FIELD THEORY WITH NONLOCAL INTERACTION. III. DIAGRAM TECHNIQUE

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Rules for construction of the matrix elements in nonlocal field theory are formulated. The rules differ from the usual ones in that a form factor is included in the vertex part of the diagram, an obligatory condition being that in evaluating the integrals by the residue method the form factor singularities should not be taken into account. Analytical properties of the matrix elements are investigated and the appearance of specific singularities whose position does not depend on the elementary length is noted. It is pointed out that the Green functions constructed from Heisenberg and 'in'-operators of the field are not identical. This leads to appearance of complex singularities in the self energy part. The validity of the Lehmann-Symanzik-Zimmermann reduction formula for scattering matrix elements in nonlocal field theory is elucidated.

1. INTRODUCTION

THIS paper, a continuation of earlier work by the author[1,2], is devoted to an analysis of the expressions for the matrix elements of the S matrix in the nonlocal field theory (NFT). These expressions differ from the usual local expressions in two respects. First, each vertex of the Feynman diagram is additionally set in correspondence in the NFT with a form factor—a function that "smears" out the interaction. In addition, the singular functions S_R, D_R and $S_\pm$, $D_\pm$, which in the local theory appear only as combinations

$$S_F = S_R - S_+, \qquad D_F = D_R - D_+,$$

are encountered in the NFT in a great variety of groupings. This circumstance, being a direct result of the violation of causality, greatly complicates the expressions for the matrix elements when they are directly calculated from the S matrix.

The Bloch diagram technique[3], which is based on an analysis of these expressions, therefore turns out to be quite cumbersome and of little practical use. This technique is characterized by the appearance of a whole assembly of diagrams, corresponding to the diagram of the local theory, and differing from one another by the substitutions $S_R \rightleftarrows S_\pm$, etc.

The analysis given in this paper (Secs. 2 and 3) shows, however, that the expressions under consideration can always be simplified in such a way as to contain only causal functions. The previous diagram technique then remains in force. The only difference from the corresponding local expressions is due to the appearance of form factors, the singularities of which must not be taken into account in the evaluation of the integrals by the residue method. This condition is a direct consequence of the unitarity of the S matrix, and is equivalent to discarding the additional degrees of freedom of the field, connected with the singularities of the form factor (Section 4). This condition constitutes therefore the necessary element in the procedure for "smearing" the interaction, which preserves the unitarity of the scattering matrix.

An investigation of the analytic properties of the matrix elements of the NFT (Sec. 5) enables us to disclose circumstances that are important from the point of view of causality. It turns out that along with the "far" complex singularities, which go to infinity with decreasing elementary length, there appear also stationary "near" singularities, which are directly connected with large virtual momenta.

Thus, even if the process under consideration is characterized by large space-time scales (accordingly, by small momenta), its matrix element differs from the corresponding local expression, in spite of the fact that the theory developed goes over formally into the local theory upon removal of the form factor. Other manifestations of this feature of NFT have been noticed earlier[4].

In Section 6 we discuss a paradox connected with the appearance of the complex singularities in the proper self energy part and with simulta-

neous satisfaction of the spectral relation for the Green's functions. In the NFT the Green's function constructed from the Heisenberg field operators and satisfying the spectral formula does not at all coincide with the Green's function constructed from the in-operators and directly connected with the scattering matrix. Only their imaginary parts, and their values near the mass shell, coincide.

The question of the applicability of the Lehmann-Symanzik-Zimmerman reduction formula in the NFT has been discussed in the literature several times, and various ideas were advanced. In Sec. 7 we justify an affirmative answer to this question. For simplicity, we consider unrenormalized theory in Secs. 2—5.

FIG. 1

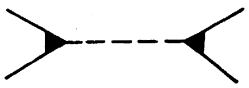
FIG. 2

FIG. 3

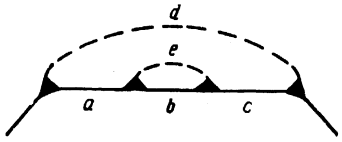

FIG. 4

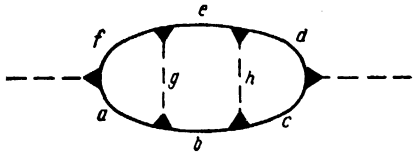

FIG. 5

2. MATRIX ELEMENTS IN THE COORDINATE REPRESENTATION

We start from the S-matrix representation introduced in [1,2], in the form of an exponential ordered with respect to the charge g:

$$S = \tilde{T}_g \exp\left[i \int_0^g \frac{dg}{g} \mathbf{L}(g)\right], \tag{2}$$

$$\mathbf{L}(g) = -\frac{g}{2}\int d(1)\{\bar{\boldsymbol{\psi}}(1')\gamma_5\boldsymbol{\varphi}(1''')\boldsymbol{\psi}(1'') - \boldsymbol{\psi}(1'')\tilde{\gamma}_5\boldsymbol{\varphi}(1''')\bar{\boldsymbol{\psi}}(1') \tag{3}$$

The Heisenberg field operators, which are designated in boldface, are defined by the equation

$$\boldsymbol{\varphi}(1) = \varphi(1) + \int d2D_R(1-2)\,\mathbf{j}(2) \tag{4}$$

etc. Here

$$d(1) \equiv d1'd1''d1'''F(1', 1'', 1'''), \quad \mathbf{j} = -\delta\mathbf{L}/\delta\varphi, \quad \varphi \equiv \varphi_{in}.$$

In accordance with the substitution $dl \to d(1)$, the Feynman diagram in the NFT should contain a complicated geometrical image of the vertex (Fig. 1), and should coincide in all other respects with the diagram of the local theory.

