ELASTICITY THEORY EQUATIONS AND DISPERSION OF SOUND IN METALS

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Elasticity equations for metals are derived by taking into account the forces which the conductivity electrons exert on the lattice. In the limiting case of high frequencies, when collisions in the electron gas are not important, the equations transform into the elasticity equations with spatial dispersion, proposed by Silin, but contain an additional force to account for nonconservation of the conductivity-electron quasimomentum. The laws of conservation of energy and momentum are examined.

A dispersion equation describing sound wave propagation in metals situated in a magnetic field is derived from the elasticity equations. The dispersion of sound velocity is considered in the absence of a magnetic field and for the limiting cases of strong and weak spatial dispersion. When $\lambda_{ac} \sim \lambda_{em}$ ($\omega \sim 10^9$, $\lambda \ll l$), where λ is the wavelength and l is the electron path length, a certain type of "resonance" should occur. Thus the relative change $\Delta s/s$ of the velocity of sound should increase sharply and become of the same order of magnitude as the absorption.

1. INTRODUCTION

QUASIPARTICLES—conduction electrons—in a deformed metal are acted upon, in addition to Lorentz forces which include macroscopic electromagnetic fields, also by forces due to direct interaction between the electrons and the lattice, owing to the change in the microscopic atomic electric fields. The resultant reaction on the lattice (averaged over the electrons) leads to terms that are functionals of the electronic distribution function in the equations of elasticity theory. These forces are far from small.

Thus, in conductors the interaction between the conduction electrons and the lattice gives rise to an appreciable influence of the electrons and the elastic properties of the body, so that in the case of sufficiently large electron free paths the closed system of equations comprises the kinetic equation for the conduction electrons, Maxwell's equations, and the equations of elasticity theory.

The forces which the deformed lattice exerts on the conduction electrons [which according to modern metal theory form an ideal gas of quasiparticles with dispersion $\epsilon_0(\mathbf{p})$] have an electromagnetic character, but only the smoothed long-range fields can be taken into account with the aid of Maxwell's equations. It is quite important that for deformations that vary over distances greatly exceeding atomic distances this interaction can be taken into account by a change in the dispersion law (Hamilton function) of the conduction electrons [1]. The conduction electrons can, as before, be regarded as an ideal gas of quasiparticles, so that the well developed formalism of modern electron theory of metals can be used [2].

This method of describing this interaction, first proposed by Akhiezer [1], is based on the fact that the dispersion law becomes applicable at distances on the order of several atomic distances (which are already affected by the periodicity of the lattice), and consequently within atomic time durations. During these times and at these distances the lattice can be regarded as stationary and periodic, but with somewhat modified periods and with suitable symmetry, so that the concept of a local dispersion law [1)] can be introduced for a conduction electron in a deformed lattice.

The electrons can exert a particularly strong influence on the elastic properties of the body (including the properties of the sound waves propagating in the body) in a magnetic field at low temperatures near the numerous magnetoacoustic resonances which exist in metals [3,5,6]. It is of interest in this connection to obtain an expression for the force contained in the equations for the

[1)] This reasoning can be used, of course, only in a local coordinate system that moves together with the considered lattice element [3,4].

elasticity of metals. This makes the indicated system of equations closed, since we know the form of the kinetic equation with allowance for the deformation interaction between the electrons and the lattice.

In the case of sufficiently high frequencies, when collisions in the electron gas can be disregarded, such a closed system of equations was proposed by Silin[7]. Our purpose is to obtain equations suitable for both high frequencies (in the classical region $\hbar\omega \ll \epsilon_0$, $\mu H \ll \epsilon_0$) and for the low frequencies. In the derivation of the equations it is necessary to take into consideration the difference between the properties of the quasimomentum of the electrons in the metal and the momentum of the free electrons. This circumstance, in particular, gives rise in the high frequency region to force terms not allowed for in Silin's expression and responsible for the additional momentum transfer to the lattice.

In addition, it follows from the derivation given below that the "energy" approach to the calculation of the sound absorption[2-6] is suitable also in the case of high frequencies. In particular, the equations of elasticity theory follow from the condition of the minimum free energy of the system, determined for the nonequilibrium state in terms of the energy and entropy of the nonequilibrium Fermi gas. The effect of the electrons reduces to changing the elasticity theory equations into nonlocal integro-differential equations, i.e., spatial dispersion begins to assume a major role.

2. KINETIC EQUATION

The form of the kinetic equation for the distribution function $f(\mathbf{r},\mathbf{p},t)$ of the quasiparticles—conduction electrons in a deformed crystal was established in the papers of Akhiezer[1], Akhiezer, Lyubarskiĭ, and Kaganov[8], V. Gurevich[3], and Blount[4] from the absorption of ultrasound in metals. We write the kinetic equation in the form

$$df/dt + \hat{\nu} f = 0, \tag{2.1}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \left(\frac{\partial \varepsilon}{\partial \mathbf{p}}\right)\frac{\partial}{\partial \mathbf{r}} + \left(\mathbf{F} - \frac{\partial \varepsilon}{\partial \mathbf{r}}\right)\frac{\partial}{\partial \mathbf{p}} \tag{2.2}$$

is the Stokes operator of the field derivative with respect to time, $\hat{\nu}$ the collision operator, and $\mathbf{F}$ the external forces, in which we include the Lorentz force:

$$\mathbf{F} = -e\mathbf{E} - \frac{e}{c}\left[\frac{\partial \varepsilon}{\partial \mathbf{p}}\mathbf{H}\right]. \tag{2.3}*$$

*$\left[\frac{\delta\epsilon}{\delta \mathbf{p}}\mathbf{H}\right] = \frac{\delta\epsilon}{\delta \mathbf{p}} \times \mathbf{H}$.

The deformation interaction with the lattice is included in the Hamiltonian of the conduction electron.

The dispersion law, i.e., the Hamilton function of the conduction electron, can be represented in the laboratory coordinate system (l.s.) in which the kinetic equation (2.1) is written in the form

$$\varepsilon(\mathbf{r}, \mathbf{p}, t) = \varepsilon_0(\mathbf{p}) + \delta\varepsilon(\mathbf{r}, \mathbf{p}, t), \tag{2.4}$$

where $\epsilon_0(\mathbf{p})$ is the law of dispersion of the electron in the undeformed lattice and $\delta\epsilon(\mathbf{r},\mathbf{p},t)$ is the local variation of the dispersion law in the case of the crystal deformation. We shall give the specific form of $\delta\epsilon$ below (3.13), but some of the derivations can be made without concretely specifying $\delta\epsilon$.

Values averaged over the quasiparticle momenta will be denoted by means of double brackets:

$$\langle\langle\psi\rangle\rangle \equiv \frac{2}{(2\pi\hbar)^3}\int \psi d\mathbf{p}, \qquad \langle\psi\rangle \equiv \frac{2}{(2\pi\hbar)^3}\int \psi \frac{dS}{v}. \tag{2.5}$$

Integration over the momentum volume $\langle\langle\ldots\rangle\rangle$ is within the limits of one unit cell of the reciprocal lattice. The surface integration $\langle\ldots\rangle$ is carried out over the unperturbed Fermi surface $\mathbf{v} = \partial\epsilon_0/\partial\mathbf{p}$ —unperturbed quasiparticle velocity over this surface.

