EQUATIONS OF MOTION AND STATE FOR MAGNETOELASTIC MEDIA

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Equations describing the dynamic behavior of magnetoelastic anistropic media (equation of motion of elasticity theory, equations of motion or of state for magnetization, equation of state for electric polarization, and equations of electrodynamics) are simultaneously deduced from the Hamilton principle. It is demonstrated that an additional force due to inhomogeneity of the rate of variation of magnetization should be taken into account in the equation of motion of elasticity theory, whereas the rotation of volume elements of the medium accompanying elastic deformations should be taken into account in the equations of motion or state for magnetization. With an isotropic ferrodielectric in the demagnetized or magnetically polarized state taken as an example, it is shown that even propagation of plane elastic waves in the range below the ferromagnetic-resonance frequency should involve some singularities such as: rotation of the plane of polarization of ultrasound, due to the combined action of gyromagnetism and magnetostriction, the appearance of forced magnetization oscillations accompanying elastic waves due to magnetostriction as well as gyromagnetism, and anisotropy of the velocity of ultrasound and the dependence of the velocity on the magnetic state of the ferromagnet.

At the present time the theory of magnetic, elastic, and magnetoelastic properties of magnetoelastic media (such as ferromagnets) has been sufficiently well developed for the case of quasistatic processes, when the magnetic properties are described thermodynamically by specifying some thermodynamic potential or equation of state[1,2]. A few recently published papers (see, for example, [3-8] and others) consider also the dynamic properties of various magnetoelastic media, with the magnetic properties described not by an equation of state but by an equation of motion. These investigations, however, either are based on specific model representations or consider in fact only some particular cases.

In the present communication we attempt to derive equations of state (in the case when the Lagrange function does not depend on the velocities of the corresponding variables) and equations of motion characterizing the dynamic behavior of magnetoelastic media, on the basis of Hamilton's principle

$$\delta S = \delta \int_V \int_t \mathcal{L} dv dt = 0 \qquad (1)$$

and using the concepts of nonequilibrium thermodynamics.

1. We consider an infinite homogeneous (in the ground state) anisotropic medium, the state of which is determined by an elastic strain tensor

$$\varepsilon_{ij} = {}^1/_2 (\partial u_i/\partial x_j + \partial u_j/\partial x_i)$$

($\mathbf{u}$ —displacement vector of the material points), electric polarization $\mathbf{P}$, and magnetization $\mathbf{M}$. Let the density of the free electric charges in the medium be ρ_e and let the density of the electric currents flowing in the medium be $\mathbf{j}$; in addition, account must be taken of the electromagnetic field, defined by a vector potential $\mathbf{A}$ and a scalar potential φ, which are connected with the intensity of the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ by the relations

$$\mathbf{E} = -\nabla\varphi - c^{-1}\partial\mathbf{A}/\partial t, \qquad \mathbf{B} = \text{rot } \mathbf{A}. \qquad (2)^*$$

The density of the Lagrangian is chosen in the form

$$\mathcal{L} = \tfrac{1}{2}\rho\dot{u}^2 + \mathbf{EP} + \mathbf{BM} + c^{-1}\mathbf{jA} - \rho_e\varphi + (E^2 - B^2)/8\pi + \mathcal{K} - \tilde{\mathcal{F}}. \qquad (3)$$

Here ρ is the density of the medium, $\tilde{\mathcal{F}}$ —the density of some thermodynamic potential characterizing the state of the medium, and $\mathcal{K}$ —the density of some quantity ("kinetic energy") character-

*rot = curl

izing the motion of the mechanical moment J connected with the magnetization M by

$$\mathbf{M} = \hat{g}\mathbf{J}, \tag{4}$$

where $\hat{g}$ is the tensor of the spectroscopic splitting factor.

Variation of the action S with respect to A and φ with constant u, P, and M yields, with allowance for (2), the equations of electrodynamics

$$\operatorname{div}\mathbf{D} = 4\pi\rho_e, \qquad \operatorname{rot}\mathbf{E} = -c^{-1}\partial\mathbf{B}/\partial t; \tag{5}$$

$$\operatorname{div}\mathbf{B} = 0, \qquad \operatorname{rot}\mathbf{H} = (4\pi/c)\,\mathbf{j} + c^{-1}\partial\mathbf{D}/\partial t. \tag{6}$$

We carry out the variation with respect to u, P, and M in the laboratory system of coordinates throughout. However, in calculating the local variations we must take into account the following circumstances. For arbitrary elastic deformations of the medium, the volume elements of the medium experience in addition to the strains ϵ_{ij} also displacements u and rotations through an angle $\omega = \frac{1}{2}\operatorname{curl}\mathbf{u}$ as a whole. Therefore the medium becomes inhomogeneous, and its general symmetry is lost. However, we shall assume that within the limits of elementary volumes there is a local thermodynamic equilibrium in that we can speak of local symmetry and of a density of the thermodynamic potential. The latter should depend on the thermodynamic variables at the points which move with velocity $\dot{\mathbf{u}}$ together with the volume element of the medium, and for the vector and tensor variables in the anisotropic media it depends on the projections of these variables on the coordinate axes which are connected with the crystallographic axes (symmetry axes) of the volume elements of the medium.