Let us consider an arbitrary matrix element $M_{n,i}$, corresponding to the n-th order of perturbation theory and described by a diagram with i internal lines. The quantity $\nu = i - n + 1$ gives the number of "degrees of freedom" of the connected diagram. The general expression for M is

$$M_{n,i} = \int d(1)\ldots d(n)\,K_{n,i}(1', \ldots n''')\,\sigma_{n,i}(1', \ldots, n'''). \tag{5}$$

The function K, which coincides with its local expression, includes the wave functions of the free ends of the diagram, the matrices γ_5 and the numerical coefficients. The function σ is a combination of the functions S_R and D_R, which appear when the field operators (4) are expanded in series, and of $S_\pm$ and $D_\pm$, which appear when (2) is reduced to normal form. Using (1), we can also express σ in terms of causal and retarded functions only.

We introduce further the concept of Feynman-diagram cycles, i.e., closed sequences of the internal lines of the diagram. The total number of cycles $N = 2^\nu - 1$ includes both the simple and compound cycles. Figures 2—5 show examples of diagrams with N = 0, 1, 3, and 7. Thus, Fig. 4 corresponds to cycles (a, b, c, d), (b, e), (a, e, c, d); among the cycles of Fig. 5 there is the constituent cycle (a, g, f), (c, d, h).

It is essential that the product of retarded (or advanced) functions corresponding to circuits over the cycle vanishes in the local limit. Thus, the circuit of the cycle of Fig. 3 corresponds to the product $S_R(2'-1'')D_R(1'-2'')$, which contains the factor $\theta(2'-1'')\theta(1'-2'')$. In the local limit, i.e., upon making the substitution $F(1', 1'', 1''') \to \delta(1'-1''')\delta(1'''-1'')$, this factor goes over into $\theta(1-2)\theta(2-1) \equiv 0$. The cyclic product of the retarded functions is the only combination of singular functions which vanishes in the local limit[1)].

Returning to (5), we impose the obvious requirement that σ go over in the local limit to the ordinary expression for σ_{loc}, which is the product of the causal functions S_F and D_F. The most general expression compatible with this requirement takes the form

$$\sigma = \sigma_{loc} - \sum_k \alpha_k \sigma_k, \tag{6}$$

[1)]The product of advanced functions corresponds to the opposite direction of circuiting around the cycle.

where σ_k differs from σ_{loc} by replacement of part of the causal functions with retarded functions, wherein each of the σ_k contains at least one closed sequence of retarded functions.

The numerical coefficients α_k satisfy the condition

$$\sum_k \alpha_k = 1, \tag{7}$$

which can be proved in the following manner. If we express $\sigma_{n,i}$ in terms of the functions S_R and D_R and $S_\pm$, $D_\pm$, then the result is at most $n-1$ retarded functions. In fact, the field operators depend only on the combinations gS_R, gD_R, whereas the Lagrangian, and consequently also the S matrix, contains one additional power of g [see (2)—(4)]. On the other hand, in relation (6) the term containing the maximum number of retarded functions (which is equal to i) will have, in accordance with (1), a coefficient $1-\sum_k \alpha_k$. Taking account of the fact that $i > n-1$ ($\nu \geq 1$) always, except for the trivial case $N = 0$, we arrive at (7).

Let us consider some of the simpler examples. In the diagrams with $N = 0$, obviously, $\sigma = \sigma_{loc}$. Therefore the scattering matrix element differs in the lowest order of perturbation theory (Fig. 2) from its local expression only in the presence of the form factor. Direct calculation yields for the self-energy of the nucleon (Fig. 3), in accordance with (6) and (7),

$$\sigma = S_F(a)D_F(b) - \tfrac{1}{2}[S_R(a)D_R(b) + S_A(a)D_A(b)]; \quad a = 2'-1'', \quad b = 1'''-2'''. \tag{8}$$

Turning to more complicated diagrams, we note that the matrix element of the non-connected diagram is equal, as in the local theory, to the product of the matrix elements of the individual blocks (with some symmetry coefficient). The proof is based on the absence of cycles which go outside the limits of each block, so that the quantity σ becomes factorized. Therefore in the NFT we can always separate the vacuum transitions, which accompany a real process, and show that these transitions are described by an inessential phase factor.

For the same reason, the matrix element of a weakly-connected diagram, which goes into a non-connected diagram by a break in one line, retains its usual form ($\sigma_1 S_F \sigma_2$ or $\sigma_1 D_F \sigma_2$). This makes it possible to reduce the non-compact self-energy parts of the S matrix to compact ones, and formulate equations analogous to the usual Dyson equations. The foregoing allows us to confine ourselves to an examination of only strongly connected diagrams.