The quasiparticle distribution function $f(\mathbf{r},\mathbf{p},t)$ is connected with the density $n(\mathbf{r},t)$ of the free electrons by the normalization condition

$$\langle\langle f\rangle\rangle = n(\mathbf{r}, t). \tag{2.6}$$

The quasiparticle flux $\langle\langle(\partial\epsilon/\partial\mathbf{p})f\rangle\rangle$ is equal, in view of the continuity equation, to the flux of the true electrons. In this connection, the density of the electric (electron) current is

$$\mathbf{j}_{el} = -e\langle\langle(\partial\varepsilon/\partial\mathbf{p})f\rangle\rangle \tag{2.7}$$

and the mass flux density, i.e., the momentum density of the free electrons, expressed in terms of the distribution function and the quasiparticle dispersion law, is ($-e$ and m are the charge and mass of the free electron)

$$\boldsymbol{\pi} = m\langle\langle(\partial\varepsilon/\partial\mathbf{p})f\rangle\rangle \tag{2.8}$$

The latter circumstance was used by Landau, in the theory of the Fermi liquid, to establish a connection between the effective mass of a quasiparticle in liquid He^3 and the mass of the helium atom. In a metal, in view of the fact that $\mathbf{p}$ is a quasi momentum, we have $\langle\langle\mathbf{p}f\rangle\rangle - m\langle\langle(\partial\epsilon/\partial\mathbf{p})f\rangle\rangle$, generally speaking, different from zero. However, as before, the quantity $\boldsymbol{\pi}$ is the free-electron momentum density. This important relation will be essentially employed in what follows.

Let us obtain an expression for the electron

energy in the laboratory system. The transformation from the coordinate system K′, which moves together with the volume element dV of the lattice in the laboratory system, is by means of the canonical transformation

$$\mathbf{r} = \mathbf{r}' + \mathbf{u}(\mathbf{r}', t), \tag{2.9}$$

where $\mathbf{u}(\mathbf{r}', t)$ is the displacement of the lattice point with coordinate $\mathbf{r}'$ at the instant t. A canonical transformation with a generating function $\Phi = (\mathbf{r}' + \mathbf{u}(\mathbf{r}', t))\mathbf{p}$ from the old canonical variables $\mathbf{r}'$ and $\mathbf{p}'$ to the new variables $\mathbf{r}$ and $\mathbf{p}$, yields, as usual,

$$\varepsilon'(\mathbf{r}', \mathbf{p}', t) = \varepsilon(\mathbf{r}, \mathbf{p}, t) - \dot{\mathbf{u}}\mathbf{p}, \tag{2.10}$$

$$\mathbf{p}' = \mathbf{p} + \nabla(\mathbf{u}\mathbf{p}). \tag{2.11}$$

We note that inasmuch as the coordinate transformation (2.9) contains the time explicitly, and ϵ' is not a quadratic function of $\mathbf{p}'$, then, in accordance with the relations known from mechanics [9], the new Hamilton function ϵ does not coincide, generally speaking, with the energy of the quasiparticle. Nonetheless, a simple connection exists between its mean over the quasiparticle distribution and the average electron energy. This connection can be obtained by multiplying (2.10) by the invariant of the canonical transformation $f\,dV\,dp = f'\,dV'\,dp'$ and integrating over the quasi momenta. It follows therefore that the electron energy in the system K′ is

$$dE' \equiv \langle\!\langle \varepsilon' f' \rangle\!\rangle\, dV' = \langle\!\langle \varepsilon f \rangle\!\rangle\, dV - \dot{\mathbf{u}} \langle\!\langle \mathbf{p} f \rangle\!\rangle\, dV. \tag{2.12}$$

On the other hand, we use the known thermodynamic identity for the energy

$$dE = dE' + \dot{\mathbf{u}} d\mathbf{P}, \tag{2.13}$$

where dE is the average energy of the electrons in the volume V (in the l.s.) and dP is the electron momentum per unit volume, which, in accordance with (2.8), is equal to

$$d\mathbf{P} = \pi dV = m \langle\!\langle (\partial\varepsilon/\partial\mathbf{p})\, f \rangle\!\rangle dV.$$

Eliminating dE′ from (2.12) and (2.13), we obtain a connection between the average energy and the average Hamilton function in the l.s.

$$dE = \langle\!\langle \varepsilon f \rangle\!\rangle\, dV - \dot{\mathbf{u}} \langle\!\langle (\mathbf{p} - m\partial\varepsilon/\partial\mathbf{p})\, f \rangle\!\rangle\, dV. \tag{2.14}$$

In the derivation of the equations of elasticity theory we shall not use any specific structure of the collision integral. Following Pippard [5], we shall assume that the collision integral is made to vanish instantaneously by an equilibrium distribution function $f_0(\epsilon - \mathbf{p}\cdot\mathbf{u} - \delta\mu)$, where $f_0(\epsilon)$ — Fermi distribution function, $f_0(\epsilon) = [\exp\{(\epsilon - \mu_0)/kT\} + 1]^{-1}$ (μ_0 — chemical potential of the electrons). This deduction is the direct consequence of the fact that the characteristic parameters of the collision a_{col} are considerably smaller than the length of the sound wave λ (or, in general, the distances over which the elastic field varies), while the collision times (but by no means the relaxation times) are smaller than the period of the elastic oscillations, the scattering centers being dragged by the moving lattice. For scattering on impurities, these conditions, obviously, are satisfied. For collisions with phonons, the parameter is the wavelength of the thermal phonons $\lambda_{ph} \sim a\Theta/T$, where Θ is the Debye temperature and a is the lattice constant. (The question of the electron-phonon collisions is more complicated, owing to the need for considering the deviation of the phonon distribution function from equilibrium [4,10].)

Thus, the collision integral is made to vanish by a Fermi function of the argument $\epsilon_0 + \delta\tilde{\epsilon}$,

$$\hat{\nu} f_0(\varepsilon_0 + \delta\tilde{\varepsilon}) = 0, \tag{2.15}$$

where

$$\delta\tilde{\varepsilon} = \delta\varepsilon - \mathbf{p}\dot{\mathbf{u}} - \delta\mu, \tag{2.16}$$

i.e., $\delta\tilde{\epsilon}$ includes both the variation of the Hamilton function and of the chemical potential of the electrons.

The total distribution function is best sought in the form

$$f(\mathbf{r}, \mathbf{p}, t) = f_0(\varepsilon_0 + \delta\tilde{\varepsilon}) + \chi(\mathbf{r}, \mathbf{p}, t)\, \partial f_0/\partial\varepsilon. \tag{2.17}$$

The distribution function χ satisfies the normalization condition

$$\langle\chi\rangle = 0, \tag{2.18}$$

if we assume that both the total and the instantaneously-equilibrium distribution functions are normalized to the electron density $n(\mathbf{r}, t)$ at the given point of the lattice. The current density j represents the sum of the electron current (2.7) and the ion current $\mathbf{j}_{lat} = en\dot{\mathbf{u}}$

$$\mathbf{j}_{el} = e\langle\mathbf{v}(\delta\tilde{\varepsilon} - \delta\varepsilon + \chi)\rangle = e\langle\mathbf{v}\chi\rangle - en\dot{\mathbf{u}}. \tag{2.19}$$

The current is therefore

$$\mathbf{j} = \mathbf{j}_{el} + \mathbf{j}_{lat} = e\langle\mathbf{v}\chi\rangle. \tag{2.20}$$

The electric neutrality condition of the metal assumes the form

$$\operatorname{div}\langle\mathbf{v}\chi\rangle = 0. \tag{2.21}$$

The continuity equation for the electrons

$$\partial n/\partial t + \operatorname{div}\langle\!\langle f\partial\varepsilon/\partial\mathbf{p} \rangle\!\rangle = 0 \tag{2.22}$$

using (2.17)—(2.21) leads to the condition

$$\langle \delta\tilde{\varepsilon} \rangle = n \operatorname{div} \mathbf{u}. \tag{2.23}$$

From this we get the expression for $\delta\mu$ [see (4.5)]. The equation which χ satisfies will be given below (4.4). These relations are already determined by the specific form of the dispersion law in the deformed crystal.

