STABILITY OF EQUILIBRIUM STATES OF THE NUCLEUS IN THE DROP MODEL

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The results of an analysis of the stability of symmetric equilibrium shapes of the nucleus with respect to asymmetric variations of the surface are presented. The results of the calculations are compared with experiment on the basis of the quasistatic model of nuclear fission.

1. STABILITY OF EQUILIBRIUM SHAPES

In view of the well-known problem of the asymmetry of fission of heavy nuclei an important place in contemporary theory of fission is occupied by the question of the stability of equilibrium and quasiequilibrium, i.e., those corresponding to conditional equilibrium [1], states of the nucleus with respect to asymmetric variations of the shape of the surface. In a rigorous formulation the problem would consist of determining the sign of the second variational derivative of the functional of the potential energy; however, usually a more restricted problem is posed which consists of determining the sign of the second derivative of the potential energy with respect to the "parameter of asymmetric deformation" (the first derivative is, evidently, always equal to zero) [2-4].

In view of the obviously limited nature of the results of preceding papers [2-4] in the course of calculations of equilibrium solutions of the drop model [5] an analysis was also carried out of the stability of equilibrium states with respect to asymmetric variations of the shape of the surface. Such an analysis was carried out for symmetric shapes corresponding to absolute and to conditional equilibrium and belonging to the usual sequence of shapes of type (1;1)—(2;2) without a neck or with a single neck [1,5] for the usual drop model with constant surface tension ($\Gamma = 0$) and $\Gamma = \pm 0.1$ [5].

For the construction of asymmetric shapes arbitrarily close to the given symmetric shape a method was used which followed naturally from the iteration method utilized for the calculation of equilibrium solutions. Along with a given symmetric figure we have also investigated three closely similar asymmetric ones: a, b, and c. For each of these, as well as for the symmetric one, a complete calculation was carried out of all the significant quantities which were compared with the corresponding quantities for the symmetric figure. An analysis of the shape of the asymmetric figures showed that case a corresponds to a simple deformation of the symmetric figure which is almost purely asymmetric with respect to the center of the neck and which consists fundamentally of a proportional change in the linear dimensions of the left hand side and the right hand side parts of the figure measured from the center of the neck. The variations of the surface b and c consist of approximately equal contributions of symmetrical and asymmetrical components with the shapes b and c being fundamentally mirror images of each other with respect to the plane perpendicular to the symmetry axis and passing through the center of the neck (cf., Fig. 1).

In the course of the calculations it turned out that the stability with respect to asymmetric variations of the shape can be characterized with the aid of the quantity

$$Q = (W_{AS} - W_S)\,\eta^{-2}, \qquad \eta = 2\,(V_l - V_r)\,(V_l + V_r)^{-1}, \tag{1}$$

where W_{AS} and W_S are the potential energies respectively of the asymmetric and of the symmetric shape, V_l and V_r are the volumes of the parts of the asymmetric figure situated to the left and to the right from the center of the neck (cf. [1,5]). The parameter η characterizes the asymmetry in a most natural manner, and it can be directly related to the asymmetry of the masses of the fission fragments if we assume that separation occurs at the "narrowest point," i.e., at the center of the neck.

The change in the potential energy accompanying asymmetric deformation is, of course, determined not only by the value of η, and therefore Q, strictly speaking, is not a constant quantity even for asymmetric variations of the surface of the same type. However, in practice it turned out that the quantity Q almost does not change even for a relatively large change in the scale of the

