POST-NEWTONIAN EQUATIONS OF MOTION AND THE HARMONICITY CONDITIONS IN THE THEORY OF GRAVITATION

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The relation between the harmonicity conditions and equations of motion in Einstein's theory of gravitation is investigated. Following Infeld, material bodies are regarded as field singularities characterized by Dirac δ-functions. It is assumed that the metric tensor can be represented as a power series in c^{-1}. It is also assumed that derivatives with respect to x^0 are of order c^{-1} relative to derivatives with respect to t. It is proved that, under these assumptions, the harmonicity conditions of zero, first, and second orders are necessary and sufficient for the derivation of the post-Newtonian equations of motion from the gravitational field equations.

1. INTRODUCTION

THE problem of deducing the equations of motion from the gravitational field equations was solved many years ago. It is well known that, with supplementary conditions, the gravitational field equations yield the Newtonian and post-Newtonian equations of motion of material points.[1-11]

In the works of Fock, Petrova, and Papapetrou [3,7,8,11], a harmonic system of coordinates is used. According to Fock, a harmonic system of coordinates is that privileged system whose properties most closely approach the properties of inertial systems of coordinates. In this connection, Fock assumes that Newtonian and post-Newtonian equations of motion can be obtained only in a harmonic coordinate system.

In the works of Infeld [9,10,12,13] it is asserted that the harmonic coordinate conditions are not necessary for the derivation of the Newtonian and post-Newtonian equations of motion. In particular, he proves [10,12] that these two sets of equations of motion can be derived in a coordinate system which is not harmonic. Therefore, according to Infeld, the harmonic system of coordinates loses its unique role in Einstein's theory of gravitation.

In the present work it is proved that harmonicity conditions of zeroth, first, and second order are necessary and sufficient for the derivation of the post-Newtonian equations of motion. In other words, the equations of motion have the Newtonian and post-Newtonian form when, and only when, the harmonicity conditions in zeroth, first, and second order approximation are satisfied. Hence, if the Newtonian and post-Newtonian equations of motion are obtained by some method, it follows that the method used contains the corresponding conditions of harmonicity in more or less hidden form. Our results point to the unique role of the harmonic system of coordinates in Einstein's theory of gravitation.

2. NOTATION AND GENERAL CONSIDERATIONS

As is well known, the gravitational field equations have the form

$$R^{\alpha\beta}-\tfrac{1}{2}Rg^{\alpha\beta}=-8\pi T^{\alpha\beta}, \tag{1}$$

where, as usual, $R^{\alpha\beta}$, R, and $T^{\alpha\beta}$ indicate, respectively, the second rank curvature tensor, the curvature invariant, and the mass tensor. The metric tensor $g_{\mu\nu}(x^\alpha)$, which characterizes the gravitational field, and the coordinates of world-points are related by the usual form

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu. \tag{2}$$

Greek indices take on the values 0, 1, 2, 3, and the summation convention is understood for repeated indices. The temporal coordinate $x^0 = ct$ is distinguished by the Hilbert conditions (cf. e.g., [14]):

$$g_{00}>0,\quad \begin{vmatrix} g_{00} & g_{01}\\ g_{10} & g_{11}\end{vmatrix}<0,\quad \begin{vmatrix} g_{00} & g_{01} & g_{02}\\ g_{10} & g_{11} & g_{12}\\ g_{20} & g_{21} & g_{22}\end{vmatrix}>0, g=\mathrm{Det}\,|g_{\alpha\beta}|<0. \tag{3}$$

With the Bianchi identities, Eq. (1) gives

$$\nabla_\alpha T^{\alpha\beta}=0, \tag{4}$$

where ∇_α is the covariant derivative corresponding to the metric tensor $g_{\mu\nu}$.

The problem we are considering here is of the astronomical type. This is to say that all masses considered are concentrated within a finite region of space separated by large distances from all other masses. With Fock, we consider the components of the mass tensor to be zero throughout space except for individual regions whose size is small compared to the distances between them. We also take it that these regions represent material bodies. Within each body the components of the mass tensor depend on the metric tensor and equations of state. One may imagine that the mass tensor depends approximately on certain over-all characteristics such as the total mass, the position and velocity of the mass center, etc.

On the other hand, one may also regard the bodies approximately as singularities of the field. Then, according to Infeld, for spherically symmetric bodies there corresponds a mass tensor in the form

$$\sqrt{-g}T^{\alpha\beta} = \sum_A m_A \delta(x^k - \xi^k_A)\, \xi^\alpha_{A|0}\xi^\beta_{A|0}, \qquad (5)$$

where $\xi^n_A(t)$ and $m_A(t)$ are the coordinates and mass of the A-th singularity. The vertical stroke denotes differentiation with respect to the coordinate indicated by the immediately following index; $\xi^0_{A|0} = 1$, $\xi^n_{A|0} = c^{-1}V^n_A$ and $\delta(x^k - \xi^k_A)$ is the Dirac δ-function as modified by Infeld and Plebanski.[15,16] Roman indices take on the values 1, 2, 3, and the summation convention is again understood for repeated indices. The treatment of bodies as singularities of the field appears less obvious but we adopt this method because we want our assumptions to be as close as possible to those of Infeld's method.