3. CONSERVATION LAWS

The electrons of a metal constitute a non-closed system that interacts with the lattice and with the electromagnetic fields. Therefore the entropy S_{el} of the non-equilibrium electron gas is not a monotonically increasing function of the time, as would be the case with a closed system. Using the definition

$$S_{el} = -k\int dV \langle\!\langle (1-f)\ln(1-f) + f\ln f \rangle\!\rangle, \tag{3.1}$$

we can readily obtain [2)]

$$T\frac{dS_{el}}{dt} = \int dV \langle (\delta\tilde{\varepsilon} + \chi)\hat{\nu}\chi \rangle. \tag{3.2}$$

For brevity we shall not write out in the intermediate formulas the terms which contribute to the fluxes through the surface. For the same reason, we take the field derivative d/dt under the sign of averaging over the momenta, since obviously for any arbitrary quantity we have by virtue of (2.2)

$$\langle\!\langle \frac{d}{dt}\psi \rangle\!\rangle \langle\!\langle \frac{\partial}{\partial t}\psi \rangle\!\rangle = \frac{\partial}{\partial x_k}\langle\!\langle \frac{\partial\varepsilon}{\partial p_k}\psi \rangle\!\rangle. \tag{3.3}$$

We shall henceforth regard the electromagnetic field and the lattice as mechanical systems, assuming the temperatures to be so low that the contribution of the phonons can be neglected, and consequently $T\,dS_{el}/dt = T\,dS/dt$, where S is the entropy of the entire system, which consists of electrons, the electromagnetic field, and the lattice. The system is in a thermostat with temperature T. In this case the direction of the physical processes, as is well known from statistical physics of non-equilibrium processes, is determined by the decrease in the total free energy of the system (for the closed system which includes the thermostat this corresponds to the increase of the total entropy). The free energy of the electrons, in accordance with (2.14), is

$$F_{el} = E_{el} - TS_{el}, \qquad E_{el} = \int dV \langle\!\langle (\varepsilon - \dot{\mathbf{u}}(\mathbf{p} - m\partial\varepsilon/\partial\mathbf{p}))f \rangle\!\rangle. \tag{3.4}$$

Using expression (2.16) for $\delta\epsilon - \delta\tilde{\epsilon}$, and recognizing that $\langle \hat{\nu}_\chi \rangle = 0$ by virtue of the conservation of the number of particles, we obtain

$$\frac{dF_{el}}{dt} = \int dV \Big\{ \langle\!\langle \frac{d\varepsilon}{dt} f \rangle\!\rangle - \ddot{\mathbf{u}} \langle\!\langle \Big(\mathbf{p} - m\frac{\partial\varepsilon}{\partial\mathbf{p}}\Big) f \rangle\!\rangle - \langle\!\langle \mathbf{p}\hat{\nu}f \rangle\!\rangle \dot{\mathbf{u}} - \dot{\mathbf{u}}\frac{\partial}{\partial t}\langle\!\langle \Big(\mathbf{p} - m\frac{\partial\varepsilon}{\partial\mathbf{p}}\Big) f \rangle\!\rangle - \langle \chi\hat{\nu}\chi \rangle \Big\}. \tag{3.5}$$

To determine the force $\mathbf{f}$ in the equations of elasticity theory

$$\rho_{lat}\ddot{u}_i = f_i, \tag{3.6}$$

we start with the law of conservation of the total momentum $\mathcal{P}$ for a system consisting of free electrons, the lattice, and the electromagnetic field. This conservation law, of course, is exact if we disregard the momentum transfer to the phonons:

$$\mathcal{P} = \int dV \Big\{ m\langle\!\langle \frac{\partial\varepsilon}{\partial\mathbf{p}} f \rangle\!\rangle + \mathbf{G} + \rho_{lat}\dot{\mathbf{u}} \Big\}. \tag{3.7}$$

The first term [see (2.8)] represents here the momentum of the free electrons, expressed in terms of the dispersion law and the quasiparticle distribution function, G is the momentum of the electromagnetic field, and ρ_{lat} is the density of the ions making up the lattice.

We obtain the momentum conservation law by equating to zero the time derivative of $\mathcal{P}$. It is convenient here to add and subtract $\langle\langle \mathbf{pf} \rangle\rangle$ from the first term, and find the derivative $(\partial/\partial t)\langle\langle \mathbf{pf} \rangle\rangle$ with the aid of the kinetic equation. According to Maxwell's equations

$$\partial G_i/\partial t = -\partial T_{ik}/\partial x_k - c^{-1}[\mathbf{jH}]_i,$$

where T_{ik} is the electromagnetic field stress tensor and $\mathbf{j}$ is the total current (2.20). It follows therefore that

$$\frac{d\mathcal{P}}{dt} = \int dV \Big\{ \rho_{lat}\ddot{\mathbf{u}} + \langle\!\langle \mathbf{p}\frac{\partial f}{\partial t} \rangle\!\rangle - \frac{1}{c}[\mathbf{jH}] - \frac{\partial}{\partial t}\langle\!\langle \Big(\mathbf{p} - m\frac{\partial\varepsilon}{\partial\mathbf{p}}\Big) f \rangle\!\rangle \Big\}. \tag{3.8}$$

Finally, using the kinetic equation and the condition $\operatorname{div}_{lat}\mathbf{F} = 0$, we obtain (in the approximation linear in $\mathbf{u}$):

$$\frac{d\mathcal{P}_i}{dt} = \int dV \Big\{ f_i - en\Big(\mathbf{E} + \frac{1}{c}[\dot{\mathbf{u}}\mathbf{H}]\Big)_i - \langle\!\langle p_i\hat{\nu}f \rangle\!\rangle - \frac{\partial}{\partial t}\langle\!\langle \Big(p_i - m\frac{\partial\varepsilon}{\partial p_i}\Big) f \rangle\!\rangle - \frac{\partial\psi_{ik}}{\partial x_k} \Big\} = 0, \tag{3.8'}$$

from which we get for the force

$$f_i = ne\Big(\mathbf{E} + \frac{1}{c}[\dot{\mathbf{u}}\mathbf{H}]\Big)_i + \langle\!\langle p_i\hat{\nu}f \rangle\!\rangle + \frac{\partial}{\partial t}\langle\!\langle \Big(p_i - m\frac{\partial\varepsilon}{\partial p_i}\Big) f \rangle\!\rangle + \frac{\partial\psi_{ik}}{\partial x_k}. \tag{3.9}$$

[2)] To prevent misunderstanding we note that in the coordinate frame connected to the moving lattice we have $T\,dS/dt = \int dV' \langle (\delta\tilde{\epsilon}' + \chi)\hat{\nu}\chi \rangle$, i.e., it is by far not equal to $\int dV' \langle \chi\hat{\nu}\chi \rangle$.

The tensor ψ_{ik} for the closed system must be obtained from the energy conservation law. In the case of a system in a thermostat, we use for this purpose the condition of the minimum of free energy of the system (3.4).