We emphasize that in NFT the well known Dyson theorem concerning the self-energy and vertex insertions, according to which the total contribution of the irreducible diagram is obtained by replacing the "free" propagation lines and the vertices by universal "exact" lines and vertices, no longer applies. The failure of this theorem to hold in the NFT is connected with the fact that the insertion no longer appears as a whole: expression (6) contains terms in which the cycle includes only part of the insertion. Thus, on the diagram of Fig. 4, the cycle (a, e, c, d) includes only part of the insertion (b, e).

3. MATRIX ELEMENTS IN THE MOMENTUM REPRESENTATION

The form of the NFT theory diagrams in the momentum representation is exactly the same as in the local theory. In particular, the energy-momentum conservation law must be satisfied at each node of the diagram. This necessary condition calls for translational invariance of the form factor $F(1', 1'', 1''') = F(1'-1''', 1'''-1'')$, or in the momentum representation $\tilde{F} = \tilde{F}(p, k, -p-k)$. Here p and p + k are respectively the nucleon momenta going in and out of the vertex, and k is the meson momentum.

Let us list the remaining conditions which the form factor must satisfy. Owing to the absence of preferred vectors we must put $F = F(p^2, k^2, (p+k)^2)$. Since the Lagrangian is hermitian, it follows that

$$\tilde{F}(p^2, k^2, (p+k)^2) = \tilde{F}^*((p+k)^2, k^2, p^2). \tag{9}$$

The principle of correspondence with the local theory stipulates that $\tilde{F} \to 1$ as $\Lambda \to \infty$, where Λ is the "cutoff" momentum. It is convenient to normalize the form factor in the following fashion: $\tilde{F}(M^2, \mu^2, M^2) = 1$. Finally, we shall assume the form factor to be an analytic function of its arguments, which falls off sufficiently rapidly for large values of the arguments, and which has singularities that are far away from the particle mass shell $p^2 = M^2$, etc.

We proceed to an investigation of the expression for the matrix element. Using (5) and (6) we have

$$M_{n,i} = \tilde{K}_{n,i} \int d^4p_1, \ldots, d^4p_\nu \underbrace{\tilde{F} \ldots \tilde{F}}_{n} \tilde{\sigma}_{n,i};$$

$$\tilde{\sigma} = \tilde{\sigma}_{loc} - \sum_k \alpha_k \tilde{\sigma}_k, \tag{10}$$

where the ~ denotes the Fourier transform of the corresponding quantity, and $\tilde{K}$ includes the δ function that expresses the law of energy-momentum conservation.

In the local limit (as $\tilde{F} \to 1$), the transition $\tilde{\sigma} \to \tilde{\sigma}_{loc}$ is realized in the momentum representation in the following fashion. The cyclic products of the retarded functions contained in $\tilde{\sigma}_k$ are characterized by an arrangement of the poles in the complex planes $p_{01}, \ldots, p_{0\nu}$ such that the path can always be completed over a large circle and a zero result obtained. In the simplest case when N = 1, the poles of $\tilde{\sigma}_k$ are located on one side of the real axis; closure on the other side leads to the vanishing of the corresponding term of (10). These questions are discussed in detail in the appendix.

In the NFT, the quantities $\tilde{\sigma}_k$ do not drop out, since it is necessary to take into account the singularities of the form factor $\tilde{F}$. It turns out, however, that these singularities make no contribution at all to the expression (10) as a whole. In fact, the functions $\tilde{\sigma}_{loc}$ and $\tilde{\sigma}_k$ differ from each other only in the replacement of some of the causal functions by retarded functions. The corresponding difference occurs only on the mass shell; in particular, according to (1)

$$D_F(k) - D_R(k) \backsim \delta(k^2 - \mu^2).$$

On the other hand, the singularities of the form factors are assumed to be far from the mass shell. Therefore in the points where these singularities are located we have $\tilde{\sigma}_k = \tilde{\sigma}_{loc}$, and by virtue of condition (7)

$$\tilde{\sigma} = \left(1 - \sum_k a_k\right)\tilde{\sigma}_{loc} = 0.$$

This circumstance enables us to reduce (10) to the usual Feynman form. Without changing this expression, we impose the condition that calls for disregarding the singularities of the form factor in the calculation of the integrals by the residue method. This gives rise to exactly the same situation as in the local limit considered above. By suitably choosing methods of closing the integrals with respect to $p_{01}, \ldots, p_{0\nu}$ in each term of $\tilde{\sigma}_k$, we cause these terms to drop out completely [2]. We ultimately get

$$M = \tilde{K}\int^{*} d^4p_1 \ldots d^4p_\nu \underbrace{\tilde{F} \ldots \tilde{F}}_{n}\tilde{\sigma}_{loc}, \tag{11}$$

where the asterisk denotes that the singularities of the form factors are not taken into account.

