DEVELOPMENT OF TURBULENCE IN THE PRESENCE OF A HEAT FLOW IN HELIUM II WITHIN A CAPILLARY AND THE CRITICAL VELOCITY PROBLEM

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Measurements have been made on the velocities of the turbulence fronts in He II within capillaries 0.4 and 1.3 mm in diameter, at various temperatures. These experiments were conducted with the principal object of analyzing the significance of the critical velocity in the presence of heat flow. Certain theoretical interpretations of the critical velocity are advanced.

SEVERAL years ago it was determined (e.g., in [1]) that with a thermal flux $W > W_{cr}$ passing through He II in a long capillary the heat transport regime can assume various characters, depending upon the prior history of the system. Ordinary laminar flow of the normal component may take place, with no interaction with the superfluid component, the latter being in potential flow; but a transport mode is also possible in which some equilibrium density of vortices is present in the superfluid. In this case, the mutual friction force between the normal and superfluid components resulting from the presence of the vortices leads to an increase in the temperature gradient within the capillary. In a paper by Peshkov and the author [2] it was shown that both heat transport regimes may exist simultaneously within the capillary. When this is so, the boundary separating the two regions—the "superfluid turbulence" front—is generally mobile. A similar situation—the motion of turbulence fronts—is observed during the flow of an ordinary viscous fluid through a tube. [3] The velocities with which the fronts move are found to be related to the critical flow velocity. It is natural to anticipate such a relation in the case of superfluid turbulence. The development of the turbulence is itself also of interest.

The apparatus with which the experiments were conducted consisted of a long capillary situated within a He^3 cryostat, as described in the paper by Peshkov, Zinov'eva, and Filimonov. [4] One end of the capillary opened into a He^4 reservoir of ~ 2 cm^3 volume in thermal contact with the He^3 bath; a heater was placed at the other end. The measuring circuit, and, in fact, the experiment itself, were wholly analogous to those described in [2]. Two capillaries were used: one was 0.4 mm in diameter and 190 cm long (capillary I), and the other, 1.3 mm in diameter and 170 cm long (capillary II). At its cold end, capillary I opened out smoothly into a ~ 1 mm diameter capillary. The transition zone extended over a length of ~ 3 mm. The exit aperture of capillary II had a sharp rim, while at the free end, the heat from the heater was fed uniformly into the capillary over a section ~ 1 cm long.

Figure 1 illustrates the development of the thermal regime in capillary I for $W = 1.19\,W_{cr}$, at 1.31°K. The thermometer R_1 was located near the outflow end of the capillary, R_4, near the heater, and R_2 and R_3, between R_1 and R_4, in such a way that the distance between neighboring thermometers was approximately uniform. The evolution of the thermal regime shown in Fig. 1 is readily explained as the result of a turbulence front advancing from the hot end. It first passes R_3, then R_2, and, finally, the turbulence fills the entire capillary. The velocity of the front v_h is found to be 7 mm/sec.

Comparison of the velocity of the front in a 1.4 mm diameter capillary [2] with that in the 0.4 mm capillary, for heat flows correspondingly exceeding the respective critical values, shows that the

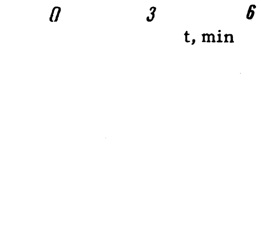


FIG. 1. Development of the thermal regime in capillary I (diameter 0.4 mm) for a thermal flux W = 0.124 W/cm² (W = 1.19 W_{cr}), T = 1.31°K.

front velocity in the smaller capillary is, to within the experimental error (10%), 3.5 times as great; i.e., by a factor which is the reciprocal of the ratio of the capillary diameters. The advance of a cold front from the cool end of the capillary was not observed when the helium was at rest, due to the smoothness of the capillary exit. When the helium was not wholly at rest, however, the presence of a heat flow gave rise to turbulence nuclei within the capillary, from which the turbulence propagated—a pattern entirely similar to that previously described[2]. Due to the short time required for establishment of the thermal pattern in this case (< 3 min) it was not possible to make reliable measurements of v_c.