It is natural to represent the total free energy of the electrons, lattice, and the electromagnetic field in the form

$$F = F_{el} + \mathcal{H}_{em} + F_{elast},$$

where F_{el} is the free energy of the conduction electrons with account of their interaction with the electromagnetic field and the lattice, $\mathcal{H}_{em}$ the energy of the electromagnetic field in the volume of the sample, and F_{elast} the free energy connected with the deformation and the lattice vibrations. Strictly speaking, such a separation is quite arbitrary in view of the appreciable interaction, but we shall be interested in the time variations of these quantities. We have already calculated $\partial F_{el}/\partial t$ in (3.5) above; for the remaining terms we obviously have

$$\frac{\partial \mathcal{H}_{em}}{\partial t} = -\int dV \mathbf{jE} - \int \frac{c}{4\pi}[\mathbf{EH}]\,ds,$$

$$F_{elast} = \int dV \left(\frac{1}{2}\rho_p \dot{\mathbf{u}}^2 + \frac{\lambda_{iklm}u_{ik}u_{lm}}{2}\right); \qquad \sigma_{ik} = \lambda_{iklm}u_{lm}.$$

It follows therefore that

$$\begin{aligned}\frac{dF}{dt} = \int dV \Big\{ &\langle\!\langle \frac{\partial \delta\varepsilon}{\partial t} f\rangle\!\rangle - \ddot{\mathbf{u}}\langle\!\langle \left(\mathbf{p} - m\frac{\partial \varepsilon}{\partial p}\right) f\rangle\!\rangle \\ &+ \dot{u}_i \Big\{ \rho_p \ddot{u}_i - enE_i - \langle\!\langle p_i \hat{v} f\rangle\!\rangle \\ &- \frac{\partial}{\partial t}\langle\!\langle \left(p_i - m\frac{\partial \varepsilon}{\partial p_i}\right) f\rangle\!\rangle - \frac{\partial \sigma_{ik}}{\partial x_k}\Big\} - \int dV \langle \hat{\chi} \hat{v} \chi\rangle. \end{aligned} \tag{3.10}$$

Now we already have to use the explicit form of the deformation correction $\delta\epsilon$ to the dispersion law, as obtained by Akhiezer, Gurevich, and Landau.

In the coordinate system moving with the lattice we can, in accordance with the foregoing, introduce a dispersion law for the conduction electrons in the deformed lattice. We note also that, owing to the acceleration $\ddot{\mathbf{u}}$, K′ is a non-inertial system in which the free electrons are acted upon by the volume inertia force $-m\langle\!\langle f\rangle\!\rangle \ddot{\mathbf{u}}$ (since $m\langle\!\langle f\rangle\!\rangle$ is the mass of the free electrons per unit volume).

Varying with respect to f, we find that in the system K′ the quasiparticle should also be acted upon by a force $-m\ddot{\mathbf{u}}$ (m is the mass of the free electron), responsible for the Stuart-Tolman effect. According to Landau, the inertial force can be taken into account by including in the energy a term $-m\dot{\mathbf{u}}\,\partial\epsilon/\partial \mathbf{p}$. Thus,

$$\varepsilon'(\mathbf{r}', \mathbf{p}', t) = \varepsilon_0(\mathbf{p}' + \lambda_{ik}(\mathbf{p}')\, u_{ik} - m\dot{\mathbf{u}}\partial\varepsilon/\partial \mathbf{p} + \Delta'\varepsilon, \tag{3.11}$$

where $\lambda_{ik}(\mathbf{p})$ is the deformation potential, introduced by Akhiezer, which satisfies [like the dispersion law $\epsilon_0(\mathbf{p})$] the condition

$$\lambda_{ik}(-\mathbf{p}) = \lambda_{ik}(\mathbf{p}), \qquad \lambda_{ik} = \lambda_{ki}, \tag{3.12}$$

$\Delta'\epsilon$ are terms quadratic in u (these terms must be taken into account, since the energy conservation law is written accurate to quadratic terms, so that the inertia force is also conveniently written in terms of $\partial\epsilon/\partial\mathbf{p}$ and not in terms of $\mathbf{v} = \partial\epsilon_0/\partial\mathbf{p}$).

Carrying out the canonical transformation (2.9) to the laboratory system K, we obtain for $\delta\epsilon$:

$$\delta\varepsilon = (\lambda_{ik} + p_i v_k)\,\partial u_i/\partial x_k + (\mathbf{p} - m\partial\varepsilon/\partial\mathbf{p})\,\dot{\mathbf{u}} + \Delta\varepsilon. \tag{3.13}$$

The asymmetrical part $p_i v_k\, \partial u_i/\partial x_k$ is the result of the fact that the local coordinate system not only executes translational motion, but rotates together with the lattice with an angular velocity $\frac{1}{2}$ curl $\dot{\mathbf{u}}$. Obviously, the antisymmetrical part $\frac{1}{2}(p_i v_k - v_i p_k)$ vanishes on going to an isotropic dispersion law.

The quadratic terms $\Delta\epsilon$ make a contribution $\langle\!\langle \Delta\epsilon f_0(\epsilon_0)\rangle\!\rangle$ to the free energy, which, obviously, reduces only to a renormalization of the moduli of elasticity μ_{iklm}, which includes only the equilibrium electron distribution function. The term $\mu_{iklm}(\mathbf{p})\,u_{ik}u_{lm}$, which is the quadratic term of the expansion of $\epsilon'(\mathbf{r}', \mathbf{p}', t)$ in powers of u_{ik}, leads to an addition $\langle\!\langle \mu_{iklm}(\mathbf{p})\, f_0(\epsilon_0)\rangle\!\rangle$ etc. This renormalization is of the same order as that of $\lambda_{ik}(\mathbf{p})$. In addition, the "nonrenormalized" moduli λ_{iklm} actually already include the interaction with the electrons, which is manifest in the transition from the free electrons to the quasiparticle with dispersion law $\epsilon_0(\mathbf{p})$. Therefore it is meaningless to take into account the terms which renormalize the moduli with the equilibrium electron distribution function [due, for example, to the terms with $\lambda_{ik}(\mathbf{p})$], and we shall omit them henceforth (stipulating this every time). We are interested only in the contribution of the non-equilibrium part of the distribution function, which is responsible for the dispersion, the dependence on the magnetic field, etc.

Using the dispersion law (3.13), we get from (3.10)

$$\begin{aligned}\frac{dF}{dt} = \int dV \dot{u}_i \Big\{ &f_i - enE_i - \frac{\partial \sigma_{ik}}{\partial x_k} - \langle\!\langle p_i \hat{v} f\rangle\!\rangle \\ &- \frac{\partial}{\partial x_k}\langle\!\langle (\lambda_{ik} + p_i v_k) f\rangle\!\rangle \\ &- \frac{\partial}{\partial t}\langle\!\langle \left(p_i - m\frac{\partial \varepsilon}{\partial p_i}\right) f\rangle\!\rangle\Big\} - \int \langle \hat{\chi} \hat{v} \chi\rangle\, dV. \end{aligned} \tag{3.14}$$

Choosing the volume V large enough to cause the surface integrals (which we have not written out

so far) to vanish, we arrive at the conclusion that the thermodynamic potential F will be a quantity of fixed sign, which decreases in any process, provided the expression for the force f_i is taken from the momentum conservation law (3.9), and the arbitrary tensor ψ_{ik} contained in it is set equal to $\langle\langle(\lambda_{ik} + p_i v_k)f\rangle\rangle + \sigma_{ik}$.

The energy conservation law takes the form

$$\frac{dF}{dt} = -\int \langle \chi \hat{\nu} \chi \rangle \, dV - \int ds_k q_k, \tag{3.15}$$

where the flux density of the free energy q is equal

$$q_k = \langle\!\langle \varepsilon f \partial\varepsilon/\partial p_k \rangle\!\rangle + \langle\!\langle \varepsilon \, (\delta\tilde{\varepsilon} + \chi) \, \partial f/\partial p_k \rangle\!\rangle - \dot{u}_i \sigma_{ik} - \dot{u}_i \langle\!\langle (\lambda_{ik} + p_i v_k) \, f \rangle\!\rangle + (c/4\pi) \, [\mathbf{EH}]_k. \tag{3.15'}$$

Accordingly, the momentum conservation law takes the form

$$d\mathcal{P}_i/dt = -\int ds_k \Pi_{ik}, \tag{3.16'}$$

where Π_{ik} is the momentum flux density tensor

$$\Pi_{ik} = T_{ik} + \langle\!\langle p_i f \partial\varepsilon/\partial p_k \rangle\!\rangle + \delta_{ik} \langle\!\langle f \delta\varepsilon \rangle\!\rangle - \sigma_{ik} - \langle\!\langle (\lambda_{ik} + p_i v_k) \, f \rangle\!\rangle \tag{3.16}$$

(T_{ik} is the Maxwell stress tensor).