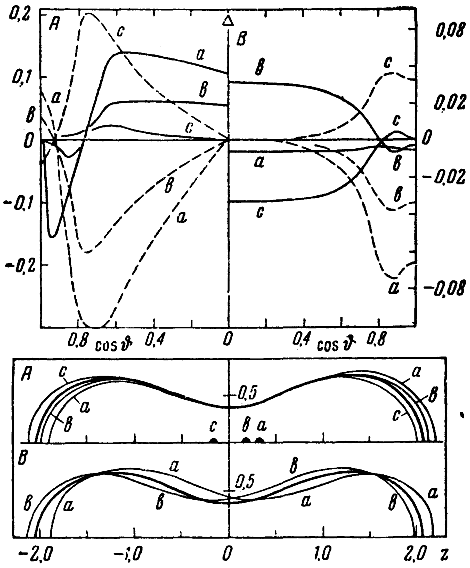

FIG. 1. Decomposition of the asymmetric variation of the surface of a figure into antisymmetric (dotted line) and symmetric (solid lines) components: A — in the center of mass system, and B — with respect to the middle of the neck of figure corresponding to an absolute extremum of type (2; 2) for x = 0.65, Γ = 0 (T = 1.7944) for variations of the surface of type a, b and c (cf. Sec. 3). Below are shown a symmetric figure (heavy solid line) and trial asymmetric figures: in case A — the centers of the necks are made to coincide, and in case B — the centers of mass are made to coincide (the dots on the horizontal axis denote the position of the centers of mass of the figures).

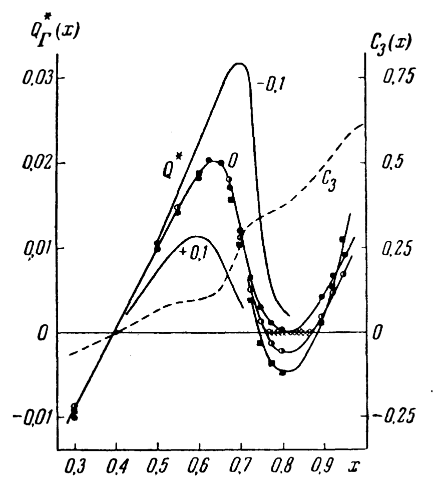

FIG. 2. The quantity $Q^*(x)$ characterizing the stability of the absolute extrema with respect to asymmetric variations of the shape of the nucleus for $\Gamma = 0, \pm 0.1$ plotted separately for variations of type a(●), b(■) and c(◆) in units of $E_S^{(sph)}$. Crosshatching along the x axis indicates the region of figures which are transitional from figures with a neck for small values of x to figures without a neck for x = 0.8. The dotted line shows the stiffness of the nuclear surface with respect to variations of shape which are octopole in the center of mass system[4].

asymmetric deformation of type a. For absolute extrema this occurs also for other asymmetric variations of the surface. The value of Q changes by not more than 10% when the numerator and the denominator in formula (1) change by more than an order of magnitude. As a control the calculation was in each case carried out for several asymmetric deformations of each type differing in magnitude.

The results obtained are illustrated in Figs. 2 and 3 where we show respectively graphs of the average value of Q for absolute extrema (saddle points), Q^* and $Q_x(\rho)$ for conditional equilibrium shapes. As in the preceding articles[1,5,6] ρ is the deformation parameter of a symmetric figure, while x is the usual parameter of the drop model.

It can be seen in Fig. 2 that Q^* for different Γ are similar to one another and differ little for deformations of types a, b, and c. This latter circumstance enables us to speak approximately of a single function $Q^*(x)$ for absolute extrema. It can be seen that the domain $x < 0.4$ corresponds to an instability of the saddle shape with respect to asymmetric variations of the shape of the type under investigation ($Q^* < 0$). For values of x in the range 0.75—0.85 the saddle shapes are unstable with respect to mixed variations of type b and c ($Q^* < 0$), and are practically unstable also for asymmetric variations of type a, for which Q^*, although positive, is still very small. For other values of x the quantity $Q^* > 0$, i.e., the saddle shapes are stable, with the maximum of stability lying at x = 0.60—0.65. Such behavior of $Q^*(x)$ can be easily understood if we take into account the change in the shape of the saddle figure depending on x. As $x \to 0$ the shape of the surface for a symmetric absolute extremum tends to two spheres touching each other. The instability for $x < 0.4$ is explained by the fact that for $x \to 0$ the asymmetric absolute extremum of type (3;1) also tends to the same figure, with the smaller energy corresponding to the latter case for $x < 0.4$[5].