In addition, we must choose a definite method of closing the integration contours [3]. Indeed, as shown in the appendix,

$$\int^{*} d^4p_1 \ldots d^4p_\nu = \frac{1}{2^\nu - 2}\left\{\int_{(+)}^{*} d^4p_1 \ldots \int_{(+)}^{*} d^4p_{\nu-1}\int_{(-)}^{*} d^4p_\nu + \ldots + \int_{(-)}^{*} d^4p_1 \ldots \int_{(-)}^{*} d^4p_{\nu-1}\int_{(+)}^{*} d^4p_\nu\right\}, \tag{12}$$

where the indices (+) and (−) at the integrals denote closing in the upper and lower half-planes, respectively. Each of the $2^\nu - 2$ terms of (12) corresponds to one of the possible combinations of the method of closing with respect to the variables $p_{01}, \ldots, p_{0\nu}$, excluding the two combinations (+), (+) . . . (+) and (−), (−) . . . (−). Thus, the matrix element of the diagram of Fig. 6 contains the integrals

$$\frac{1}{2}\left\{\int_{(+)}^{*} d^4p_1 \int_{(-)}^{*} d^4p_2 + \int_{(-)}^{*} d^4p_1 \int_{(+)}^{*} d^4p_2\right\}.$$

A special case is $\nu = 1$ (see Fig. 7), for which, according to (8),

$$\int^{*} d^4p = \frac{1}{2}\left\{\int_{(+)}^{*} d^4p + \int_{(-)}^{*} d^4p\right\}.$$

In the formulation of these rules we assumed that the indices of all the lines of the diagram are the arithmetic sums of the integration variables: p_1, $p_1 + p_2$, $p_1 + p_2 + p_3$, . . .

4. CONNECTION WITH THE UNITARITY CONDITION

The fact that the singularities of the form factor make no contribution to the matrix element is a direct consequence of the unitarity condition. Let us consider by way of an example the self-energy of the nucleon (Fig. 7)

$$M \backsim \frac{1}{2i}\left\{\int_{(+)}^{*} d^4p_1 + \int_{(-)}^{*} d^4p_1\right\} \times |\tilde{F}(p^2, p_1^2, (p+p_1)^2)|^2 S_F(p + p_1) D_F(p_1). \tag{13}$$

For simplicity we have left out here the spinors, the γ_5 matrices, and the δ function that expresses the conservation law.

[2] Thus, the contributions of the singularities that are specific for the NFT, connected with the form factor and the functions $\tilde{\sigma}_k$, cancel each other completely.

[3] In usual contour integrals, where all the singularities of the integrand are taken into account, the result does not depend on the method of closure. In our case, however, there is such a dependence.

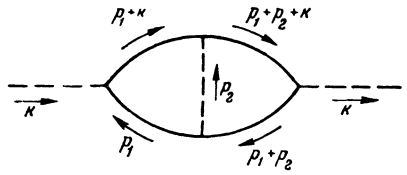

FIG. 6

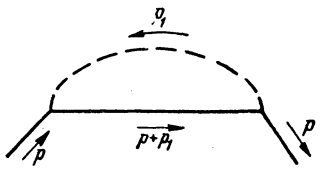

FIG. 7

The unitarity condition in second order perturbation theory yields

$$\langle p | S_2 + S_2^+ + S_1^+ S_1 | p \rangle = 0, \qquad (14)$$

where $S_{1,2}$ are the parts of the S matrix of first and second order in g, $|p\rangle$ —amplitude of the single-nucleon state. Recognizing that S_2 is expressed in terms of M and S_1 in terms of the interaction Lagrangian, and using Wick's theorem, we find that the left half of (14) is proportional to

$$\left\{ \int_{(+)}^{*} d^4 p_1 S_A (p + p_1) D_A (p_1) + \int_{(-)}^{*} d^4 p_1 S_R (p + p_1) D_R (p_1) \right\} \times |\tilde{F}(p^2, p_1^2, (p + p_1)^2)|^2.$$

This quantity reduces to residues at the form-factor poles and discontinuities across cuts, and vanishes only owing to the condition that these singularities not be taken into account. In other words, inasmuch as the term $S_1^+ S_1$ in (14) contains form factors referred to the mass shell $p_1^2 = \mu^2$ and $(p + p_1)^2 = M^2$, the hermitian part of the matrix element (13) should possess the same property. It is obvious that this property is due to the omission of the form-factor singularities.

We note that in the known Feynman regularization procedure, which corresponds to the substitution

$$|\tilde{F}|^2 \to -\mathfrak{M}^2/(p_1^2 - \mathfrak{M}^2 + i\delta), \qquad \mathfrak{M} \to \infty,$$

the "cutoff" of the integral is exclusively due to the form-factor singularities, i.e., to violation of the unitarity condition. It is easy to verify that the converging integral is the difference between two infinite residues at the points μ^2 and $\mathfrak{M}^2$.

It is well known that this violation of unitarity manifests itself in the appearance of an indefinite metric. It can be stated that in general the discussed requirement that the singularities of the form factor not be taken into account is tantamount to discarding additional degrees of freedom, connected with these singularities and leading to an indefinite metric. If these singularities were to be taken into account along with the ordinary singularities at the points $p^2 = \mu^2$, etc., then the main and additional degrees of freedom of the field would obviously manifest themselves in identical fashion. The foregoing is in full agreement with the previously made statement[2] that the selection of the solutions of the Heisenberg field equations, on which the developed theory is based, does not lead to the appearance of additional degrees of freedom in the field (see also[5]). A reflection of this selection is the disappearance of the contribution of the form-factor singularities.