4. EQUATIONS OF ELASTICITY THEORY

As a consequence of the energy and momentum conservation laws, the equations of motion of the lattice assume the form

$$\rho_{\text{lat}} \ddot{u}_i = en \left(\mathbf{E} + \frac{1}{c} [\dot{\mathbf{u}}\mathbf{H}] \right)_i + \frac{\partial}{\partial x_k} \langle\!\langle (\lambda_{ik} + p_i v_k) \, f \rangle\!\rangle + \frac{\partial \sigma_{ik}}{\partial x_k} + \langle\!\langle p_i \hat{\nu} f \rangle\!\rangle + \frac{\partial}{\partial t} \langle\!\langle \left(p_i - m \frac{\partial \varepsilon}{\partial p_i} \right) f \rangle\!\rangle. \tag{4.1}$$

The physical meaning of the first term is obvious: ne is the density of the equilibrium charge of the lattice. The second term is essentially connected with the deformation interaction between the electrons and the lattice. It coincides with the corresponding term introduced by Silin[7] in the equation of motion of the lattice, if we take the tensor $\Lambda_{ik}^{(c)}$ introduced by Silin, to mean $\lambda_{ik} + p_i v_k$. The last two terms are missing from Silin's equations. The first is connected with the momentum transferred from the electrons to the lattice as by the collisions, while the second[3] takes into account the non-conservation of the conduction electron quasimomentum, connected with the momentum exchange between the free electrons and the lattice. The expression for the force can be transformed to a more convenient form, if we use the quasimomentum transport equation. We note by way of introduction that

$$\langle\!\langle \left(\mathbf{p} - m \frac{\partial \varepsilon}{\partial \mathbf{p}} \right) f \rangle\!\rangle = -\langle (\mathbf{p} - m\mathbf{v}) \, \chi \rangle = -\frac{\partial}{\partial t} \langle \mathbf{p}\chi \rangle + \frac{m}{e} \frac{\partial \mathbf{j}}{\partial t}. \tag{4.2}$$

This can be readily demonstrated if we recognize that $\lambda_{ik}(\mathbf{p}) = \lambda_{ik}(-\mathbf{p})$, and we use the expressions for $\delta\tilde{\epsilon}$ and $\delta\tilde{\epsilon} - \delta\epsilon$. On the other hand, we can renormalize σ_{ik}, including in it the term $-\langle (\lambda_{ik} + p_i v_k)(\Lambda_{lm} + p_l v_m) \rangle \partial u_l/\partial x_m$, after which

$$\frac{\partial}{\partial x_k} \langle\!\langle (\lambda_{ik} + p_i v_k) \, f \rangle\!\rangle \to -\frac{\partial}{\partial x_k} \langle (\lambda_{ik} + p_i v_k) \, \chi \rangle. \tag{4.3}$$

We multiply the equation for χ, which is of the form

$$(d/dt + \hat{\nu}) \chi = g \equiv e\mathbf{v}\tilde{\mathbf{E}} - \Lambda_{lk} \dot{u}_{lk}, \qquad \Lambda_{lk} = \lambda_{lk} - \langle \lambda_{lk} \rangle / \langle 1 \rangle, \tag{4.4}$$

$$\tilde{\mathbf{E}} = \mathbf{E} + c^{-1} [\dot{\mathbf{u}}\mathbf{H}] + e^{-1} \nabla \delta\mu + m\ddot{\mathbf{u}}/e, \qquad \delta\mu = u_{ik} \langle \lambda_{ik} \rangle / \langle 1 \rangle, \tag{4.5}$$

by p and average over the Fermi surface. We obtain, using (2.20),

$$\frac{\partial}{\partial t} \langle \mathbf{p}\chi \rangle + \frac{\partial}{\partial x_k} \langle \mathbf{p} v_k \chi \rangle + \langle \mathbf{p} \hat{\nu} \chi \rangle = -\frac{1}{c} [\mathbf{jH}] + en\tilde{\mathbf{E}}. \tag{4.6}$$

Substituting (4.2) and (4.3) into the expression for the force and using (4.6) and (2.18) we obtain (carrying out one more renormalization of σ_{ik}, namely including in it the term $-\delta_{ik} n \delta\mu$) the following equation of elasticity theory:

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k} + \frac{1}{c} [\mathbf{jH}]_i + \frac{m}{e} \frac{\partial j_i}{\partial t} - \frac{\partial}{\partial x_k} \langle \Lambda_{ik} \chi \rangle. \tag{4.7}$$

Here ρ is the total density of matter, $\rho = \rho_{\text{lat}} + nm$. The complete system of equation now comprises the equation of motion (4.7), the kinetic equation, (4.4), (4.5), the expression for the current (2.20) in terms of the distribution function, and Maxwell's equations

$$\text{rot rot } \mathbf{E} = -4\pi c^{-2} \partial \mathbf{j}/\partial t. \tag{4.8}*$$

The boundary conditions for this system are the boundary condition on the distribution function χ (the condition of reflection of the electrons from the boundary), the continuity of the components $\Pi_{ik} n_k$ of the momentum flux, and the continuity of the tangential components of the electric field and of the normal component of the magnetic field. The system of equations obtained above now enables us to solve those problems of the linear theory of elasticity in metals, in which an appreciable role can be played by electrons and electromagnetic fields; in particular, the problem of the mutual transformation of electromagnetic and sound waves[11].

[3] This term was obtained also by K. B. Vlasov and V. I. Filippov (private communication).

*rot = curl.

5. THE DISPERSION EQUATION

We introduce the Green's operator $\hat{\mathbf{R}}(\mathbf{p})$ of the kinetic equation and its parts that are symmetrical and antisymmetrical in $\mathbf{p}$

$$\hat{R}^{s,a} \equiv \tfrac{1}{2}(\hat{R}(\mathbf{p}) \pm \hat{R}(-\mathbf{p})):$$

$$\hat{R}(\mathbf{p}) = [d/dt + \hat{\nu}]^{-1}. \tag{5.1}$$

The formal solution of the kinetic equation is of the form

$$\chi = \chi^s + \chi^a = \hat{R}g,$$
$$\chi^a = \hat{R}^s e\mathbf{v}\tilde{\mathbf{E}} - \hat{\mathbf{R}}^a\Lambda_{lm}\dot{u}_{lm}, \qquad \chi^s = \hat{R}^a e\mathbf{v}\tilde{\mathbf{E}} - \hat{R}^s\Lambda_{lm}\dot{u}_{lm}. \tag{5.2}$$

The current and the deformation force [the last term in (4.8)] are expressed respectively in terms of the antisymmetrical (χ^a) and symmetrical (χ^s) parts of the distribution function:

$$j_i = e^2\langle v_i\hat{R}^s v_k\rangle\tilde{E}_k - e\langle v_i\hat{R}^a\Lambda_{lm}\rangle\dot{u}_{lm},$$
$$f_i^{\mathrm{d}} = -\nabla_k\{e\langle\Lambda_{ik}\hat{R}^a v_l\rangle\tilde{E}_l - \langle\Lambda_{ik}\hat{R}^s\Lambda_{lm}\rangle\dot{u}_{lm}\}. \tag{5.3}$$

In the case of weak spatial dispersion, we obtain from (5.3) Ohm's law, which includes also the deformation currents produced not by the fields but by the deformation of the crystal. If the source of the fields is sound excited in the crystal, then by solving Maxwell's equations we can express the effective electric field $\mathbf{E}$ in terms of the deformations

$$\tilde{E}_k = \hat{\varepsilon}_{kl}u_l. \tag{5.4}$$

The operator $\hat{\epsilon}_{kl}$ is connected with the Green's tensor of Maxwell's equations. Expressing the current and the force with the aid of (5.2)—(5.4) in terms of the displacement, we obtain

$$j_i = \hat{j}_{il}u_l, \qquad f_i^{\mathrm{d}} = \hat{f}_{il}u_l,$$
$$\hat{j}_{il} = e^2\langle v_i\hat{R}^s v_k\rangle\hat{\varepsilon}_{kl} - e\langle v_i\hat{R}^a\Lambda_{lm}\rangle\partial^2/\partial t\partial x_m,$$
$$\hat{f}_{il} = -\nabla_k\{e\langle\Lambda_{ik}\hat{R}^a v_n\rangle\hat{\varepsilon}_{nl} - \langle\Lambda_{ik}\hat{R}^s\Lambda_{lm}\rangle\partial^2/\partial t\partial x_m. \tag{5.5}$$