It is therefore advisable to introduce three coordinate systems: (1) the laboratory system, (2) a system moving in translation together with the volume element of the medium, (3) a system moving and rotating with the volume element of the medium.

All the quantities calculated in the second and third coordinate frames will be designated by one and two asterisks, respectively. In this connection we can introduce different variations and different velocities of the corresponding quantities. By δ and $\partial/\partial t$ we shall mean a change (variation) and a rate of change of some quantity in the laboratory system of coordinates (local variations and rate of change); by δ_1 and d/dt we mean the variation and rate of change in the coordinate system that moves in translation, while by δ_2 and D/Dt we mean the change and the rate of change in the system that moves and rotates. The following relations hold true here

$$\delta N = \delta_1 N - (\delta\mathbf{u}\nabla)N, \qquad \partial N/\partial t = dN/dt - (\dot{\mathbf{u}}\nabla)N, \tag{7}$$

where N is a scalar, vector, or tensor.

For a scalar

$$\delta_2 N = \delta_1 N, \qquad DN/Dt = dN/dt, \tag{8}$$

for a vector

$$\delta_2\mathbf{N} = \delta_1\mathbf{N} - [\delta\boldsymbol{\omega}\mathbf{N}], \qquad D\mathbf{N}/Dt = d\mathbf{N}/dt - [\dot{\boldsymbol{\omega}}\mathbf{N}] \tag{9}*$$

and for a second-rank tensor

$$(\delta_2\hat{N})_{ij} = (\delta_1\hat{N})_{ij} - N_{il}\delta\omega_{jl} + N_{lj}\delta\omega_{li},$$

$$(D\hat{N}/Dt)_{ij} = (d\hat{N}/dt)_{ij} - N_{il}\dot{\omega}_{jl} + N_{lj}\dot{\omega}_{li}, \tag{10}$$

where ω_{ij} is the rotation tensor which is the dual of the rotation vector ω:

$$\omega_{ij} = -\varepsilon_{ijk}\omega_k, \tag{11}$$

ϵ_{ijk} is an absolutely antisymmetrical unit tensor of third rank. In the variation it is necessary to take into account also the relation

$$\delta_1\dot{\mathbf{u}} = d\delta\mathbf{u}/dt. \tag{12}$$

In the calculations we shall assume the strains ϵ_{ij} and the velocities $\dot{\mathbf{u}}$ to be small, i.e., we neglect terms proportional to ϵ_{ij}^2 and $(\dot{\mathbf{u}}/c)^2$. In these approximations the only variables that are different in the laboratory and in the translational-motion frames are

$$\mathbf{P}^* = \mathbf{P} - c^{-1}[\dot{\mathbf{u}}\mathbf{M}], \qquad \mathbf{M}^* = \mathbf{M} + c^{-1}[\dot{\mathbf{u}}\mathbf{P}],$$

$$\mathbf{B}^* = \mathbf{B} - c^{-1}[\dot{\mathbf{u}}\mathbf{E}], \qquad \mathbf{E}^* = \mathbf{E} + c^{-1}[\dot{\mathbf{u}}\mathbf{B}]; \tag{13}$$

$$\mathbf{j}^* = \mathbf{j} - \rho_e\dot{\mathbf{u}}, \tag{14}$$

where j* is the density of the conduction current.

When varying the density of the free charges ρ_e and the current density j we assume, following Tamm[9], that the electric charge Q in any isolated volume V, bounded by a closed surface S (which moves together with the medium) remains constant

$$\delta Q = \delta\int_{\dot{V}}\rho_e\,dv = \int_{\dot{V}}\delta\rho_e\,dv + \oint_S\rho_e\delta\mathbf{u}\,d\mathbf{S}$$

$$= \int_{\dot{V}}[\delta\rho_e + \operatorname{div}(\rho_e\delta\mathbf{u})]\,dv = 0$$

and the total conduction current J* flowing through any surface S (which moves together with the medium), bounded by a closed contour L, remains unchanged

$$\delta J^* = \delta\int_{\dot{S}}\mathbf{j}^*\,d\mathbf{S} = \int_{\dot{S}}\delta\mathbf{j}^*\,d\mathbf{S} + \oint_L\mathbf{j}^*\,[\delta\mathbf{u}\,d\mathbf{l}]$$

$$= \int_{\dot{S}}(\delta\mathbf{j}^* + \operatorname{rot}[\mathbf{j}^*\delta\mathbf{u}])\,d\mathbf{S} = 0.$$

*$[\omega\,\mathbf{N}] = \omega\times\mathbf{N}$.