Frequently the form factor is introduced into the theory by formal substitution in the matrix element, without stipulating that its singularities not be taken into account. Such a procedure, as is clear from the foregoing, is unsatisfactory and contradicts the unitarity condition.

This condition is also violated if the form factor is introduced directly in the "euclidized" expression for the matrix element. We rewrite (11) in the form

$$M = M' - \delta M,$$

where M′ is expression (11) in which the integral has its usual meaning, δM is the contribution of the form-factor singularities. Going over in M′ to the imaginary axes $p_{01}, \ldots, p_{0\nu}$, we have

$$M' = M_{eu} + \delta M',$$

where M_{eu} is the "euclidized" expression and δM′ the contribution of the singularities of the form factor in the first and third quadrants. The quantities δM and δM′, not only do not cancel each other out, but, to the contrary, reinforce each other. Therefore M_{eu} does not coincide with the unitary expression (11) and contradicts the unitarity conditions.

5. ANALYTIC PROPERTIES OF THE MATRIX ELEMENTS

The rules formulated above can be used to advantage to clarify the analytic properties of the matrix elements of the NFT in the complex plane of their arguments. The principal interest attaches to the singularities which appear in the NFT in addition to the ordinary local singularities, and which are due to the singularities of the form factor. These additional singularities (a.s.) can be divided into "far" and "near." The far a.s., like the form-factor singularities, move away from the

particle mass shell as $\Lambda \to \infty$, while the near singularities remain at a finite distance, determined by the masses of the particles.

The most important property of the matrix element in the NFT is that along with the far a.s., which are directly transferred from the form factor, there appear unavoidably (in the case when $\nu \neq 0$) also near a.s. The latter are the "integrated" singularities of the form factor and are the result of high virtual momenta.

Let us consider by way of very simple examples the scattering diagrams (Fig. 2) and self-energy diagrams (Fig. 7). In the first case ($\nu = 0$), denoting the momentum transfer by k, we have

$$M(k) \backsim |\tilde{F}(M^2, k^2, M^2)|^2 D_F(k). \quad (15)$$

Only far a.s. appear here.

The situation is more complicated in Fig. 7. We represent (13) in the form

$$M(p) \backsim i \int^{*} d^4 p_1 \frac{|\tilde{F}(p^2, p_1^2, (p+p_1)^2)|^2 (\hat{p}(pp_1)/p^2 + M)}{(p_1^2 - \mu^2 + i\delta)((p+p_1)^2 - M^2 + i\delta)} = A(p^2) + \hat{p} B(p^2). \quad (16)$$

Taking residues only at the zeros of the denominators and taking the closure conditions into account, we arrive at a system consisting of four terms which contain the form factors

$$\tilde{F}[p^2, \mu^2, p^2 + \mu^2 - 2(\pm \omega_{p_1} p_0 + \mathbf{p}_1 \mathbf{p})], \quad (17)$$
$$\tilde{F}[p^2, p^2 + M^2 + 2(\pm E_{\mathbf{p}_1+\mathbf{p}} p_0 + (\mathbf{p}_1 + \mathbf{p})\mathbf{p}), M^2].$$

The first arguments yield the far a.s. Let us investigate from this point of view the arguments which depend on the integration variable p_1.

As can be readily shown, the functions $A(p^2)$ and $B(p^2)$ in (16) do not contain any divergences and are analytic functions of p^2. It is therefore necessary to consider only the case $\mathbf{p} = 0$, $p^2 = p_0^2$. Assuming that the singularities of $\tilde{F}$ in the second and third arguments lie respectively at the points ξ and ξ', we find that the singularities of (17) lie at the points

$$p^2 = 2p_1^2 + M^2 + \xi \pm 2\sqrt{(p_1^2 + M^2)(p_1^2 + \xi)}, \quad (18)$$
$$p^2 = 2p_1^2 + \mu^2 + \xi' \pm 2\sqrt{(p_1^2 + \mu^2)(p_1^2 + \xi')}.$$

The singularities of (16) correspond to the end points of the integration interval, $p_1 = 0, \infty$. It follows therefore that there are four far a.s. corresponding to $p_1 = 0$. The point $p_1 = \infty$ gives two infinitely remote singularities [the upper sign in (18)] and one close a.s. $p^2 = O(p_1^{-4})$, corresponding to the choice of the lower sign. Thus, the high virtual momenta lead to the appearance of a near a.s.

$$p^2 = 0. \quad (19)$$

The origin of this singularity can be readily interpreted by taking account of the fact that, in accordance with (17), the form factor becomes ineffective at the point $p = 0$, where it changes into a number. At this point, consequently, the local (infinite) value of $M(p)$ is restored.[4)]