Writing out Eq. (4), we obtain

$$M^\beta_{\alpha|\beta} - \tfrac{1}{2} M^{\mu\nu} g_{\mu\nu|\alpha} = 0, \qquad (6)$$

where we have introduced the notation

$$\sqrt{-g}M^{\alpha\beta} = T^{\alpha\beta}, \qquad M^\beta_\alpha = g_{\alpha\sigma}M^{\beta\sigma}. \qquad (7)$$

Integration of (6) over small three-dimensional regions containing the singularities gives

$$\Omega_{A\alpha} \equiv d\overline{M}^0_{A\alpha}/dx^0 - \tfrac{1}{2}\overline{M^{\mu\nu}_A}\,\overline{g_{A\mu\nu|\alpha}} = 0, \qquad (8)$$

where for any quantity $\Phi(x^k, t)$ we have

$$\overline{\Phi_A(\xi^k_A, t)} = \int \Phi(x^k, t)\, \delta(x^k - \xi^k_A)\, d_3x.$$

Introducing the notation

$$\overline{M^{\alpha\beta}_A} = m_A \overline{N^{\alpha\beta}_A}, \qquad \overline{N^{\alpha\beta}_A} = \xi^\alpha_{A|0}\xi^\beta_{A|0}, \qquad (9)$$

and separating the spatial and temporal parts of Eq. (8), we find after several transformations

$$\overline{\varepsilon_{A\alpha}} \equiv -\tfrac{1}{2}\overline{N^{\mu\nu}_A}\,\overline{g_{A\mu\nu|\alpha}} + c^{-1}[\overline{N^0_{A\alpha|t}} + \overline{N^0_{A\alpha|n}}\,v^n_A + \overline{N^0_{A\alpha}}\,(\overline{N^0_{A0}})^{-1}$$
$$+ (\tfrac{1}{2}\overline{N^{\mu\nu}_A}\,\overline{g_{A\mu\nu|t}} - \overline{N^0_{A0|t}} - \overline{N^0_{A0|n}}\,v^n_A)] = 0, \qquad (10)$$

where $N_{|t} = \partial N/\partial t$. In the following we shall drop the label A and the bar where this does not lead to misunderstanding.

We assume that the metric tensor can be represented in the form of a power series:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + c^{-1}g^{(1)}_{\mu\nu} + c^{-2}g^{(2)}_{\mu\nu} + \ldots . \qquad (11)$$

From (7), (9), and (11) we obtain two formulas which we shall need further on:

$$N^0_\alpha = g^{(0)}_{\alpha 0} + c^{-1}(g^{(1)}_{\alpha 0} + g^{(0)}_{\alpha n}v^n) + c^{-2}(g^{(2)}_{\alpha 0} + g^{(1)}_{\alpha n}v^n) + \ldots, \qquad (12)$$

$$(N^0_0)^{-1} = (g^{(0)}_{00})^{-1} + c^{-1}(g^{(0)}_{00})^{-2}(g^{(1)}_{00} + g^{(0)}_{0n}v^n)$$
$$+ c^{-2}(g^{(0)}_{00})^{-2}[(g^{(0)}_{00})^{-1}g^{(1)}_{00}g^{(1)}_{00} - g^{(2)}_{00} + v^n(2(g^{(0)}_{00})^{-1}g^{(1)}_{00}g^{(0)}_{0n}$$
$$- g^{(1)}_{0n}) + v^n v^m (g^{(0)}_{00})^{-1}g^{(0)}_{0n}g^{(0)}_{0m}] + \ldots . \qquad (13)$$

We also assume that derivatives with respect to t and x^k are of the same order of smallness as the quantity differentiated. It follows that derivatives with respect to x^0 are of order c^{-1} compared to derivatives with respect to t.

3. THE HARMONICITY CONDITIONS

As already noted, $x^0 = ct$ in Infeld's method $x^0 = t$ in Fock's method. Consequently the components of the metric tensors denoted by $g^*_{\mu\nu}$ in Fock's notation and by $g_{\mu\nu}$ in Infeld's notation are related according to

$$\overset{*}{g}_{00} = c^2 g_{00}, \qquad \overset{*}{g}_{0a} = c g_{0a}, \qquad \overset{*}{g}_{ab} = g_{ab}. \qquad (14)$$

In the works of Fock the harmonic system of coordinates is defined by: (a) the equations of deDonder and Lanczos [11]

$$\Gamma^\alpha \equiv -(-g^*)^{-1/2}(\sqrt{-g^*}\,g^{*\alpha\beta})_{|\beta} = 0, \qquad (15)$$

(b) the conditions at infinity

$$(\overset{*}{g}_{\alpha\beta})_\infty = \overset{*}{\eta}_{\alpha\beta}, \qquad \lim_{n\to\infty}(\overset{*}{g}_{\mu\nu} - \overset{*}{\eta}_{\mu\nu}) = \lim_{r\to\infty}\frac{\alpha^*}{r}, \qquad (16)$$

where

$$r^2 = \delta_{ik}x^i x^k, \qquad \overset{*}{\eta}_{00} = c^2,$$
$$\overset{*}{\eta}_{0a} = 0, \qquad \overset{*}{\eta}_{ab} = -\delta_{ab}, \quad \alpha^* = \text{const};$$