The equation for the displacement $\mathbf{u}$ assumes the form

$$\rho\ddot{u}_i = \lambda_{iklm}\partial^2 u_l/\partial x_k\partial x_m + \hat{D}_{il}u_l, \tag{5.6}$$

$$\hat{D}_{il} = (c^{-1}\varepsilon_{imn}H_n + \delta_{im}(m/e)\,\partial/\partial t)\,\hat{f}_{ml} + \hat{f}_{il}. \tag{5.7}$$

Here ϵ_{ikl} is a unit antisymmetrical tensor.

In our case $\hat{\mathbf{D}}_{il}$ is an integro-differential operator with a rather complicated kernel. In a homogeneous unbounded medium, the kernel becomes of the difference type in the coordinates, and the Fourier method can be employed. For plane waves, in which the quantities vary as $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, the Fourier components will be denoted in the same manner as in the coordinate representation, leaving out the indices $\mathbf{k}$ and ω.

Equations (5.4) and (5.5) assume the form

$$\tilde{E}_i = \varepsilon_{il}u_l, \qquad j_i = (\sigma_{ik}\varepsilon_{kl} + \partial_{il})\,u_l,$$
$$f_i^{\mathrm{d}} = (c_{ik}\varepsilon_{kl} + b_{il})\,u_l. \tag{5.8}$$

Here σ_{ik} is the conductivity, ∂_{il} the "deformation conductivity," and c_{in} and b_{il} characterize the changes in the elastic moduli connected with the non-equilibrium nature of the electron gas: c_{in} is due to the field and b_{il} is directly due to the deformation interaction

$$\sigma_{ik} = e^2\langle v_i R^s v_k\rangle, \qquad \partial_{il} = -e\omega k\,\langle v_i R^a\Lambda_{lx}\rangle,$$
$$c_{in} = -iek\,\langle\Lambda_{ix}R^a v_n\rangle, \qquad b_{il} = i\omega k^2\,\langle\Lambda_{ix}R^s\Lambda_{lx}\rangle. \tag{5.9}$$

We use the notation $a_{il\kappa l} \equiv a_\kappa$, where $\kappa = \mathbf{k}/k$ and a_{il} is an arbitrary tensor.

Let $\mathbf{p}$ and $\mathbf{q}$ be the principal axes of the tensor $\lambda_{i\kappa lK}$. We denote by $\eta_{\mathbf{p}}$ the eigenvalues of this tensor in units of ρs^2, where s is a certain characteristic velocity of sound. Then

$$\eta_{\bar{p}}\delta_{pq} = \lambda_{p\varkappa q\varkappa}/\rho s^2. \tag{5.10}$$

No summation is carried out over the barred index.

Equation (5.6) for the Fourier components of the displacement in terms of the principal axes $\mathbf{p}$ and $\mathbf{q}$ assumes in the dimensionless variables the form

$$\{(\eta_{\bar{p}} - \zeta)\,\delta_{pq} + d_{pq}\}\,u_q = 0. \tag{5.11}$$

Here

$$\zeta \equiv (\omega/ks)^2, \qquad d_{il} = -D_{il}/\rho s^2 k^2. \tag{5.12}$$

The eigenvalues ζ are determined from the dispersion equation

$$\mathrm{Det}\,|(\eta_{\bar{p}} - \zeta)\,\delta_{pq} + d_{pq}| = 0. \tag{5.13}$$

If $|d_{pq}| \ll 1$, then for $\eta_1 \neq \eta_2 \neq \eta_3$ the eigenvalues ζ in the approximation that is linear in d_{pq} are

$$\zeta_i = \eta_i + d_{i\bar{i}} \qquad (\eta_1 \neq \eta_2 \neq \eta_3). \tag{5.14}$$

In the case of twofold degeneracy (for example in an isotropic medium) $\eta = \eta_1 = \eta_2 \neq \eta_3$ in the same approximation, ζ_3 is again expressed with the aid of (5.14), and

$$\zeta_{1,2} = \eta + \tfrac{1}{2}[d_{11} + d_{22} \pm \sqrt{(d_{11} - d_{22})^2 + 4d_{12}d_{21}}\,]. \tag{5.15}$$

If $d_{pq} \gtrsim 1$, then it is necessary to solve (5.13). The tensor d_{lm} has in accordance with (5.7) and (5.12) the form

$$d_{lm} = -(\rho k^2 s^2)^{-1}\{(c_{lt} + \frac{m}{e}(\varepsilon_{lik}\omega_k^c - i\omega\delta_{li})\,\sigma_{it})\,\varepsilon_{tm}$$
$$+ b_{lm} + \frac{m}{e}(\varepsilon_{lik}\omega_k^c - i\omega\delta_{li})\,\partial_{im}\}. \tag{5.16}$$

We have introduced here the notation

$$\omega^c = eH/mc. \tag{5.17}$$

Let us now find the tensor ϵ_{ik}. We introduce axes orthogonal to κ, which we denote by the Greek indices α, β, γ, etc. From $j_\kappa = 0$ follows an expression for the longitudinal field in terms of the transverse field

$$\tilde{E}_\varkappa = -\tilde{E}_\alpha\sigma_{\varkappa\alpha}/\sigma_{\varkappa\varkappa} - \partial_{\varkappa l}u_l/\sigma_{\varkappa\varkappa},$$

obtained from Maxwell's equations (4.9):

$$j_\alpha = \sigma^*_{\alpha\beta}\tilde{E}_\beta + j^d_\alpha, \qquad \tilde{E}_\alpha = -\rho_{\alpha\beta}j^d_\beta + c^{-1}[\dot{u}H]_\alpha + m\ddot{u}_\alpha/e. \tag{5.18}$$

The current j^d is equal to

$$j^d_\alpha = \sigma^*_{\alpha\beta}(c^{-1}[\dot{u}H]_\beta + m\ddot{u}_\beta/e) + \partial^*_{\alpha l}u_l.$$

The renormalized tensors σ^β_α and $\partial^*_{\alpha l}$ are expressed in terms of σ_{ik} and ∂_{ik} in the following manner:

$$\sigma^*_{\alpha\beta} = \sigma_{\alpha\beta} - \sigma_{\alpha\varkappa}\sigma_{\varkappa\beta}/\sigma_{\varkappa\varkappa}, \qquad \partial^*_{\alpha l} = \partial_{\alpha l} - \sigma_{\alpha\varkappa}\partial_{\varkappa l}/\sigma_{\varkappa\varkappa}, \tag{5.19}$$

while the resistivity $\rho_{\alpha\beta}$ is given

$$\hat{\rho} = [\hat{\sigma}^* + i\Omega_s\sigma\hat{I}]^{-1}, \qquad \Omega_s = k^2c^2/4\pi\sigma\omega,$$
$$\hat{\sigma}^* \equiv (\sigma^*_{\alpha\beta}), \qquad \hat{\rho} \equiv (\rho_{\alpha\beta}), \qquad \hat{I} \equiv (\delta_{\alpha\beta}). \tag{5.20}$$

Finally, using the relations already written, we obtain for ϵ_{ik} the expression

$$\varepsilon_{tq} = -(\delta_{t\alpha} - \delta_{t\varkappa}\frac{\sigma_{\varkappa\alpha}}{\sigma_{\varkappa\varkappa}})\rho_{\alpha\beta}\{i\sigma\Omega_s(m\omega/e)(\omega\delta_{\beta q} + i\omega^c_s\varepsilon_{\beta qs}) + \partial^*_{\beta q}\} - \delta_{t\varkappa}\partial_{\varkappa q}/\sigma_{\varkappa\varkappa}. \tag{5.21}$$

The phase velocity of the sound wave in the metal can be obtained in accordance with (5.12) from the formula

$$(\omega/k)_i = s\sqrt{\zeta_i}, \tag{5.22}$$

where ζ_i is the i-th root of the dispersion equation. In the case of (5.14) we obtain, for example,

$$(\omega/k)_i = s\sqrt{\eta_i}(1 + d_{i\bar{i}}/2\eta_i). \tag{5.23}$$

Thus, the real part of the element $d_{i\bar{i}}/2\eta_i$ is a dimensionless addition to the velocity of sound (the imaginary part of $d_{i\bar{i}}/2\eta_i$ in units of the unperturbed velocity of sound $s\sqrt{\eta_i}$ describes the absorption).