The singularity (19) is obviously a branch point. It is of interest to discuss the method of going around the pole whose integration yields this singularity. It is easy to show that the contour around the pole should be taken not in the Feynman sense but in the principal-value sense. To this end it is sufficient to verify that the imaginary part of the functions A and B in (16) vanishes at $p^2 = 0$. Taking (8) into account, we consider the integral in (16) in the usual sense, subtracting from the product of the causal functions the half-sum of the products of the advanced and retarded functions. It is then easy to see that the imaginary part of A and B will contain the product $\delta(p_1^2 - \mu^2)\delta((p_1 + p)^2 - M^2)$, which transforms the form factors into $|\tilde{F}(p^2, \mu^2, M^2)|^2$. Therefore the imaginary parts of A and B will differ from their local expressions (which vanish at the point $p^2 = 0$) only by the factor $|\tilde{F}|^2$. Direct calculation shows that at the point $p^2 = 0$

$$M(p) \backsim \ln |p^2|. \quad (20)$$

If the form factor is chosen in the form

$$\tilde{F}(p_1, p_2, p_3) = \frac{\Lambda^4}{(p_1^2 - M^2)^2 + \Lambda^4} \frac{\Lambda^4}{(p_2^2 - \mu^2)^2 + \Lambda^4} \frac{\Lambda^4}{(p_3^2 - M^2)^2 + \Lambda^4},$$

then we obtain the following expression for $M(p)$ in the region $p^2 \gg M^2, \mu^2$

$$M(p) \backsim (\eta^2 + 1)^{-2} [\ln(1 + \eta^{-4}) + \eta^{-1} \tan^{-1} \eta], \quad (21)$$

where $\eta = p^2/\Lambda^2$. The far a.s. are located at the points $p^2 = \pm i\Lambda^2$.

6. GREEN'S FUNCTIONS IN THE NFT

The appearance of additional singularities (including complex ones) of the self-energy part contradicts at first glance the existence of the Kallen-Lehmann representation for the Green's

[4)] In this connection, the inequality $M \neq M_{eu}$ is important (see Section 4), since M_{eu} is finite everywhere.

function (see [2]). Considering for simplicity the Green's function of the meson field, we write this representation in the form [5]

$$D'_F(1-2) = -i\langle\Psi_0|T(\tilde{\varphi}(1)\,\tilde{\varphi}(2))|\Psi_0\rangle,$$

$$D'_F(k) = D_F(k) + \int \frac{d\varkappa^2 \rho(\varkappa^2)}{k^2 - \varkappa^2 + i\delta}. \qquad (22)$$

We have here only a cut along the real axis.

Actually, the foregoing contradiction disappears if we take account of the fact that in the NFT the Green's function (22) is not directly related to the S matrix. In the local theory there occurs a coincidence between (22) and the Green's function.

$$D''_F(1-2) = -i\langle\Psi_0|S^+T(\varphi(1)\varphi(2)S)|\Psi_0\rangle. \qquad (23)$$

The latter quantity, in which the ordering must be taken in the sense of the Wick theorem, is already expressed directly in terms of the self-energy part of the S matrix:

$$D''_F(1-2) = D_F(1-2) + \int d3\,d4\,D_F(1-3)\,\Sigma''(3-4)\,D_F(4-2),$$

$$\Sigma''(1-2) = \hat{K}_1\hat{K}_2\,(D''_F(1-2) - D_F(1-2)) = i\langle\Psi_0|S^+\delta^2 S/\delta\varphi(1)\,\delta\varphi(2)|\Psi_0\rangle;$$

$$\hat{K} = \Box - \mu^2.$$

It is important that in the non-causal NFT the introduced Green's functions do not coincide:

$$D''_F \neq D'_F. \qquad (24)$$

To prove this we introduce a quantity $\Sigma' = \hat{K}_1\hat{K}_2(D'_F - D_F)$, which, accurate to quasi local terms that contain $\delta(t_1 - t_2)$ etc., is equal to $-i\langle\Psi_0|T(\tilde{j}(1)\tilde{j}(2))|\Psi_0\rangle$. Using the identity

$$S^+\delta^2 S/\delta\varphi(1)\,\delta\varphi(2) = -\tilde{j}(1)\,\tilde{j}(2) - i\delta\tilde{j}(1)/\delta\varphi(2), \qquad (25)$$

we have

$$\Sigma''(1-2) - \Sigma'(1-2) = \langle\Psi_0|\theta(1-2)\,\delta\tilde{j}(1)/\delta\varphi(2) + \theta(2-1)\,\delta\tilde{j}(2)/\delta\varphi(1)\,\Psi_0\rangle.$$

Thus, for D'_F and D''_F to coincide, it is necessary in any case to satisfy the causality conditions of Bogolyubov [8] [6].

In the NFT there exist, by the same token, two different Green's functions. One has all the spectral properties but has no direct bearing on the S matrix. The second, on the contrary, is contained in the expansion of the S matrix in normal products, but on the other hand can have arbitrary singularities, as is observed also by direct calculation.

The author has noted at one time [4] that the Green's function in the NFT depends strongly on the form of the form factor and in particular may contradict relation (22). Inasmuch as in [4] we referred to the function D''_F, this result can be readily explained by means of inequality (24). For a similar reason, the occurrence of the unphysical pole of the Green's function in the theory that makes use of the form factor does not constitute as undisputed evidence of the difficulties of the local theory as is frequently assumed. This pole is possessed by the function D''_F, which is entitled to have an "anti-Lehmann" behavior.