The front velocity v_h in the capillary of diameter $d_1 = 0.4$ mm depends upon the thermal flux (solid curve in Fig. 2, capillary diameter 0.4 mm). This dependence is found to be the same for capillaries of various diameters. The dashed curve in Fig. 2 represents a function of the form

$$v(W) = \frac{d_2}{d_1} v_h\left(\frac{W_{cr}(d_2)}{W_{cr}(d_1)} W\right). \quad (1)$$

Here $v_h(W)$ is the velocity of the hot front as a function of the thermal flux in a capillary having a diameter $d_2 = 1.4$ mm[2], and $W_{cr}(d)$ is the critical thermal flux for a capillary of diameter d.

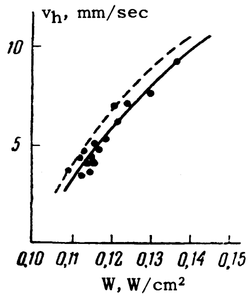


FIG. 2. Dependence of the velocity of the hot front upon thermal flux at T = 1.31°K in capillary I.

The small displacement of the solid and dashed curves from one another may be attributed to inaccuracy in the determination of the critical thermal flux, and to the small difference in the temperatures at which the two curves were recorded (1.31 and 1.34°K).

The relation thus found for the dependence of the turbulence front velocity at equivalent values of the thermal flux upon the capillary diameter—inverse proportionality—can readily be interpreted. When the capillary diameter is reduced by a factor k and the thermal current density is increased in correspondence with the increased value of the critical flux, as a result of which the mean distance between the vortex lines is also reduced by the factor k, the geometrical distribution of the vortices in the capillary must remain the same.[5,6] When, however, all of the vortex separations are reduced by the factor k, the constancy of the circulation about each vortex line requires that all of the velocities be increased to k times their previous values. In particular, the velocities v_h and v_c must increase by the factor k. These considerations lead us to the conviction that the inverse proportionality of the front velocities and the capillary diameters must hold for all diameters of the capillary, provided that the critical regime within the capillary can be treated as one of superfluid turbulence.

Figure 3 illustrates the evolution of the thermal regime typical of capillary II. A single front is observed to move within the capillary—from the cold end. Turbulence does not arise at the hot end owing to the distributed nature of the heat input. The difference in the thermometer readings, which appears as soon as the thermal current is initiated, is due to viscous flow of the normal component.

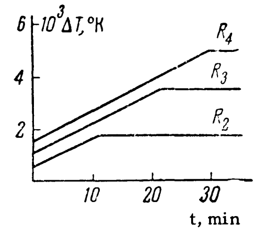


FIG. 3. Development of the thermal regime in capillary II (diameter 1.3 mm) for $W = 2.52 \times 10^{-2}$ W/cm², T = 1.1°K.

It is of interest to trace in greater detail the behavior of the curves $v_h(W)$ and $v_c(W)$ near $W = W_{cr}$. Let us take the front velocity to be positive when it lies in the same direction as the flow of heat. It is readily shown that as W approaches W_{cr}, the function $v_c(W)$ must change sign. As an actual case, let a turbulence front travel away from the hot end of a reasonably long capillary, in which a slightly super-critical thermal current $W > W_{cr}(T)$ is flowing. As the front advances, the temperature at the hot end will rise, accompanied by an increase in the value of the critical thermal flux. When the latter becomes larger than W, the turbulence at the hot end must vanish. As a result, it is found that as the temperature at the hot end is raised further, the region of laminar flow in this vicinity becomes more and more extensive, which is interpreted as the retreat of a cold front.

We were able to carry out several experiments to measure the velocity with which the cold front retreats. Figure 4 shows a plot of the thermometer readings for the case of a retreating turbulence

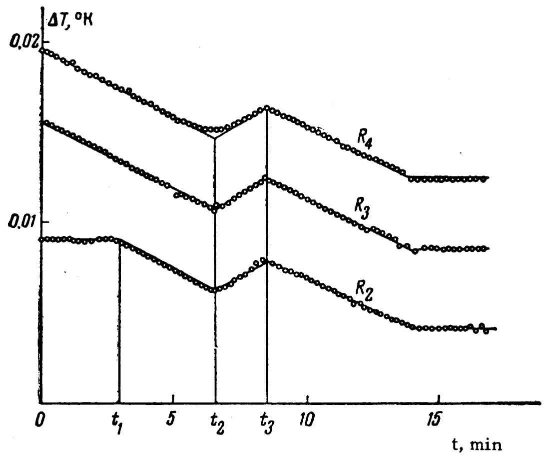


FIG. 4. Disappearance of turbulence in capillary I for $W = 5.3 \times 10^{-2}$ W/cm², T = 1.1°K.

front. The time t = 0 corresponds to turbulent flow over the segment of the capillary from R_1 to R_3.