As x is increased the saddle shape tends to the critical shape which corresponds to the maximum deformation and which is unstable with respect to separation into two fragments[6,5]. For $\Gamma = 0$ the saddle shape corresponds to the greatest deformation for $x = x^* = 0.55$. As can be seen from Fig. 2, the maximum stability of the saddle shape with respect to asymmetric variations of the shape occurs at values of x somewhat greater than x^*. For still greater values of x the deformation decreases, since as $x \to 1$ the shape of the saddle figure tends to a sphere.

The transition of absolute equilibrium (AE) shapes from shapes without a neck to shapes with a neck occurs at $x \approx 0.8$; this particular value of x is characterized by maximum instability. For $x \gtrsim 0.85$ the saddle shape again becomes stable. The spherical shape, with which the absolute extremum under consideration coincides at $x = 1$, is stable for all $x < 7/4$ [3]. In Fig. 2 we have shown for comparison the curve for the stiffness C_3 with respect to variations of the surface which are octupole in the center of mass system calculated for absolute extrema in the paper by Cohen and Swiatecki [4]. It can be seen that for $x \gtrsim 0.7$ there is no correspondence whatever between the curves for $Q^*(x)$ and $C_3(x)$.

An analogous analysis of stability was also carried out for the usual sequence of conditional equilibrium shapes described in [5,6]. In these calculations it turned out that unique values of Q independent of the scale of the deformation are obtained only for the "purely asymmetric" deformation of type a. This should have been expected, since variations of the surface of type b and c also contain a symmetric component, as a result of which in the case of conditional equilibrium shapes there also occurs an increment in the potential energy linear in the deformation.

Figure 3 shows data for the asymmetric variation of type a ($\Gamma = 0$). It can be seen that the stability increases as the deformation of the symmetric shape is increased and the critical shape is approached which corresponds to $\rho = \rho_{crit} = 1.17$, with the quantity Q for a given x being the larger the greater ρ. These results are in direct contradiction to the results of the papers of Nosov [2] and of Bussinaro and Gallone [3]. This contradiction is, possibly, explained by the fact that in the papers indicated an investigation was made essentially of arbitrary ellipsoidal shapes which for large deformations have little in common with the sequence of conditional equilibrium shapes.

Asymmetric variations of shape for figures consisting of two spheres in contact were investigated by Frankel and Metropolis [7] who showed that such a figure is stable for all $x > 0.6$. Utilizing formula (31) from [7] one can obtain the following expression for Q in the limiting case of two spheres:

$$Q_\infty = 0.035\,(5/3\,x - 1). \qquad (2)$$

The value of Q should tend to this limiting value for $\Gamma = 0$ when the figure tends to two spheres in contact.

This occurs for the second (upper) branch of the sequence of symmetric conditional equilibrium

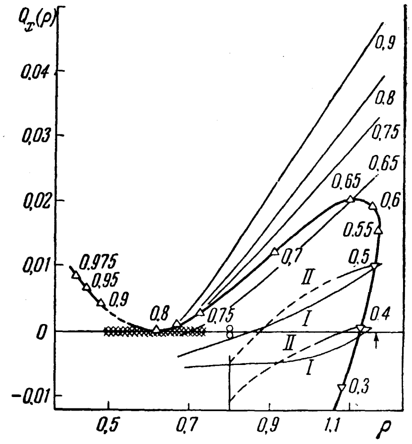

FIG. 3. The quantity $Q_x(\rho)$ characterizing the stability of conditional equilibrium shapes of type (2; 2) (with a neck) with respect to asymmetric variations of the shape of type a (light lines) $\Gamma = 0$. On the right are shown values of the parameter x. Triangles correspond to absolute extrema, the values of x are shown alongside. Along the $Q = 0$ axis cross-hatching indicates the region corresponding to transitional figures, the arrow denotes $\rho = \rho_{cr} \approx 1.17$, two circles correspond to $\rho = 0.795$ for spheres in contact. The Roman numeral I corresponds to the usual (lower) branch, II corresponds to the upper branch of the sequence of conditional extrema. The dotted line joins the calculated intervals of $Q_x(\rho)$ for the upper branch with points for two spheres in contact.