Using (25) and the condition of stability $\langle\Psi_0|\tilde{j}|\Psi_k\rangle = 0$, which is the consequence of the renormalization, we can show that the principal parts of the functions D'_F and D''_F coincide near the mass shell, i.e., as $k^2 \to \mu^2$ we have

$$(k^2 - \mu^2)(D'_F - D''_F) \to 0. \qquad (26)$$

Thus, the renormalization of these functions occurs simultaneously.

The imaginary parts of the functions D'_F and D''_F coincide for all values of k:

$$\mathrm{Im}\,D'_F(k) = \mathrm{Im}\,D''_F(k) = -\pi(\delta(k^2 - \mu^2) + \rho(k^2)). \qquad (27)$$

From the equality $D'_F = -(1/2)\,iD'_1 + \epsilon D'$ and (25) it follows that [7]

$$\mathrm{Im}\,\Sigma'(k) = \mathrm{Im}\,\Sigma''(k) = \langle\Psi_0|\{\tilde{j}(k), \tilde{j}(-k)\}|\Psi_0\rangle,$$

and from Dyson's theorem and (26) we get

$$\mathrm{Im}(D'_F - D''_F) = (k^2 - \mu^2)^{-2}\,\mathrm{Im}(\Sigma' - \Sigma'') = 0.$$

Thus, to calculate D'_F we can employ the following procedure. We use the diagram technique

[5] This definition is invariant inasmuch, as indicated in [2], the averages $\langle\Psi_0|[\tilde{\varphi}, \tilde{\varphi}]|\Psi_0\rangle$, $\langle\Psi_0|\{\tilde{\psi}, \bar{\tilde{\psi}}\}|\Psi_0\rangle$ vanish outside the light cone. A simple proof of this statement is contained also in Wigner's discussions of the paper by Kallen (see [9]). In practice, the property of "weak locality" appears here while only the character of the symmetry of the theory is important, and not its locality [7].

[6] The elementary derivation of the equality $D'_F = D''_F$, based on the condition $\tilde{\varphi}(1) = S^+(t_1)\varphi(1)S(t_1)$ is not applicable in NFT, owing to the obvious violation of this condition (see [1,2]).

[7] We note that in the lowest order of perturbation theory $\tilde{j}(k) = \tilde{F}(M^2, k^2, M^2)\tilde{j}(k)_{loc}$ and the formula presented gives $\rho(\kappa^2) = |\tilde{F}(M^2, \kappa^2, M^2)|^2\rho_{loc}$. Therefore by choosing the form factor it is always possible to obtain convergence of spectral integrals of the type $\int d\kappa^2\rho(\kappa^2)$, etc.

and determine $\mathrm{Im}D''_F$ and simultaneously $\mathrm{Im}D'_F$; we then reconstitute D'_F with the aid of the spectral formula (22). By the same token our method incorporates the known Redmond procedure, which in this case, however, is free of all the objections connected with its ambiguity[9].

7. REDUCTION FORMULA FOR THE MATRIX ELEMENTS

The fact that the Green's functions constructed from the Heisenberg and in field operators coincide on the mass shell [see (26)] is not an accident, but is a manifestation of the correctness of the general reduction formula[10] in the NFT which expresses the matrix element in the physical region in terms of the mean values of the Heisenberg field operators. This formula takes the form

$$M = \int d\xi \ldots d\eta\, f^*_\beta(\xi) \ldots f_\alpha(\eta)(-i\hat{K}_\xi) \ldots (-i\hat{K}_\eta) \times \langle \Psi_0 | T(\tilde{\varphi}(\xi) \ldots \tilde{\varphi}(\eta)) | \Psi_0 \rangle, \quad (28)$$

where f_α and f_β are the states of the incoming and scattered particles, respectively, and $\hat{K}f = 0$.

To prove this it is sufficient to recognize first, that from the relation

$$\tilde{\varphi}(1) = \varphi(1) + \int d2D_R(1-2)\tilde{j}(2)$$

and from its equivalent

$$\tilde{\varphi}(1) = S^+T(\varphi(1)S)$$

(ordered in the sense of Wick's theorem) it follows that the following asymptotic condition is satisfied[11] (see also [12])

$$\lim_{t\to-\infty} \langle \Psi' | \tilde{\varphi}^f(t) - \varphi^f(t) | \Psi \rangle = 0, \quad (29)$$

where for any operator a(x) we have

$$a^f(t) = i \int\limits_{x_0=t} d^3x\, a(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f(x). \quad (30)$$

Satisfaction of the asymptotic condition implies the correctness of the reduction formula (28) without any assumptions on the causality[10].

Strictly speaking, the last deduction calls for some stipulation with respect to the ambiguity of the transition from (29) to (28); this circumstance is particularly important in the axiomatic theory [13]. In the physical region, importance is attached to ambiguity of the expression (28), an ambiguity connected with the possibility of replacing the commutators in the T product by anti-commutators and vice versa. It is clear, however, that in the dynamic NFT considered here, which is subject to the obligatory requirement that a correct local limit exist, this ambiguity does not play any role, in view of the continuity considerations. In other words, the correct definition of the T-product in (28) is the one which coincides with the analogous local definition.