This regime was generated by gradually reducing the heat flow from a highly supercritical value, for which the flow was turbulent over the whole capillary. Under these conditions the turbulence in the hot portion of the capillary vanished, due to the fact that the thermal flux was near the critical value for this temperature. The turbulence lasted somewhat longer in the cold part of the capillary. The decrease in the temperatures of the thermometers R_3 and R_4 corresponds to a retreat of the turbulence front. At t_1 the front passed R_2, and this thermomenter likewise began to cool down. At t_2, however, there appeared in the vicinity of R_2 a turbulence nucleus from which two fronts proceeded to move, both in the direction of the heat flow, but with $v_h > v_c$. A similar phenomenon—the random appearance of a new source of turbulence—has previously been described.[2] The reason for the appearance of such turbulence nuclei is not as yet completely clear. At t_3 the newly-formed internal hot front encountered the previously existing cold front, and the turbulent regions merged. As the new cold turbulence front retreated, the turbulence eventually vanished from the whole capillary—the remaining temperature drop along the capillary corresponds to the viscous flow of the normal component.

Figure 5 shows the relation of the front velocities v_h and v_c at T = 1.1°K. Measurements of v_c for $v_c < 0$ were not conducted with the 0.4 mm diameter capillary due to the short equilibrium time at high flux values (< 3 min). The values for $v_c < 0$ given in Fig. 5 represent velocities v_c in capillary II, recomputed with the aid of Eq. (1).

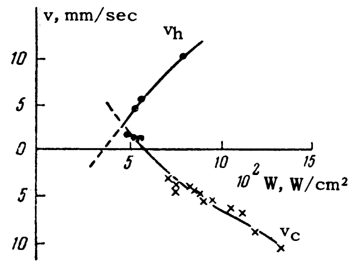
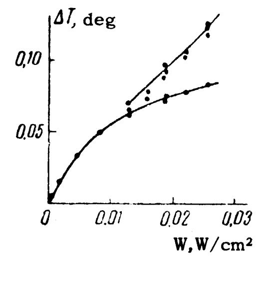


FIG. 5. Dependence of the front velocities v_h and v_c upon thermal flux at T = 1.1°K for capillary I [the crosses correspond to experimental points from capillary II, recomputed with the aid of Eq. (1)].

Turbulence fronts were observed over the range 0.8—1.5°K. For T > 1.5°K, the sensitivity of the circuit was found to be insufficient for the detection of fronts in the 0.4 and 1.3 mm capillaries. For T < 1°K difficulties of another sort arise. When a thermal current near the critical value is passed through the capillary, the temperature drop along the latter becomes extremely large (~ 0.1°), and it is therefore no longer possible to specify the temperature to which the measured velocity corresponds. Moreover, the magnitude of the mutual friction in the transport of heat through helium becomes smaller and smaller as the temperature is reduced. While for T ~ 1.3 and $W \sim W_{cr}$, ∇T increases by a factor of ten at the transition to the supercritical regime, and for T ~ 1.1°K ($W \sim W_{cr}$) ∇T is approximately doubled, at T = 0.8°K the relative increase in ∇T is found to be very small (not over 10%).

Figure 6 illustrates the dependence of the temperature drop along the capillary upon the thermal flux. The cold end of the capillary was held at 0.8°K. The lower branch corresponds to laminar flow; the upper, to complete filling of the capillary with turbulence, which can be detected from the motion of the fronts (v_h and v_c were found to vary from 6 to 9 mm/sec, and from −0.4 to −1.1 mm/sec, respectively, as W increased from 0.018 to 0.025 W/cm²). The points lying between the curves correspond to partial filling of the capillary with turbulence. Determination of the critical thermal current under these conditions is found to be difficult.

FIG. 6. Dependence of the temperature drop along capillary II upon thermal flux for a cold-end temperature T = 0.8°K.



Let us pause to consider what is meant by the critical velocity. Generally speaking, this is a velocity v_{cr} such that for $v < v_{cr}$, the flow is laminar, while for $v > v_{cr}$, it is turbulent. These statements are valid, however, only when the character of the flow can be described in terms of a single velocity, and is independent of prior history. Such a dependence is, however, present in the flow of heat through helium. An analogous situation also prevails in the ordinary viscous flow of a liquid in a tube.