The absorption coefficient γ per unit length is equal to

$$\gamma = -\omega\,\mathrm{Im}\,d_{i\bar{i}}/2s\eta_i^{3/2} \qquad (\mathrm{Im}\,d_{i\bar{i}} < 0). \tag{5.24}$$

6. DISPERSION OF THE VELOCITY OF SOUND

Owing to the interaction with the electrons, appreciable dispersion of the velocity of sound should be observed in metals. We consider below the dependence of the velocity of sound on the frequency in the absence of a magnetic field. We shall present also a new result pertaining to the absorption of sound.

We introduce the relaxation time $1/\nu(p)$. Then the inverse operator of the kinetic equation has the form

$$R(p) = [i(kv_\varkappa - \omega) + \nu]^{-1}, \qquad v_\varkappa = (v\varkappa). \tag{6.1}$$

In the case of strong spatial dispersion ($kv_\kappa \gg \nu$) we have

$$R^s = \frac{\pi}{k}\delta(v_\varkappa), \qquad R^a = -i\frac{P}{kv_\varkappa},$$
$$v_\varkappa R^s = \frac{\nu - i\omega}{k^2}\frac{P}{v_\varkappa}, \qquad v_\varkappa R^a = \frac{1}{ik} + \frac{\pi(\omega + i\nu)}{k}\delta(v_\varkappa), \tag{6.2}$$

here P is the symbol for the principal value of the integral and will be left out henceforth.

From (5.9) we obtain the principal terms of the tensors, for which we present the estimates

$$\sigma_{\alpha\beta} \sim \sigma(1 + is/v), \quad \sigma_{\varkappa i} = \sigma_{i\varkappa} \sim \sigma(\nu + i\omega)/kv, \quad \sigma = ne^2/mkv,$$
$$c_{i\alpha} \sim en(1 + is/v), \quad c_{i\varkappa} \sim en(\nu + i\omega)/kv, \quad \partial_{li} = i\omega c_{li},$$
$$b_{il} \sim inm\omega kv(1 + is/v), \quad \sigma^*_{\alpha\beta} \sim \sigma_{\alpha\beta}. \tag{6.3}$$

Further estimates are essentially connected with the quantity Ω_s (5.20). By definition $\Omega_s = (\lambda_{em}/\lambda_{ac})^2$, where $\lambda_{em} = c/\sqrt{4\pi\sigma\omega}$ is the length of the electromagnetic wave in the medium (with account of spatial dispersion), and $\lambda_{ac} = s/\omega$ — the wavelength of sound. All these lengths differ appreciably, i.e., we have either $\Omega_s \ll s/v$ or $\Omega_s \gg v/s$, we obtain for d_{pq} according to (5.16), (5.20), (5.21), using the estimate (6.3) (m and M are the masses of the electron and ion):

$$d_{pq} \sim (mv/Ms)(i + s/v) \qquad (kv \gg \nu) \tag{6.4}$$

The sound absorption coefficient increases in accordance with (5.24) linearly with the frequency, as shown by Akhiezer, Lyubarskiĭ, and Kaganov [8]. The variation of the velocity of sound $\Delta s \sim sm/M$ is, in accordance with (5.23), s/v times smaller than the absorption.

It is easy to see that when $\Omega_s \gg v/s$ the contribution to the absorption from the electric fields can be neglected, while when $\Omega_s \ll s/v$ the electric fields make a noticeable contribution to the absorption, comparable with the deformation absorption (see, for example, [3]). In this connection, when $\Omega_s \sim 1$ (i.e., $\lambda_{em} \sim \lambda_{ac}$, which corresponds to frequencies $\omega \sim 10^9$), a transition should take place from a linear dependence for $\Omega_s \ll s/v$ to a linear dependence (but with a different coefficient) for $\Omega_s \gg s/v$. It turns out that in this region there is a strong dispersion of the velocity of

sound, and the variation of the velocity of sound becomes of the same order as the absorption. In this region of frequencies we can obtain rather simple formulas for the velocity of sound, but when it comes to absorption it will be shown below that when $s/v \ll \Omega_s \ll 1$ there should exist one more region that is linear with the frequency, and simple formulas can be obtained for the entire frequency region $s/v \ll \Omega_s$, including the transition region near $\Omega_s \sim 1$. Indeed, for $\Omega_s \sim 1$,

$$\rho_{\alpha\beta} \sim \sigma^{-1}(1+i), \qquad \varepsilon_{il} \sim (m/e)\,\omega k v\,(1+i). \tag{6.5}$$

It is seen even from this that the electric fields at $\Omega_s \sim 1$ should make a contribution of the same order both to the absorption and to the dispersion.

Using the estimates (6.3), we find that only the deformation force f^d makes a contribution to d_{pq}, and the contribution made to it by the longitudinal electric fields can also be neglected. Then

$$(s/v \ll \Omega_s \ll v/s)$$

$$d_{pq} = -(\rho\omega^2)^{-1}\,[c_{p\alpha}\varepsilon_{aq} + b_{pq}], \qquad \varepsilon_{\alpha q} = -\rho_{\alpha\beta}\partial_{\beta q}. \tag{6.6}$$

In this frequency region the estimate for d_{pq} has a form which differs noticeably from (6.4):

$$d_{pq} \sim (mv/Ms)\,(i+1), \qquad kv \gg \nu, \qquad \Omega_s \sim 1. \tag{6.7}$$

It is sufficient to retain in the transverse specific resistivity $\rho_{\alpha\beta}$ only the real part of the tensor $\sigma^*_{\alpha\beta}$, which simply coincides with $\sigma_{\alpha\beta}$. The answer is conveniently written by choosing for the coordinate axes the principal axes μ and ν of the tensor $\sigma_{\alpha\beta}$. In this case

$$\rho_{\mu\nu} = \delta_{\mu\nu}/(\sigma_\mu + i\sigma\Omega_s), \qquad \sigma_{\mu\nu} = \sigma_\mu\delta_{\mu\nu},$$
$$2\sigma_{1,2} = \mathrm{Sp}\hat{\sigma} \pm \sqrt{\mathrm{Sp}^2\hat{\sigma} - 4\,\mathrm{Det}\,\hat{\sigma}}, \qquad \hat{\sigma} = (\sigma_{\alpha\beta}). \tag{6.8}$$

According to (6.6), (5.9), and (6.2) we obtain

$$d_{pq} = -\frac{i}{\rho\omega}\left\{e^2\sum_{\mu=1}^{2}\frac{\langle\Lambda_{px}v_\mu/v_x\rangle\langle\Lambda_{qx}v_\mu/v_x\rangle}{\sigma_\mu + ik^2c^2/4\pi\omega} + \pi k\langle\Lambda_{px}\Lambda_{qx}\delta(v_x)\rangle\right\}. \tag{6.9}$$