(c) by the radiation conditions

$$\lim_{r\to\infty} [\partial (r\chi^*)/\partial r + c^{-1}\, \partial (r\chi^*)/\partial t], \tag{17}$$

where χ^*—all the functions $g^*_{\mu\nu} - \eta^*_{\mu\nu}$

In Infeld's notation Eqs. (15) take the form

$$\Gamma^0 = -(-g)^{-1/2} [c^{-2} (\sqrt{-g}\, g^{00})_{|t} + c^{-1} (\sqrt{-g}\, g^{0b})_{|b}] = 0, \tag{18}$$

$$\Gamma^a = -(-g)^{-1/2} [c^{-1} (\sqrt{-g}\; g^{0a})_{|t} + (\sqrt{-g}\; g^{ab})_{|b}] = 0. \tag{19}$$

For the conditions at infinity, (16), we have

$$(g_{\alpha\beta})_\infty = \eta_{\alpha\beta}, \qquad \lim_{r\to\infty} (g_{\mu\nu} - \eta_{\mu\nu}) = \lim_{r\to\infty}(\alpha/r), \tag{20}$$

with

$$\eta_{00} = 1, \qquad \eta_{0a} = 0, \qquad \eta_{ab} = -\delta_{ab}.$$

Substitution of (11) into (18) and (19) gives the deDonder-Lanczos conditions in the form of power series

$$\Gamma^0 = \Gamma^{(0)0} + c^{-1}\Gamma^{(1)0} + c^{-2}\Gamma^{(2)0} + \dots,$$
$$\Gamma^a = \Gamma^{(0)a} + c^{-1}\Gamma^{(1)a} + c^{-2}\Gamma^{(2)a} + \dots . \tag{21}$$

Setting the coefficients of these power series equal to zero we obtain the harmonicity conditions of the corresponding orders.

4. THE GENERAL SOLUTIONS OF THE GRAVITATIONAL FIELD EQUATIONS

The physical system which we are studying is characterized by the functions $g_{\mu\nu}(x^\alpha)$ and $\xi^k_A(t)$. For the determination of these functions we have Eqs. (1) and (10). These equations must be solved simultaneously. The procedure for obtaining the solutions is based on the method of successive approximations which consists of the expansion of the metric tensor in the power series (11).

It turns out that the structure of the field equations is such that we can determine the quantities $g^{(0)}_{\mu\nu}$, $g^{(1)}_{\mu\nu}$, $g^{(2)}_{\mu\nu}$, $g^{(3)}_{\mu 0}$, and $g^{(4)}_{00}$ for arbitrary velocities $v^k_A(t)$. The solution of the field equations (1) in the corresponding approximation is given in Appendix A.

For the zeroth approximation we have

$$g^{(0)}_{00} = 1, \qquad g^{(0)}_{0a} = S^{(1)0}_{|a}, \qquad g^{(0)}_{ab} = S^{(1)0}_{|a} S^{(1)0}_{|b} - \delta_{nm} S^{(0)n}_{|a} S^{(0)m}_{|b}. \tag{22}$$

For the first approximation we have

$$g^{(1)}_{00} = 2S^{(1)0}_{|t}, \qquad g^{(1)}_{0a} = S^{(2)0}_{|a} + S^{(1)0}_{|a} S^{(1)0}_{|t} - \delta_{nm} S^{(0)n}_{|a} S^{(0)m}_{|t},$$
$$g^{(1)}_{ab} = S^{(2)0}_{|a} S^{(1)0}_{|b} + S^{(1)0}_{|a} S^{(2)0}_{|b} - \delta_{nm} (S^{(1)n}_{|a} S^{(0)m}_{|b} + S^{(0)n}_{|a} S^{(1)m}_{|b}). \tag{23}$$

ans for the second approximation the more complicated form

$$g^{(2)}_{00} = 2S^{(2)0}_{|t} + S^{(1)0}_{|t} S^{(1)0}_{|t} + \varphi - \delta_{nm} S^{(0)n}_{|t} S^{(0)m}_{|t},$$
$$g^{(2)}_{0a} = S^{(3)0}_{|a} + S^{(2)0}_{|a} S^{(1)0}_{|t} + S^{(1)0}_{|a} S^{(2)0}_{|t} + \varphi S^{(1)0}_{|a} - \delta_{nm} (S^{(1)n}_{|a} S^{(0)m}_{|t} + S^{(0)n}_{|a} S^{(1)m}_{|t}),$$
$$g^{(2)}_{ab} = S^{(3)0}_{|a} S^{(1)0}_{|b} + S^{(2)0}_{|a} S^{(2)0}_{|b} + S^{(1)0}_{|a} S^{(3)0}_{|b} + \varphi S^{(1)0}_{|a} S^{(1)0}_{|b} - \delta_{nm} (S^{(2)n}_{|a} S^{(0)m}_{|a} + S^{(1)n}_{|a} S^{(1)m}_{|b} + S^{(0)n}_{|a} S^{(2)m}_{|b} - \varphi S^{(0)n}_{|a} S^{(0)m}_{|b}), \tag{24}$$

where

$$\varphi = -2 \sum_A \frac{m_A}{|\xi^k_A - x^k|}. \tag{25}$$

In the third approximation the forms are still more complicated but we do not need all the components of the metric tensor. The functions of interest to us take the form