A more important question is the apparent relativistic invariance of (28). The point is that the concept of the T product presupposes that the corresponding operators commute (anti-commute) outside the light cone. In the NFT this is correct only for the average of the product of two operators (see footnote [5]), but not for a larger number of factors. This is precisely why Lehmann, Zimmermann, and Symanzik[10] stipulated that the causality condition be fulfilled.

However, the derivation of the reduction formula itself, according to which it coincides with the certainly invariant expression (11), suggests that although the mean value in (28) is not invariant, nevertheless the expression as a whole is no longer subject to this shortcoming. A direct proof of this statement is contained in the paper by Prokhorov[14] in which it is indicated that if the commutators of the field operators vanish sufficiently rapidly outside the cone, then the space-like intervals make practically no contributions to (28). More accurately speaking, it is shown that without changing the result it is possible to replace integration over the entire hyperplane $x_0 = t$ in (28) and (30) by integration over the part of this plane inside a cone close to the light cone, but having a spread larger than the light cone.

The correctness of the reduction formula in the NFT indicates positively that the known system of equations of Lehmann-Symanzik, and Zimmermann[10], which is based on this formula, is patently nonunique.

In conclusion we note that in the NFT it is not possible to go over from (28) to a T product of currents, for the terms usually discarded thereby play an important role from the point of view of the analytic properties of the matrix element.

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APPENDIX

Let us consider the question of the method of closing the integration contours. The simple cycles of the diagram can be broken up into two classes. Each of the cycles of the first class (the number of which is ν) contains a common integration variable in the indices of all its lines, and has a single circuiting direction. For cycles of the second class, both these properties are violated. Of the three cycles of the diagram of Fig. 6 ($\nu = 2$), two belong to the first class, namely (p_1, $p_1 + k$, $p_1 + p_2 + k$, $p_1 + p_2$) and (p_2, $p_2 + p_1 + k$, $p_2 + p_1$), while one belongs to the second class, (p_1, $p_1 + k$, p_2). We introduce the symbols $\{+p_k\}$ (circuit in the direction of the arrows on the diagram) and $\{-p_k\}$ (opposite direction) for the cycles of the first class and $\{s_1p_1, s_2p_2, \ldots\}$, where $s = \pm 1$, for the cycles of the second class [8] (the circuit of the cycle has the same direction as the arrows of the cycles $\{s_1p_1\}$, $\{s_2p_2\}$, etc.) In particular, the cycles of the diagrams of Fig. 6 are designated $\{+p_1\}$, $\{+p_2\}$, $\{+p_1, -p_2\}$.

Let us establish a method for closing the contour in the integral (10) such as to annihilate the terms $\tilde{\sigma}_k$ when the singularities of the form factors are not taken into account. To each term $\tilde{\sigma}$ we can set in correspondence the symbols of the cycles within which the substitution $S_F, D_F \to S_R, D_R$ is made. For the cycle $\{\pm p_k\}$, the singularities of $\tilde{\sigma}_k$ lie respectively in the upper (lower) half plane of the variable p_{0k}. Therefore the required method of closing is given by the relation

$$\{\pm p_k\} \sim \int\limits_{(\mp)}^{*} d^4p_k;$$

For the cycle $\{s_1p_1, s_2p_2, \ldots\}$ it is necessary to have a consistent method of closure in the variable $p_{01}, p_{02}, \ldots$:

$$\{s_1p_1, s_2p_2, \ldots\} \sim \int\limits_{(-s_1)}^{*} d^4p_1 \int\limits_{(-s_2)}^{*} d^4p_2 \ldots$$

If the foregoing substitution of functions involves several cycles (and also in the case of a constituent cycle), it is sufficient to choose a closure method which leads to the vanishing of one of the cycles subject to the substitution.

We break down, furthermore, $\sum \alpha_k \tilde{\sigma}$ into $m = 2^\nu - 2$ groups,

$$\sum_k \alpha_k \tilde{\sigma}_k \equiv \sum_{l=1}^{m} \xi_l,$$

in such a way that each group contains the $\tilde{\sigma}_k$ corresponding to the cycles $\{s_1p_1, \ldots, s_\nu p_\nu\}$, $\{s_2p_2, \ldots, s_\nu p_\nu\}, \ldots, \{s_1p_1\}, \ldots, \{s_\nu p_\nu\}$, and also all the constituent cycles and the aggregates of cycles which include at least one of the foregoing. The closure method

$$\int\limits_{(-s_1)}^{*} d^4p_1 \ldots \int\limits_{(-s_\nu)}^{*} d^4p_\nu$$

leads to the vanishing of all the terms of the given group. The number of such groups is indeed equal to $2^\nu - 2$, in accordance with the number of combinations $s_k = \pm 1$, with the exception of two combinations, referred to in footnote [8]. Representing $\tilde{\sigma}$ in (10) in the form

$$\tilde{\sigma} = \sum_{l=1}^{m} \left\{ \frac{1}{m} \tilde{\sigma}_{1oc} - \xi_l \right\}$$

and choosing in each term the proper closure method, we arrive at the rule formulated in the text.

[8] The cases $s_1 = s_2 = \ldots = \pm 1$ should be excluded in view of the absence of a single circuiting direction for a cycle of the second class.

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