shapes of type (2;2) [5,6] which connects the critical shape ($\rho \approx 1.17$) and two spheres in contact ($\rho = 0.795$). Unfortunately, calculations for shapes close to two spheres in contact can be carried out only for small values of x, and therefore in Fig. 3 the graphs of $Q_X(\rho)$ are given for all values of x with the exception of $x = 0.4$ and $x = 0.5$ only for the usual branch connecting the initial sphere and the critical shape. For $x = 0.4$ and $x = 0.5$ we have also shown $Q_X(\rho)$ for that part of the second branch which is close to the critical shape. These segments are denoted by the Roman numeral II to distinguish it from the usual branch denoted for these values of x by I. The segments for which these calculations have been made are joined by a dotted line to the limiting points for $\rho = 0.795$ calculated in accordance with (2). As can be seen from Fig. 3 the situation for two spheres in contact has very little to do with conditional equilibrium shapes corresponding to a large deformation, and in particular for a deformation equal to the critical deformation.

The transition shapes ($\rho \approx 0.6$) are unstable for all values of x, but the magnitude of $|Q|$ diminishes as x is increased. Apparently, this is connected with the fact that the shape of the transition figure in the region of the neck is the closer to cylindrical shape the larger the value of x, while for a cylindrical shape it is natural to assume that $Q \approx 0$, i.e., absolute equilibrium exists with respect to the position of the neck.

In Fig. 3 triangles denote points corresponding for a given value of x to an absolute extremum. The heavy curve drawn through them corresponds to $Q^*(x)$ in Fig. 2 and is another representation of this quantity. It can be seen that the characteristic variation of Q^* in Fig. 2 is related to the variation in the shape of the saddle figure. An increase in stability with increasing ρ explains why the stability maximum in Fig. 2 lies at $x > x^*$.

2. COMPARISON WITH EXPERIMENT

From data given in Fig. 2 it follows that fission of nuclei which, depending on Γ, correspond to $x = 0.55-0.75$ must be symmetric. Indeed, due to the fact that in this case the (symmetric) fission barrier almost coincides with the critical point for the loss of stability, the second phase of the fission process, the descent from the barrier, is in fact absent. Since for these values of x the saddle shape is stable with respect to asymmetric variations of the surface, the greatest probability corresponds to fission into two identical fragments.

Within the framework of the quasiequilibrium statistical model[1)] of fission one can with the aid of data shown in Fig. 2 also give an estimate of the width of the distribution of the fragments with respect to mass. For $x \approx 0.55-0.65$ the distribution of the fragments with respect to mass will have the form of a Gaussian distribution, and by utilizing (1) the probability of a given mass ratio may be written in the form

$$P(\eta) \sim \exp\{-T^{*-1} E_S^{(\mathrm{sph})} Q_\Gamma^*(x)\, \eta^2\}, \qquad (3)$$

where T^* is the temperature at the saddle point; while $E_S^{(\mathrm{sph})}$ is the surface energy of the initial spherical nucleus.

The half-width of the Gaussian distribution (3) is equal to

$$\eta^{(G)} = (T^*/E_S^{(\mathrm{sph})} Q_\Gamma^*)^{1/2}. \qquad (4)$$

Relation (4) agrees relatively well with available experimental data on the distribution of fragments with respect to mass if for Q^* we adopt values corresponding to the maxima of the curve of $Q^*(x)$ in Fig. 1. The table gives for some reactions the values of T^* calculated from the experimental

Target nucleus	Incident particle	T^*, MeV	
		$\Gamma = 0$	$\Gamma = -0.1$
Bi^{209}	He^4 (26 MeV)	0.6	0.9
Pb^{206}	He^4 (43 MeV)	1.0	1.3
Au^{197}	C^{12} (112 MeV)	1.3	1.7
Bi^{209}	d (190 MeV)	2.2	3.0

width of the mass distribution with the aid of formula (4) for $\Gamma = 0$ $(x \approx 0.65)$ and $\Gamma = -0.1$ $(x \approx 0.70)$. The values of T^* are quite reasonable if we adopt for the energy of the saddle point (the fission threshold) a quantity of the order of 10–15 MeV. In the case of the last two reactions we must take into account the possibility of evaporation of particles prior to fission. The value of T^* for $\Gamma = -0.1$ agrees better with the value of the excitation energy at the saddle, and also with T^* obtained from the value of the fluctuation of the charge of the fragment of a given mass in the reaction $Bi^{209} + C^{12}$[9].