$$g^{(3)}_{00} = 2 (S^{(3)0}_{|t} + S^{(2)0}_{|t} S^{(1)0}_{|t} + \varphi S^{(1)0}_{|t} - \delta_{nm} S^{(1)n}_{|t} S^{(0)m}_{|t}),$$
$$g^{(3)}_{0a} = S^{(4)0}_{|a} + S^{(3)0}_{|a} S^{(1)0}_{|t} + S^{(2)0}_{|a} S^{(2)0}_{|t} + S^{(1)0}_{|a} S^{(3)0}_{|t} + \varphi S^{(2)0}_{|a} + \varphi S^{(1)0}_{|a} S^{(1)0}_{|t} + \varphi \delta_{nm} S^{(0)n}_{|a} S^{(0)m}_{|t} + A_n S^{(0)n}_{|a} - \delta_{nm} (S^{(2)n}_{|a} S^{(0)m}_{|t} + S^{(1)n}_{|a} S^{(1)m}_{|t} + S^{(0)n}_{|a} S^{(2)m}_{|t}), \tag{26}$$

where

$$A_n = 4 \sum_A \frac{m_A}{|\xi^k_A - x^k|} v^n_A. \tag{27}$$

We must still determine $g^{(4)}_{00}$. Complicated calculations yield

$$g^{(4)}_{00} = 2S^{(4)0}_{|t} + 2S^{(3)0}_{|t} S^{(1)0}_{|t} + S^{(2)0}_{|t} S^{(2)0}_{|t} + 2\varphi S^{(2)0}_{|t} + \varphi S^{(1)0}_{|t} S^{(1)0}_{|t} + \varphi \delta_{nm} S^{(0)n}_{|t} S^{(0)m}_{|t} + 2A_n S^{(0)n}_{|t} + \psi - \delta_{nm} (2S^{(2)n}_{|t} S^{(0)m}_{|t} + S^{(1)n}_{|t} S^{(1)m}_{|t}), \tag{28}$$

where

$$\psi = 2 \sum_A \frac{m_A^2}{|\xi^k_A - x^k|^2} - \sum_A m_A |\xi^k_A - x^k|_{|nm} v^n_A v^m_A - \sum_A m_A |\xi^k_A - x^k|_{|n} v^n_{A|t} - 3 \sum_A \frac{m_A}{|\xi^k_A - x^k|} v^n_A v^n_A + 2 \sum_A \sum_B \frac{m_A m_B}{|\xi^k_A - \xi^k_B|\,|\xi^k_A - x^k|}. \tag{29}$$

Thus we have the most general solutions of the field equations. In Eqs. (22)—(28) the functions $S^{(i)\alpha}_\beta$ must be such that the components of the metric tensor satisfy the conditions at infinity. They are otherwise arbitrary.

5. PROOF OF THE SUFFICIENCY OF THE HARMONICITY CONDITIONS

From assumption (11) it follows that the quantities (10) can be expressed in the power series

$$\varepsilon_a = \varepsilon_a^{(0)} + c^{-1}\varepsilon_a^{(1)} + c^{-2}\varepsilon_a^{(2)} + \dots . \quad (30)$$

If we assume that $v_A^n \sim 0$, then the equations of motion are obtained from the equations

$$\varepsilon_a^{\to(i)} = 0, \quad (31)$$

where the arrow indicates that the ε_α are not quantities of a definite order, but range up to a definite order. [13]

The coefficients in Eqs. (30) are obtained from very involved calculations which give

$$\varepsilon_a^{(0)} = -\tfrac{1}{2} g_{00|a}^{(0)},$$
$$\varepsilon_a^{(1)} = g_{0a|t}^{(0)} - \tfrac{1}{2} g_{00|a}^{(1)} + v^n (g_{0a|n}^{(0)} - g_{0n|a}^{(0)}),$$
$$\varepsilon_a^{(2)} = g_{0a|t}^{(1)} - \tfrac{1}{2} g_{0a}^{(0)} g_{00|t}^{(1)} - \tfrac{1}{2} g_{00|a}^{(2)}$$
$$+ v^n (g_{0a|n}^{(1)} - g_{0n|a}^{(1)} + g_{an|t}^{(0)} - g_{0a}^{(0)} g_{00|n}^{(1)})$$
$$+ v^n v^m (g_{an|m}^{(0)} - g_{0a}^{(0)} g_{0n|m}^{(0)} - \tfrac{1}{2} g_{nm|a}^{(0)}) + w^n (g_{an}^{(0)} - g_{0a}^{(0)} g_{0n}^{(0)}), \quad (32)$$

where $w^n = v^n_{|t}$. The remaining quantities are more complicated. We shall express them in the forms

$$\varepsilon_a^{(3)} = B_a^{(3)} + v^n B_{an}^{(3)} + v^n v^m B_{anm}^{(3)}$$
$$+ v^n v^m v^r B_{anmr}^{(3)} + w^n C_{an}^{(3)} + w^n v^m C_{anm}^{(3)},$$
$$\varepsilon_a^{(4)} = B_a^{(4)} + v^n B_{an}^{(4)} + v^n v^m B_{anm}^{(4)} + v^n v^m v^r B_{anmr}^{(4)}$$
$$+ v^n v^m v^r v^p B_{anmrp}^{(4)} + w^n C_{an}^{(4)} + w^n v^m C_{anm}^{(4)} + w^n v^m v^r C_{anmr}^{(4)}, \quad (33)$$

where the coefficients B and C are known functions of $\overset{(i)}{g}_{\alpha\beta}$ and $\overset{(i)}{g}_{\alpha\beta|\gamma}$. For our purposes the point of importance is what combinations of the quantities v^n and w^n appear in these formulas.