Substituting in (6.9) the value of σ_μ,

$$\sigma_\mu = \pi e^2 k^{-1}\langle v_\mu^2\delta(v_x)\rangle, \tag{6.10}$$

we obtain for the diagonal element d_{pp} [assuming that case (5.14) is realized]

$$\mathrm{Im}\,d_{p\bar{p}} = -\frac{1}{\rho}\left\{\sum_{\mu=1}^{2}\frac{\pi e^4 s\langle v_\mu^2\delta(v_x)\rangle\langle\Lambda_{px}v_\mu/v_x\rangle^2}{[\pi e^2 s\langle v_\mu^2\delta(v_x)\rangle]^2 + \omega^4(c^2/4\pi s^2)^2} + \frac{\pi}{s}\langle\Lambda_{px}^2\delta(v_x)\rangle\right\}, \tag{6.11}$$

$$\mathrm{Re}\,d_{p\bar{p}} = -\left(\frac{ec}{\sqrt{4\pi\rho}\,s}\right)^2\sum_{\mu=1}^{2}\frac{\langle\Lambda_{px}v_\mu/v_x\rangle^2\omega^2}{[\pi e^2 s\langle v_\mu^2\delta(v_x)\rangle]^2 + \omega^4(c^2/4\pi s^2)^2}. \tag{6.12}$$

We note once more that (6.11) describes not the entire transition region from the low-frequency linear law to the high-frequency region. The linear region for the absorption with $s/v \ll \Omega_s \ll 1$ corresponds to the fact that contributions to the absorption are made only by those transverse electric fields, which determine the deformation force f^d. For $\Omega_s \sim s/v$, there is still another transition region to the linear law with $\Omega_s \ll s/v$. The corresponding explicit expressions become cumbersome and will not be presented here.

According to (6.12), the change in the velocity of sound is a sum of two terms ($\mu = 1, 2$) which differ from each other because of anisotropy. Each term is a symmetrical curve with a maximum at

$$\omega^2 = \omega_\mu^2 \equiv (2\pi e)^2 s^3 c^{-2}\langle v_\mu^2\delta(v_x)\rangle. \tag{6.13}$$

Inasmuch as ω_1 and ω_2 do not equal each other in the presence of anisotropy, the dependence of the velocity of sound on the frequency will represent a double-humped curve with two minima, the positions of which can be readily obtained from (6.12). They are determined by the roots of the equation $d\,\mathrm{Re}\,d_{p\bar{p}}/d\omega^2 = 0$, which is cubic in ω^2.

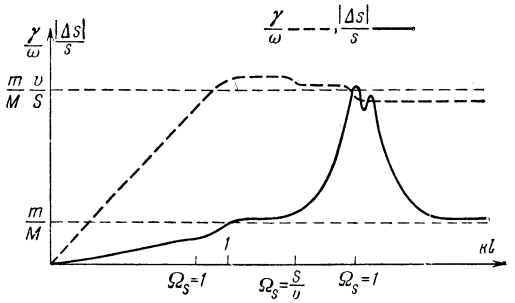
In the case of weak spatial dispersion $kv_x \ll \nu$, obviously,

$$R^s = \nu^{-1} + i\omega/\nu^2, \qquad R^a = -ikv_x/\nu + 2k\omega v_x/\nu^3. \tag{6.14}$$

The estimates for the tensors are:

$$\sigma_{ik} \sim \sigma(1 + i\omega/\nu), \quad \sigma = ne^2/m\nu, \quad c_{il} \sim ne\,(kv/\nu)^2(1 + i\omega/\nu),$$
$$b_{ik} = imn\,(kv)^2(\omega/\nu)(1 + i\omega/\nu), \quad \partial_{il} = i\omega c_{li}. \tag{6.15}$$

In the case of weak spatial dispersion Ω_s is proportional to the frequency, in accord with (5.20)



Absorption and dispersion of sound in the absence of a magnetic field. Relative variation of the velocity of sound $\Delta s/s$ for $\Omega_s \sim 1$ ($\lambda_{em} \sim \lambda_{ac}$) in the region $kl \ll 1$ becomes comparable with the relative absorption γ/ω. The curve with the two maxima corresponds to a sufficiently strong anisotropy in the plane orthogonal to the wave vector (otherwise there is only one maximum). If the "resonance" ($\Omega_s = 1$) turns out to be in the region of weak spatial dispersion ($kl \ll 1$), then a transition occurs to a steeper quadratic dependence of $\Delta s/s$ on the frequency.

and (6.15), and it is convenient to represent Ω_s in the form

$$\Omega_s = \omega/\omega_s, \qquad \omega_s = 4\pi\sigma s^2/c^2. \tag{6.16}$$

The estimate for d_{pq} assumes the form ($kv \gg \nu$)

$$d_{pq} \sim \frac{m}{M}\left(\frac{v}{s}\right)^2 \frac{\omega}{\nu}\left\{i + \frac{\omega}{\nu}\right\} \quad \text{for } \omega \ll \omega_s, \tag{6.17}$$

$$d_{pq} \sim \frac{m}{M}\left(\frac{v}{s}\right)^2 \frac{\omega}{\nu}\left\{i + \frac{\omega}{\nu}\frac{\omega_v}{\nu}\right\} \quad \text{for } \omega \gg \omega_s. \tag{6.18}$$

Here $\omega_v = 4\pi\sigma v^2/c^2$ and $\omega_v/\nu = (\omega_p v/\nu c)^2 \sim (10^{14}/\nu)^2 \gg 1$.

In the transition region $\Omega_s \sim 1$ we obtain for d_{pq}:

$$d_{pq} \sim \frac{m}{M}\left(\frac{v}{s}\right)^2 \frac{\omega}{\nu}\left\{i + \left[\frac{\omega}{\nu} + \frac{\omega/\nu}{1+\Omega_s^2} + \frac{\Omega_s (kv/\nu)^2}{1+\Omega_s^2}\right]\right\}. \tag{6.19}$$

We thus see from (6.19) that the correction to the velocity of sound is always much smaller than the damping when $kv \ll \nu$, including the case when $\Omega_s \sim 1$. Outside this region [formulas (6.18), (6,17)] it increases quadratically with the frequency, and when $\Omega_s \gg 1$ the parabola is steeper than when $\Omega_s \ll 1$.

When $\Omega_s \gtrsim 1$, and only the last term in (6.19) is significant, we can write for d_{pq}, the following explicit expressions which determine the absorption and dispersion

$$\operatorname{Im} d_{p\bar{p}} = -(\omega/\rho s^2)\langle \Lambda_{p\varkappa}^2/\nu\rangle, \tag{6.20}$$

$$\operatorname{Re} d_{p\bar{p}} = -\frac{4\pi e^2}{\rho s^2 c^2}\sum_{\mu=1}^{2}\frac{\omega^4\langle \Lambda_{p\varkappa} v_\varkappa v_\mu^*/\nu^2\rangle^2}{(4\pi\sigma_\mu^* s^2/c^2)^2+\omega^2}. \tag{6.21}$$

Here $v^* = v_\mu - \sigma_{\kappa\mu}v_\kappa/\sigma_{\kappa\kappa}$; σ_μ^* is the principal value of the renormalized tensor $\sigma_{\alpha\beta}^*$.

If we disregard the frequency region where $\Omega_s \sim 1$, then on the whole the picture of the velocity of sound dispersion is as follows. The addition to the velocity of sound increases in proportion to the square of the frequency at low frequencies, after which saturation sets in when $kv \gg \nu$. The transition region is $kv \sim \nu$, or $l \sim \lambda_{ac}$, where $l = v/\nu$ is the mean free path. As in the case of absorption[8], there is no boundary $\omega\tau \sim 1$ in dispersion.

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