In the quasistatic model of fission the value K of the average total kinetic energy of the fission fragments is approximately equal to the energy of the mutual Coulomb repulsion of the two "halves" of the nucleus at the critical point $E_{\mathrm{mut}}^{(C)}$. The quantity $E_{\mathrm{mut}}^{(C)}$ is proportional to x[5]:

$$E_{\mathrm{mut}}^{(C)} \approx k_\Gamma x, \qquad (5)$$

where with sufficient accuracy we have

$$k_\Gamma = 0.43\,(1 + 1.2\,\Gamma). \qquad (6)$$

On substituting into (5) the value of x in accordance with its definition[5] we obtain

$$K = E_{\mathrm{mut}}^{(C)} = {}^1/_2 E_C^{(\mathrm{sph})} k_\Gamma,$$

where $E_C^{(\mathrm{sph})}$ is the Coulomb energy of the initial spherical nucleus. From this we obtain

$$K \approx a_\Gamma Z^2 A^{-1/3}, \qquad (7)$$

where

$$a_\Gamma = 0{,}3\, k_\Gamma e^2 r_0^{-1} = 0{,}42\, k_\Gamma r_0^{-1}\,\mathrm{F}. \qquad (8)$$

Relation (7) has been well checked experimentally. The "experimental" value is $a_\Gamma = 0.12$ MeV [10]. This value agrees with formula (8), although for $\Gamma = 0$ the numerical agreement is obtained only for $r_0 = 1.5$. Formula (8) does not contain x and, therefore, it is convenient to utilize it for the determination of Γ. It can be seen that for $r_0 = 1.2$ one should take $\Gamma \sim -0.15$. This value agrees with the theoretical estimate[2)] [11] for the

[1)]This model is confirmed particularly by data on the angular anisotropy of the fragments which are well described by the statistical theory[8].

[2)]In[11] the drop model is obtained as an approximation to the statistical model of the nucleus.

density distribution of the nucleons near the edge of the nucleus described by the Fermi function.

Other quantities which can be evaluated in the quasistatic approximation are the height of the fission threshold E_f and the mass defect ΔM. In making comparisons with experimental data it is convenient to express these quantities in units of the Coulomb energy of the initial nucleus, since the latter contains only the one parameter r_0. For ΔM we obtain

$$\Delta M = E_C^{(\mathrm{sph})} \Delta m,$$

$$\Delta m = 0.5x^{-1}\,[-0.260 + 0.740\,x + 0.33\,(1 + x)\,\Gamma_0],$$

where Γ_0 is the value of Γ for the initial nucleus ($\Gamma \sim A^{-1/3}$). In the specific case of the fission of the Po^{220} nucleus the best agreement with experiment for all three quantities (E_f, ΔM, K) is obtained for $\Gamma \approx -0.1$ and $x \approx 0.72-0.74$ if $r_0 = 1.2$, and $\Gamma = 0$ and $x \approx 0.67$ if $r_0 = 1.24$.