We assume that the harmonicity conditions are satisfied to the second order inclusive, so that we have

$$\Gamma_a^{(0)} = 0, \quad \Gamma_a^{(1)} = 0, \quad \Gamma_a^{(2)} = 0. \quad (34)$$

With Eq. (19), the first of Eqs. (34) gives us the zeroth approximation to the harmonicity conditions

$$(\sqrt{-g^{(0)}}\, g^{(0)\alpha\beta})_{|\beta} = 0. \quad (35)$$

After some rearrangements we get

$$g_{00|b}^{(0)} (\tfrac{1}{2} g^{(0)ab} g^{(0)00} - g^{(0)0a} g^{(0)0b}) + g_{0n|b}^{(0)} (g^{(0)ab} g^{(0)0n}$$
$$- g^{(0)an} g^{(0)0b} - g^{(0)bn} g^{(0)0a})$$
$$+ g_{nm|b}^{(0)} (\tfrac{1}{2} g^{(0)ab} g^{(0)nm} - g^{(0)an} g^{(0)bm}) = 0. \quad (36)$$

The covariant components of the metric tensor in zeroth approximation are given by (22); for the contravariant components we have

$$g^{(0)00} = 1 - \gamma^{ab} S_{|a}^{(1)0} S_{|b}^{(1)0}, \quad g^{(0)0a} = \gamma^{an} S_{|n}^{(1)0}, \quad g^{(0)ab} = -\gamma^{ab}, \quad (37)$$

where

$$\gamma_{ab} = S_{|a}^{(0)k} S_{|b}^{(0)k}, \quad \gamma^{ab}\gamma_{ac} = \delta_c^b. \quad (38)$$

Substitution of (22) and (37) into (36) gives

$$\gamma^{nm} S_{|nm}^{(0)k} = 0. \quad (39)$$

Referring to the Hilbert conditions (3), it follows that the form

$$dl^2 = \gamma_{ab} dx^a dx^b \quad (40)$$

is positive definite. Hence, Eqs. (39) are elliptic and their solutions, satisfying the conditions at infinity, are

$$S_{|a}^{(0)k} = c_a^k, \qquad S_{|t}^{(0)k} = 0, \quad (41)$$

where the constants c_a^k are such that

$$S_{|a}^{(0)k} S_{|b}^{(0)k} = \delta_{ab}. \quad (42)$$

Therefore, if the harmonicity conditions are satisfied in zeroth approximation the metric tensor has the form

$$g_{00} = 1 + O(c^{-1}), \quad g_{0a} = S_{|a}^{(1)0} + O(c^{-1}),$$
$$g_{ab} = -\delta_{ab} + S_{|a}^{(1)0} S_{|b}^{(1)0} + O(c^{-1}). \quad (43)$$

Let us examine the first approximation in more detail. Substitution of (22) and (23) into (18) gives, with required accuracy,

$$S_{|nn}^{(1)0} = 0. \quad (44)$$

We have taken note here also of (41). Noting the conditions at infinity:

$$\lim_{r\to\infty} g_{0a}^{(0)} = \lim_{r\to\infty} S_{|a}^{(1)0} = 0, \quad \lim_{r\to\infty} g_{00}^{(1)} = \lim_{r\to\infty} 2S_{|t}^{(1)0} = 0, \quad (45)$$

which follow from (20), the solution of (44) has the form

$$S_{|a}^{(1)0} = 0, \qquad S_{|t}^{(1)0} = 0. \quad (46)$$

Proceeding similarly for (19) we obtain

$$S_{|a}^{(1)k} = 0, \qquad S_{|t}^{(1)k} = 0. \quad (47)$$

Equations (46) and (47) are consequences of the harmonicity conditions of first order.

Simple calculations for the second order yield

$$S_{|a}^{(2)0} = 0, \quad S_{|t}^{(2)0} = 0, \quad S_{|a}^{(2)k} = 0, \quad S_{|t}^{(2)k} = 0. \quad (48)$$

Finally, we find the metric tensor satisfying the harmonicity conditions to the second order inclusive:

$$g_{00} = 1 + c^{-2}\varphi + 2c^{-3} S_{|t}^{(3)0} + c^{-4}(\psi + 2S_{|t}^{(4)0}) + O(c^{-5}),$$
$$g_{0a} = c^{-2} S_{|a}^{(3)0} + c^{-3}(A_a + S_{|a}^{(4)0}) + O(c^{-4}),$$
$$g_{ab} = -\delta_{ab} + c^{-2}\delta_{ab}\varphi + O(c^{-3}). \quad (49)$$

Substitution of (12) and (13) into (10) gives, to the second approximation,

$$w^n S^{(0)k}_{|a} S^{(0)k}_{|n} - \tfrac{1}{2}\varphi_{|a} = v^n v^m S^{(0)k}_{|a} S^{(0)k}_{|nm} + 2v^n S^{(0)k}_{|a} S^{(0)k}_{|nt} + S^{(0)k}_{|a} S^{(0)k}_{|tt} \quad (50)$$

without using the harmonicity conditions. Now, if we assume the harmonicity conditions of zeroth order (41) to be satisfied, then (50) takes the form

$$w^a - \tfrac{1}{2}\varphi_{|a} = 0. \quad (51)$$

These are the Newtonian equations of motion.

