 0$ we then have

$$-\frac{2}{3}C^2\int_{0}^{\tau} g(0, \tau-\tau')\,\ddot{\xi}_\mu(\tau')\,d\tau' = f^0_\mu(\tau), \quad \tau > 0. \qquad (20)$$

Since according to Eq. (18)

$$g(0, \omega) = \int_0^\infty g(0, \tau)\,e^{i\omega\tau}d\tau,$$

then

$$g(0, ip) = \int_0^\infty g(0, \tau)\,e^{-p\tau}d\tau \quad \operatorname{Re} p > 0,$$

is the Laplace transform of the function $g(0, \tau)$. The solution of Eq. (20) has, therefore, the form

$$\ddot{\xi}_\mu(\tau) = \frac{3}{4\pi i C^2}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{p\tau}\frac{f^0_\mu(p)}{g(0, ip)}\,dp, \quad \tau > 0, \qquad (21)$$

where $f^0_\mu(p)$ is the Laplace transform of the function $f^0_\mu(\tau)$ and $\sigma > 0$ lies to the right of all the zeros of the function $g(0, ip)$, considered as a function of p.

It follows from Eq. (21) that for the stability of uniform and rectilinear motion it is necessary that the function $g(0, \omega)$ have no zeros in the upper half plane of the variable ω.

Before proceeding to the proof of the absence of zeros of $g(0, \omega)$ in the upper half plane, we note that from the equation satisfied by the function $z(\eta)$

$$z\chi(^1/_2 z^2, 0) = C/\eta^2, \qquad \chi(0, 0) = 1,$$

follows the absence of zeros of $z(\eta)$ in the interval $0 < \eta < \infty$. Therefore the function $z(\eta)$ is of constant sign in the interval $0 < \eta < \infty$.

And so let ω lie in the upper half plane. Then, upon multiplication of Eq. (16) by g^*/z and integration over η using the boundary condition (17), it is easy to obtain the equality

$$\omega^2 = \int_0^\infty \frac{1}{z}g'g'^*d\eta \Big/ \int_0^\infty \frac{1}{z}gg^*d\eta + g^*(0, \omega)\lim_{\eta\to 0}\frac{1}{z}g'(\eta, \omega) \Big/ \int_0^\infty \frac{1}{z}gg^*d\eta. \qquad (22)$$

From here it is seen that $g(0, \omega)$ cannot vanish in the upper half plane because if it did then there would follow the relation (z has constant sign!)

$$\omega^2 = \int_0^\infty \frac{1}{z}g'g'^*d\eta \Big/ \int_0^\infty \frac{1}{z}gg^*d\eta > 0,$$

which contradicts the assumption that ω lies in the upper half plane.

Thus, for any choice of the Lagrangian of the nonlinear electromagnetic field, uniform and rectilinear motion of the singularity turns out to be stable.

The function $g(\eta, \omega)$ may be expanded in powers of the variable ω (see Appendix):

$$g(\eta, \omega) = \sum_{n=0}^\infty \omega^n g^{(n)}(\eta),$$

$$g^{(0)}(\eta) = \frac{1}{C}\int_\eta^\infty z\,d\eta, \qquad g^{(1)}(\eta) = i, \ldots \qquad (23)$$

The expansion (23) is valid if the radius of curvature of the 4-trajectory (proportional to ω^{-1}) is large in comparison with the "singularity radius."

Keeping in the expansion (23) only the first two terms:

$$g(0, \omega) \approx \frac{3}{8\pi C^2}\,m + i\omega, \qquad m = \frac{8\pi C}{3}\int_0^\infty z\,d\eta,$$

where m is the singularity mass,[1] we find that $g(0, \omega)$ has an apparent zero in the upper half plane. One can, however, attempt to obtain an exact expression for $g(0, \omega)$ in the linear theory by means of renormalization of the mass of the singularity. Denoting by m_0 the mechanical mass we obtain for the function $g(0, \omega)$ the expression

$$g(0, \omega) = \frac{3}{8\pi C^2} m_0 + \sum_{n=0}^{\infty} \omega^n g^{(n)}(0).$$

Choosing m_0 so as to make in the limit of the linear theory $m_0 + m$ equal to the given quantity $M > 0$ ($m_0 + m \to M$), we find

$$\lim_{L \to J_1} g(0, \omega) = \frac{3}{8\pi C^2} M + i\omega = \frac{6\pi}{e^2} M + i\omega .$$

In this fashion mass renormalization leads to the appearance of a zero for $g(0, \omega)$ in the upper half plane, and consequently to self-acceleration of the singularity.

In conclusion the authors express their gratitude to A. I. Akhiezer for interest in the work and to P. I. Fomin for discussion of results.