For $\Gamma \neq 0$ the proportionality between x and the quantity Z^2/A is approximate, since due to the dependence of the surface tension on A the quantity x contains the factor $(1-\Gamma_0)^{-1}$, where $\Gamma_0 \sim A^{-1/3}$. Within the narrow domain of fissionable nuclei one can, nevertheless, write x in the form

$$x \approx (Z^2/A)/(Z^2/A)_{\mathrm{cr\,(eff\,)}}.$$

The values of x given above for the symmetric fission of Po^{210} correspond to $(Z^2/A)_{\mathrm{cr(eff)}} = 45$ for $r_0 \sim 1.2$ and $(Z^2/A)_{\mathrm{cr(eff)}} = 50$ for $r_0 = 1.4$. For the fission threshold for the U^{238} nucleus for $r_0 = 1.2$, $\Gamma = -0.1$ and $(Z^2/A)_{\mathrm{cr(eff)}} = 45$ we then obtain the value 6.5 MeV[5] which agrees well with experiment. In this case we have the values $E_C^{(\mathrm{sph})} = 960$ MeV, $E_S^{(\mathrm{sph})} = 600$ MeV. The last value corresponds to $4\pi r_0 O_{\mathrm{eff}} = 16$ MeV, if we represent $E_S^{(\mathrm{sph})}$ in the form

$$E_S^{(\mathrm{sph})} = 4\pi r_0^2 O_{\mathrm{eff}}\, A^{2/3}.$$

When the term ΓH is present in the surface energy one should redetermine all the constants of the semiempirical formula for the nuclear masses and, therefore, it is difficult to say whether this value of $E_S^{(\mathrm{sph})}$ is in agreement with the mass formula.

One should also note that for the absolute extremum for $x \approx 0.65$ there is good agreement between the value of the effective moment of inertia J_{eff} evaluated in [6] and the experimental value determined from the angular distribution of fission fragments of nuclei with A = 200—210[11]. Unfortunately, it is not possible to give an estimate of the magnitude of the fluctuation of the mean kinetic energy of the fragments since the latter is determined by specific dynamical conditions at the point of separation. One can only assume that, in view of the special properties of the critical point where stability with respect to breakup is lost, the probability distribution for the kinetic energy will not be symmetric.

The estimates given above for symmetric fission are based on the description of fission as a thermodynamically quasiequilibrium statistical process. Therefore, it would be better to speak of the statistical approximation to the theory of fission, since the "drop model" is utilized here only for the evaluation of potential energy. Since in the statistical model the question is posed of the probability of one or another type of asymmetry of shape with respect to the point of separation, it is natural to choose for the asymmetric variations of the surface those which are asymmetric with respect to the assumed point of separation, i.e., the middle of the neck (type a).

In the case of heavier nuclei the instability of the saddle shape with respect to asymmetric deformations for x = 0.77—0.78 suggests that this is to a certain extent related to the asymmetry of fission of U, Pu, and other nuclei which correspond to x = 0.74—0.77 ($\Gamma = 0$). The available data from the calculations give evidence of the possibility of asymmetric descent from the saddle; however, it is not possible to express in advance a preference for asymmetric deformations, since the saddle shape is also unstable with respect to symmetric changes of shape. Since the deformation of the saddle shape is still insufficiently great, then, evidently, the question of what type of fission, symmetric or asymmetric, will be more probable will be decided at a later nonequilibrium stage of the process.

In this case for the description of fission we require additional physical assumptions which will determine the model for the descent from the barrier. Such a model must take into account the kinetic properties of the fission process, since by considering only the potential energy we cannot say anything about the "unprofitable" character of the asymmetric descent from the barrier. In addition to the magnitude of the potential energy we must also prescribe at least one other physical quantity which directly characterizes the fission process.

The simplest quantity of such a kind could be an element of length which determines the metric of the space of generalized variables describing the shape of the nucleus[1]. It can be shown that the sequence of conditional equilibrium shapes investigated in this paper and in [5] is indeed the "most advantageous" one in the sense that it corresponds to "steepest descent" in the space in which the distance between two "points" is defined as the change in the deformation parameter. By prescribing two or an even greater number of deformation parameters one can in principle provide a model of asymmetric descent from the saddle.

In view of the fact that in neighboring deformations the asymmetry is almost inconsequential from the point of view of the quasiclassical ("drop") potential energy, the position of the maximum of the asymmetric peak in the distribution of fragments with respect to mass can be determined by relatively small factors which are accidental from the point of view of the quasiclassical model, and in particular by the possibility of formation of shells in the fragments (cf., for example, [12]).

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