We emphasize that the above considerations show that the Newtonian equations of motion follow from the harmonicity conditions of zeroth order; it is not necessary to invoke the stricter conditions

$$g^{(0)}_{00} = 1, \quad g^{(0)}_{0a} = 0, \quad g^{(0)}_{ab} = -\delta_{ab}. \quad (52)$$

If we now assume the harmonicity conditions to the second order inclusive, that is, (41), (46), (47), and (48), to be satisfied, or that the metric tensor is of the form (49), then the equations up to the fourth order give

$$w^a - \tfrac{1}{2}\varphi_{|a} = c^{-2}\,[w^a\varphi - \tfrac{1}{2}v^n v^n \varphi_{|a} + v^n (A_{a|n} - A_{n|a} + \tfrac{3}{2}\delta_{na}\varphi_{|t} + 2v^a\varphi_{|n}) + A_{a|t} - \tfrac{1}{2}\psi_{|a}]. \quad (53)$$

In Appendix B it is shown that these are the post-Newtonian equations of motion.

6. PROOF OF THE NECESSITY OF THE HARMONICITY CONDITIONS

We shall now prove the converse assertion: to obtain the post-Newtonian equations of motion from (10) it is necessary to satisfy the harmonicity conditions to the second order inclusive, that is, Eqs. (41), (46), (47), and (48).

In order to simplify the proof we shall use successively results already obtained. Equation (31) in zeroth order gives

$$\varepsilon^{(0)}_a = -\tfrac{1}{2}N^{(0)00} g^{(0)}_{00|a} = -\tfrac{1}{2} g^{(0)}_{00|a} = 0. \quad (54)$$

These equations are satisfied identically. For the first order we find

$$\varepsilon^{(1)}_a = (S^{(1)0}_{|a})_{|t} - \tfrac{1}{2}(2S^{(1)0}_{|t})_{|a} + v^n (S^{(1)0}_{|a|n} - S^{(1)0}_{|n|a}) = 0. \quad (55)$$

These are likewise identically satisfied.

Substitution of (32) into (31) gives, for the second approximation,

$$w^n S^{(0)k}_{|a} S^{(0)k}_{|n} - \tfrac{1}{2}\varphi_{|a} = v^n v^m S^{(0)k}_{|a} S^{(0)k}_{|nm} + 2v^n S^{(0)k}_{|a} S^{(0)k}_{|nt} + S^{(0)k}_{|a} S^{(0)k}_{|tt} \quad (56)$$

By hypothesis these should be the Newtonian equations of motion. Comparison of (56) with (51) leads to

$$S^{(0)k}_{|a} S^{(0)k}_{|n} = \delta_{an}, \quad S^{(0)k}_{|nm} = 0, \quad S^{(0)k}_{|tt} = 0. \quad (57)$$

To simplify further calculations, let us note that in the post-Newtonian equations of motion (53) there are no terms involving the products

$$v^n v^m v^r, \quad v^n v^m v^r v^p, \quad w^n v^m, \quad w^n v^m v^r. \quad (58)$$

Consequently, the coefficients multiplying the indicated products must be equal to zero.

For the equation

$$\overset{\rightarrow}{\varepsilon}{}^{(3)}_a = 0 \quad (59)$$

these terms have the form

$$v^n v^m v^r \delta_{ar} S^{(1)0}_{|nm}, \quad w^n v^m \delta_{am} S^{(1)0}_{|n}. \quad (60)$$

Setting the coefficients of $w^n v^m$ and $v^n v^m v^r$ equal to zero and taking account of the conditions at infinity, we obtain

$$S^{(1)0}_{|n} = 0, \quad S^{(1)0}_{|t} = 0. \quad (61)$$

Taking account of (57) and (61), we get from (59)

$$w^a - \tfrac{1}{2}\varphi_{|a} = c^{-1}\,[w^n (S^{(1)n}_{|a} + S^{(1)a}_{|n}) + v^n v^m S^{(1)a}_{|nm} + 2v^n S^{(1)a}_{|nt} + S^{(1)a}_{|tt}]. \quad (62)$$

Comparison of these equations with the post-Newtonian equations of motion (53), gives

$$S^{(1)a}_{|n} = 0, \quad S^{(1)a}_{|t} = 0. \quad (63)$$

In the equation $\overset{\rightarrow}{\varepsilon}{}^{(4)}_\alpha = 0$ the terms involving the products (58) have the form

$$v^n v^m v^r \delta_{ar} S^{(2)0}_{|nm}, \quad w^n v^m \delta_{am} S^{(2)0}_{|n}. \quad (64)$$