APPENDIX

Consider the expansion of the function $g(\eta, \omega)$ in powers of ω:

$$g(\eta, \omega) = \sum_{n=0}^{\infty} \omega^n g^n(\eta).$$

It follows from Eqs. (16), (17) that $g^{(n)}(\eta)$ satisfies the following equations and boundary conditions:

$$z\frac{d}{d\eta}\frac{1}{z}\frac{dg^{(n)}}{d\eta} + g^{(n-2)} = 0, \qquad g^{(-2)} = g^{(-1)} = 0,$$
$$g_n(\eta) \to \frac{1}{n!}\frac{1}{\eta}(i\eta)^n + O\left(\frac{1}{\eta}\right), \qquad \eta \to \infty, \quad n = 0, 1, 2, \ldots \tag{A.1}$$

where $O(1/\eta)$ are terms of smaller order than $1/\eta$ when $\eta \to \infty$. From Eq. (A.1) it is easy to find the explicit form of the functions $g^{(0)}$, $g^{(1)}$,

$$g^{(0)}(\eta) = \frac{1}{C}\int_\eta^\infty z\,d\eta, \qquad g^{(1)}(\eta) = \pm i,$$
$$g^{(2)}(\eta) = \frac{1}{C}\int_\eta^\infty z\,d\eta \int_\eta^\infty d\eta \left(\eta - \frac{1}{z}\int_\eta^\infty z\,d\eta\right) + \frac{1}{2C}\int_\eta^\infty (\eta^2 z - C)\,d\eta. \tag{A.2}$$

The + sign for $g^{(1)}$ in the formula corresponds to retarded, and the − sign to advanced, boundary conditions. The expansion (23) in powers of ω is valid if the radius of curvature of the 4-trajectory is large in comparison with the "singularity radius."

Making use of Eqs. (8), (14), (A.2), it is easy to find expressions for the quantities $g_1(\eta, \omega)$, $g_2(\eta, \omega)$, $g_3(\eta, \omega)$ for $\omega = 0$:

$$g_1(\eta, 0) = \int_\eta^\infty z\,d\eta - \frac{z}{\eta}\left\{\int_\eta^\infty d\eta \left(\eta - \frac{1}{z}\int_\eta^\infty z\,d\eta\right) + \frac{1}{2}\eta^2\right\},$$
$$g_2(\eta, 0) = \frac{1}{\eta^2}\int_\eta^\infty z\,d\eta + \frac{1}{\eta}\left(\frac{z}{\eta}\right)'\left\{\int_\eta^\infty d\eta \left(\eta - \frac{1}{z}\int_\eta^\infty z\,d\eta\right) + \frac{1}{2}\eta^2\right\},$$
$$g_3(\eta, 0) = 0. \tag{A.3}$$

These formulas coincide with Eqs. (16), (18), (19) of [1], in which, however, the constant of integration C_1 has now been determined.

The quantity $g^{(0)}(0)$ is related to the mass of the singularity, m, by the formula[1]

$$g^{(0)}(0) = \frac{3}{8\pi C^2} m = \frac{1}{C}\int_0^\infty z\,d\eta.$$

Taking into account Eqs. (23), (A.2) we find

$$g(0, \omega) = \frac{3}{8\pi C^2} m \pm i\omega + \ldots \tag{A.4}$$

The equations of motion (15′), according to (23), have the form

$$\frac{2}{3}C^2 \sum_{n=0}^{\infty} i^n g^{(n)}(0)\, \xi^{(n+2)}(\tau) = C f^0_{\mu\nu}\dot{\xi}_\nu.$$

If one keeps in the expansion of the equations of motion in the curvature of the 4-trajectory the term following the radiation reaction force, and if one takes into account that $\dot{\xi}_\mu^2 = -1$ for arbitrary τ, then, according to (A.2), one obtains

$$m\ddot{\xi}_\mu = e f^0_{\mu\nu}\dot{\xi}_\nu \pm \frac{e^2}{6\pi}(\dddot{\xi}_\mu - \dot{\xi}_\mu\ddot{\xi}^2) + b(\ddddot{\xi}_\mu - 3\dot{\xi}_\mu\ddot{\xi}_\nu\dddot{\xi}_\nu) + d\ddot{\xi}_\mu\ddot{\xi}^2, \tag{A.5}$$

where

$$b = \frac{2}{3}e\left\{\int_0^\infty z\,d\eta \int_\eta^\infty d\eta \left(\eta - \frac{1}{z}\int_\eta^\infty z\,d\eta\right) + \frac{1}{2}\int_0^\infty \left(\eta^2 z - \frac{e}{4\pi}\right)d\eta\right\},$$

and the constant d can not be determined from the formulas obtained here.

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