Under such circumstances the critical velocity may be defined as that velocity beyond which laminar flow cannot exist.[5] The occurrence of such a velocity, however, for $v \ll c$ (c is the velocity of sound) is dubious, in the absence of any experimental or theoretical evidence to demonstrate that at some velocity the flow becomes absolutely unstable relative to small perturbations. Experiment shows that as the perturbations are reduced, laminar flow is observed to progressively higher velocities. This is true of both superfluid and ordinary turbulence.

Inasmuch as laminar flow is found to be possible even at velocities considerably beyond the value usually regarded as critical, one might define the critical velocity as the minimum velocity at which turbulence is still possible. True, the existence of such a velocity does not appear to have been fully demonstrated. We shall see below, however, that this definition of the critical velocity is more fruitful than that given above (i.e., the maximum velocity at which laminar flow is still possible).

Let us consider the conditions for the existence of turbulence in the capillary, on the basis of our data on the velocity of the turbulence front (Fig. 5). The destruction of the turbulence in capillary I for $v_c > 0$ (Fig. 4) demonstrates that it is not always possible to speak of the thermal flux as either subcritical or supercritical. If this were so, then if vortices were to be continuously generated at the hot end of the capillary, the thermal regime within the capillary would always be turbulent, while if not, the turbulence would vanish. As the thermal flux is reduced, however, we encounter the situation where $v_c > v_h$. In this case a local interruption of turbulence occurring at some point within the capillary would always tend to grow. Specifically, the turbulence might vanish in the presence of mechanical vibration.[6] Finally, it may be found that $v_h < 0$ and $v_c > 0$. In this event, the capillary can never be filled with turbulence; the thermal flux for which $v_h = 0$ is therefore to be regarded as the critical value.

We shall now discuss in greater detail the processes taking place within the capillary for a thermal flux near the critical value. It has already been shown repeatedly that the velocity profiles of the normal and superfluid components must be nearly parabolic (see, for example, [7]), with the depth of the v_s profile of the same order as v_n. The minimal density of vortices in the superfluid components necessary for the existence of a profile of this depth is $L = v_n/r(h/m)$, where r is the radius of the capillary and h/m is the circulation of the velocity about a vortex line. The mutual friction force $F_{sn}^{(1)}$ arising from the interaction of these oriented vortices will be[8]

$$F_{sn}^{(1)} = B\frac{\rho_s\rho_n}{\rho}(v_s - v_n)\frac{v_n}{2r}. \quad (2)$$

On the other hand, the experimentally observed mutual friction force can be described by the formula

$$F_{sn}^{(2)} = A\rho_s\rho_n(v_s - v_n)^3. \quad (3)$$

The values for the coefficient A determined from our experiments (Fig. 7), using the formula $F_{sn} = \rho S \nabla T$ and Eq. (3) agree closely with the data found in the literature. It is evident that $F_{sn}^{(2)} \geq F_{sn}^{(1)}$. This inequality is fulfilled when, in addition to the oriented vortices, vortices of arbitrary orientation exist within the capillary. Since $F_{sn}^{(2)}$ decreases more rapidly with decreasing thermal flux than $F_{sn}^{(1)}$, there then exists some value of the flux for which the inequality is no longer fulfilled and, consequently, no equilibrium density of vortices can exist within the capillary—at this point the transition to the subcritical regime takes place. The corresponding velocity is

$$v_{n\,cr} = \frac{B}{A\rho\cdot 2r(1+\rho_n/\rho_s)^2}. \quad (4)$$

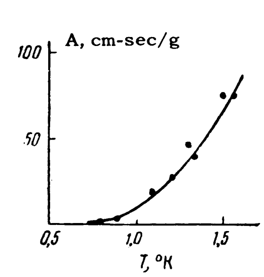


FIG. 7. Values of the mutual friction coefficient A as a function of temperature.

The value of the critical thermal flux from our experiments with capillary I (0.4 mm diameter) are presented in Fig. 8. The accuracy to which W_{cr} was determined amounts to 7—10% for T > 1.2°K, while at T = 0.9°K it falls to ~ 20%. To within this degree of accuracy, the critical thermal fluxes in capillaries I and II are inversely proportional to their diameters, and fully agree with the results of Chase.[9]

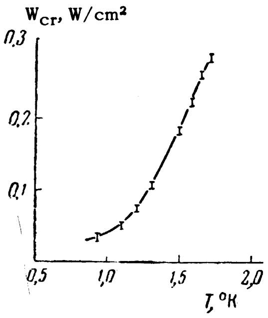


FIG. 8. Temperature dependence of the critical thermal flux for capillary I.