We have taken (61) into account. However, as we have noted, terms like (64) must vanish. Noting the conditions at infinity, we find

$$S^{(2)0}_{|n} = 0, \quad S^{(2)0}_{|t} = 0. \quad (65)$$

In consequence of (57), (61), (63), and (65), the equations $\overset{\rightarrow}{\varepsilon}{}^{(4)}_\alpha = 0$ can be written in the form

$$w^a - \tfrac{1}{2}\varphi_{|a} = c^{-2}\,[w^a\varphi + v^n v^m (2\delta_{an}\varphi_{|m} - \tfrac{1}{2}\delta_{nm}\varphi_{|a}) + v^n (\tfrac{3}{2}\delta_{an}\varphi_{|t} + A_{a|n} - A_{n|a}) + A_{a|t} - \tfrac{1}{2}\psi_{|a}] + c^{-2}\,[w^n (S^{(2)a}_{|n} + S^{(2)n}_{|a}) + v^n v^m S^{(2)a}_{|nm} + 2v^n S^{(2)a}_{|nt} + S^{(2)a}_{|tt}]. \quad (66)$$

Comparison of (66) with the post-Newtonian equations (53) gives

$$S^{(2)a}_{|n} = 0, \quad S^{(2)a}_{|t} = 0. \quad (67)$$

Equations (57), (61), (63), and (67) constitute the harmonicity conditions to second order inclusive. Hence, these conditions are necessary for the derivation of the post-Newtonian equations of motion.

7. CONCLUSIONS

As we have shown, the derivation of the post-Newtonian equations of motion requires the harmonicity conditions to the second order inclusive. It follows from our calculations that if we assume the existence of equations of motion of higher orders then we must also assume the existence of higher order harmonicity conditions.

In the work of Infeld[10] it is stated that the post-Newtonian equations of motion can also be obtained without satisfying the harmonicity conditions. Thus, Infeld obtains these equations in another non-harmonic coordinate system termed "isotropic." In this connection we make the following observations:

The harmonicity conditions as expressed in the second of Infeld's [10] Eqs. (9.2) are, indeed, not satisfied by the solutions (49). But these conditions of harmonicity are of the third order and, as indicated by our proof, are not required. The first of Infeld's Eqs. (9.2) is also not satisfied by the solutions (49). But these equations again do not coincide with the harmonicity conditions of second order which, according to (19), take the form

$$\left(g^{(2)\,ab} + \sqrt{-g^{(2)}}g^{(0)\,ab}\right)_{|b} = 0. \quad (68)$$

In particular, from these conditions it follows that $a_m^{(2)} = 0$ (in Infeld's notation [10]). Accordingly, the "isotropic" system is harmonic in the appropriate approximation.

In our work we did not concern ourselves with the problem of accuracy of transformations as we did for the determination of the harmonic system of coordinates. This problem is investigated in Fock's work [17] where it is also proved that the coordinate conditions used by Einstein and Infeld [1,4] coincide with the harmonic conditions.

APPENDIX A

We need to find the most general functions $g_{\alpha\beta}^{(0)}$, $g_{\alpha\beta}^{(1)}$, $g_{\alpha\beta}^{(2)}$, $g_{0\beta}^{(3)}$, $g_{00}^{(4)}$, satisfying the field equations (1) and the conditions at infinity (20). The field equations are covariant. Hence, if $g_{\alpha\beta}(x^n)$ are solutions of the field equations then the functions

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \quad (69)$$

also satisfy the field equations.

On the other hand, as is well known [11], by appropriate transformation of coordinates it is always possible to satisfy the conditions

$$(\sqrt{-g}\,g^{\mu\nu})_{|\nu} = 0. \quad (70)$$

Hence, if we obtain the most general solution of the field equations (1) subject to the conditions (70) then with (69) we also obtain the most general solution without these conditions.

In the work of Jankiewicz[18] it is shown that, in zeroth approximation, the solution satisfying the field equations (1), the equations (70), and the conditions at infinity (20), has the form

$$g_{00}^{(0)} = 1, \quad g_{0a}^{(0)} = 0, \quad g_{ab}^{(0)} = -\delta_{ab}. \quad (71)$$

For the first approximation Eqs. (1) and (70) give

$$g_{\mu\nu|aa}^{(1)} = 0 \quad (72)$$

from which, with the conditions at infinity, we find

$$g_{\mu\nu}^{(1)} = 0. \quad (73)$$

For the second approximation Eqs. (1) and (70) give

$$g_{00|aa}^{(2)} = 8\pi\sum_A m_A\delta(x^k - \xi_A^k), \quad g_{0n|aa}^{(2)} = 0,$$
$$g_{nm|aa}^{(2)} = 8\pi\delta_{nm}\sum_A m_A\delta(x^k - \xi_A^k). \quad (74)$$

The sought for solutions take the form

$$g_{00}^{(2)} = \varphi = -2\sum_A \frac{m_A}{|\xi_A^k - x^k|}, \quad g_{0n}^{(2)} = 0, \quad g_{nm}^{(2)} = \delta_{n,n}\varphi. \quad (75)$$