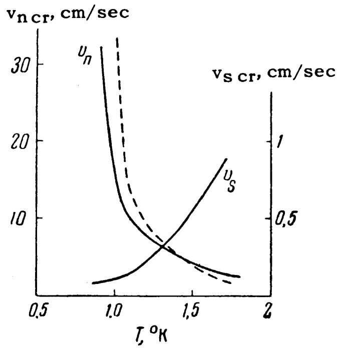


FIG. 9. Temperature dependence of the velocities v_n and v_s at the critical thermal flux (capillary I).

Figure 9 shows the values of the velocities v_s and v_n at the critical thermal flux, as a function of temperature. The dashed curve corresponds to Eq. (4). The values for the coefficient B were taken from the paper by Hall and Vinen[8] (below 1.2°K we have assumed B = 1.5). As is evident, the agreement is quite satisfactory—the order of magnitude and the temperature dependence are similar. It would be difficult to anticipate any closer agreement, in view of the approximate nature of the model. Thus, we see that the critical thermal flux is that flux at which the number of vortices formed under relative motion of the normal and superfluid components is comparable with the minimum number of vortices necessary for the establishment of a deep superfluid velocity profile. For lower heat flow rates no turbulence can exist.

We should note that the interpretation presented above for the critical velocity is applicable only in the case of a thermal flux such that the velocity profiles in the supercritical regime are nearly parabolic. If we consider purely superfluid flow, for which the superfluid velocity profile remains linear, in the first approximation, in the supercritical regime, the number of oriented vortices required to support this profile is zero. In this case, a weaker condition emerges for the existence of turbulence—that the mean distance between the vortices $l = 1/\sqrt{L}$ must be less than the dimension d of the channel. The vortex density for relative motion of the normal and superfluid components is readily determined from the relation[5]

$$\frac{B}{3}\frac{\rho_s\rho_n}{\rho}(v_s - v_n) L \frac{h}{m} = F_{sn} = A\rho_s\rho_n (v_s - v_n)^3$$

For $v_n = 0$. The condition $1/\sqrt{L} \sim d$ then yields

$$v_{s\,cr} \sim d^{-1}\sqrt{Bh/3\rho Am}. \qquad (5)$$

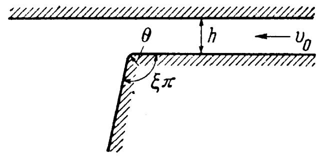
The value computed from this formula agrees well with the data of Kidder and Fairbank[10] on the critical velocity for pure superfluid flow in a capillary 1.1 mm in diameter.

None of the foregoing discussions treat of the process by which vortices arise in initially vortex-free helium. In an ideal fluid the theorems concerning the conservation of velocity circulation prohibit the generation of vortices. In order, therefore, to resolve this question we must take into account the possibility of cavitation and jet-streaming resulting in the formation of velocity discontinuity surfaces.[11]

Our experiments show that the generation of vortices is influenced by the character of the outflow aperture and general non-uniformity of the capillary, which evidently serve as vortex sources. Using the methods of the theory of the functions of a complex variable,[12] the maximum velocity multiplication factor k as the liquid flows around an angle $\pi\xi$ $(0 \le \xi \le 1)$ (see Fig. 10), assuming $\theta \ll h$ (where θ is the radius of curvature of the edge), is:

$$k \sim (h/\theta)^{(1-\xi)/(2-\xi)}. \qquad (6)$$

FIG. 10. Flow around an angle in an ideal fluid.



A flow configuration of this sort, with $\xi \sim \frac{1}{2}$ (a right angle), can occur at the capillary exit. If we assume $\theta \sim 10^{-8}$ cm, h ~ 0.1 cm, and $\xi \sim \frac{1}{2}$, we then obtain k ~ 200. For flow through a capillary 1 mm in diameter, with a mean velocity of the order of the critical value, $v_s \sim$ 1 mm/sec, the maximum flow velocity will be ~ 20 cm/sec. In flow around angles the conditions for the generation of vortices are more favorable than for the case of flow within a smooth tube. The process of vortex formation, however, is still not understood, and requires further study from all sides.

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