For the third approximation we shall use only the equations necessary for the determination of the functions $g_{00}^{(3)}$ and $g_{0n}^{(3)}$. The equations have the form

$$g_{00|aa}^{(3)} = 0, \quad g_{0n|aa}^{(3)} = -16\pi\sum_A m_A\delta(x^k - \xi_A^k)\,v_A^n. \quad (76)$$

The corresponding solutions are

$$g_{00}^{(3)} = 0, \quad g_{0n}^{(3)} = A_n = 4\sum_A \frac{m_A}{|\xi_A^k - x^k|}\,v_A^n. \quad (77)$$

We must still find $g_{00}^{(4)}$ which is determined by the corresponding

$$\left(g_{00}^{(4)} - \tfrac{1}{2}\varphi^2\right)_{|aa} = \varphi_{|00} + \varphi\varphi_{|aa}$$
$$+ 12\pi\sum_A m_A\delta(x^k - \xi_A^k)\,v_A^n v_A^n - 4\pi\varphi\sum_A m_A\delta(x^k - \xi_A^k). \quad (78)$$

The solution is given by the rather complex expression:

$$g_{00}^{(4)} = \psi = 2\sum_A \frac{m^2{}_A}{|\xi^k_A - x^k|^2} - \sum_A m_A|\xi^k_A - x^k|_{|nm} v^n_A v^m_A - \sum_A m_A|\xi^k_A - x^k|_{|n} w^n_A - 3\sum_A \frac{m_A}{|\xi^k_A - x^k|} v^n_A v^n_A + 2\sum_A\sum_B \frac{m_A m_B}{|\xi^k_A - \xi^k_B|\,|\xi^k_A - x^k|}. \tag{79}$$

Finally, we have the most general solution satisfying (1), (70), and the conditions at infinity,

$$g_{00} = 1 + c^{-2}\varphi + c^{-4}\psi + O(c^{-5}),$$
$$g_{0n} = c^{-3}A_n + O(c^{-4}),$$
$$g_{nm} = -\delta_{nm} + c^{-2}\delta_{nm}\varphi + O(c^{-3}). \tag{80}$$

As we have already noted, the general solution of just Eq. (1) follows from Eq. (80) using Eq. (69). Introducing the change of notation $x \to x'$, $x' \to S$, $g'_{\alpha\beta} \rightleftarrows g_{\alpha\beta}$ we obtain the solutions (22), (23), (24), (26), and (28).

To conclude, we note that if

$$\psi^* = 2\sum_A \frac{m^2_A}{|\xi^k_A - x^k|^2} - \sum_A m_A|\xi^k_A - x^k|_{|nm} v^n_A v^m_A - 3\sum_A \frac{m_A}{|\xi^k_A - x^k|} v^n_A v^n_A + \sum_A\sum_B \frac{m_A m_B}{|\xi^k_A - \xi^k_B|\,|\xi^k_A - x^k|}, \tag{81}$$

then

$$\overline{\psi^*_{|n}} = \overline{\psi}_{|n}. \tag{82}$$

APPENDIX B

We shall now show that the equations (53) coincide with the post-Newtonian equations of motion, as derived by Einstein and Infeld and also Petrova and Papapatrou. For simplicity we examine the case of two particles. If we introduce the notation $\xi^n_1 = \xi^n$, $\xi^n_2 = \eta^n$, and $|\xi^n - \eta^n| = r$, then equations (53) take on the form

$$\ddot\xi^a - \tfrac{1}{2}(\varphi_{|a})_1 = c^{-2}\{\ddot\xi^a(\varphi)_1 - \tfrac{1}{2}\dot\xi^n\dot\xi^n(\varphi_{|a})_1 + \dot\xi^n[(A_{a|n})_1 - (A_{n|a})_1 + \tfrac{3}{2}\delta_{na}(\varphi_{|t})_1 + 2\dot\xi^a(\varphi_{|n})_1] + (A_{a|t})_1 - \tfrac{1}{2}(\psi_{|a})_1\}, \tag{83}$$

where the dot denotes differentiation with respect to time.

Eqs. (25) and (27) give

$$(\varphi)_1 = -2m_2 r^{-1}, \quad (A_a)_1 = 4m_2 r^{-1}\dot\eta^a \tag{84}$$

Instead of ψ we shall use ψ^*. For two particles this function has the form (cf. Appendix A)

$$\bar\psi^* = 2m_2^2 r^{-2} - 3m_2\dot\eta^n\dot\eta^n r^{-1} + m_1 m_2 r^{-2} - m_2\dot\eta^n\dot\eta^m r_{|nm}. \tag{85}$$

Subsitution of (84) and (85) into (83) gives

$$\ddot\xi^a - m_2(r^{-1})_{|a} = c^{-2}m_2\{(r^{-1})_{|a}[\dot\xi^n\dot\xi^n + \tfrac{3}{2}\dot\eta^n\dot\eta^n - 4\dot\xi^n\dot\eta^n - 4m_2 r^{-1} - 5m_1 r^{-1}] + (r^{-1})_{|n}[4\dot\xi^n(\dot\eta^a - \dot\xi^a) + 3\dot\xi^a\dot\eta^n - 4\dot\eta^a\dot\eta^n] + \tfrac{1}{2} r_{|anm}\dot\eta^n\dot\eta^m\}. \tag{86}$$

These are the well known post-Newtonian equations of motion.

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