An analogous requirement ($\delta\int_V \rho\,dv = 0$) should be imposed also when varying the density ρ. Then the variations $\delta\rho$, $\delta\rho_e$, and $\delta\mathbf{j}^*$ will be expressed in terms of $\delta\mathbf{u}$ in the following fashion:

$$\delta\rho = -\operatorname{div}(\rho\delta\mathbf{u}), \qquad \delta\rho_e = -\operatorname{div}(\rho_e\delta\mathbf{u}); \tag{15}$$

$$\delta\mathbf{j}^* = -\operatorname{rot}[\mathbf{j}^*\delta\mathbf{u}]. \tag{16}$$

In similar fashion, we obtain the continuity equations

$$\partial\rho/\partial t = -\operatorname{div}(\rho\dot{\mathbf{u}}), \qquad \partial\rho_e/dt = -\operatorname{div}(\rho_e\dot{\mathbf{u}}). \tag{17}$$

From (16), (14), (17), (15), (12), and (7) it follows that

$$\delta\mathbf{j} = -\operatorname{rot}[\mathbf{j}\delta\mathbf{u}] + \partial(\rho_e\delta\mathbf{u})/\partial t. \tag{18}$$

We note also that sometimes it is more convenient to use not the densities of some quantities, but the per-unit values of these quantities. In this connection we introduce the notation

$$\mathbf{M} = \rho\mathbf{m}, \qquad \mathbf{J} = \rho\mathbf{I},$$

$$\mathbf{P} = \rho\mathbf{p}, \qquad \mathcal{K} = \rho K, \qquad \mathcal{F} = \rho F, \tag{19}$$

and also take account of the fact that

$$\delta\mathcal{F} = \rho\delta_1 F - \nabla(\delta\mathbf{u}F), \ldots;$$

$$(\partial/\partial t + \nabla\dot{\mathbf{u}})\,\mathcal{F} = \rho\,dF/dt, \ldots, \tag{20}$$

where $\mathbf{m}$, $\mathbf{I}$, $\mathbf{p}$, K, and F are the per-unit magnetic and associated mechanical moments, electric polarization, "kinetic energy," and free energy.

For periodic processes both the form of the thermodynamic potential and the form of the equations of motion (for example, for the magnetization) may be different, depending on the relation between the relaxation frequency (for example, different components of the magnetization vector) and the frequency of the periodic process. Accordingly we consider two cases: (a) substances in which the modulus of the vector of the specific magnetic moment does not have time to relax and remains constant at frequencies close to the frequencies of magnetic resonance; (b) substances and frequencies when the modulus of the specific magnetic moment vector has time to relax, i.e., assume its equilibrium value.

2. We consider first the case when the modulus of the specific mechanical moment $|\mathbf{I}|$, connected with the magnetization, remains constant ($|\mathbf{I}| = \text{const}$). Since the Lagrange function should be relativistically invariant in all the coordinate systems, the quantity $\mathcal{K} - \widetilde{\mathcal{F}}$ in (3) should be relativistically invariant (inasmuch as the sum of the remaining terms remains relativistically invariant accurate to $(\dot{u}/c)^2$). This requirement can be satisfied by assuming

$$\mathcal{K} - \widetilde{\mathcal{F}} = \rho(K^* - F^*) - 2\pi M^{*2}, \tag{21}$$

where F* and K* are respectively the specific free and "kinetic" energies in the coordinate system that moves in translation with the volume element of the medium. F* can depend, for example, on the per-unit magnetic moment m* (in this case only on its orientation), the electric polarization $\mathbf{p}^*$, the strain tensors $\hat{\epsilon}$, and the gradients of the magnetic moment $\partial\mathbf{m}^*/\partial\mathbf{r}$, calculated in the translational coordinate system but projected on the axis of the coordinate system which moves and rotates together with the volume element of the medium. In other words, the variation of F* depends on the variation of the vectors and tensors which define it relative to the volume elements of the medium. Thus, we have for the variation $\delta_1 F^*$

$$\delta_1 F^* = \delta_2 F^* = \frac{\partial F^*}{\partial \mathbf{m}^*}\delta_2\mathbf{m}^* + \frac{\partial F^*}{\partial \mathbf{p}^*}\delta_2\mathbf{p}^* + \frac{\partial F^*}{\partial\hat{\epsilon}}\delta_2\hat{\epsilon} + \frac{\partial F^*}{\partial(\partial\mathbf{m}^*/\partial\mathbf{r})}\delta_2\left(\frac{\partial\mathbf{m}^*}{\partial\mathbf{r}}\right). \tag{22}$$

For the per-unit "kinetic energy" we can choose the expression[10]

$$K^* = \mathbf{I}^*[\mathbf{I}_\perp^*\dot{\mathbf{I}}_\perp^*]/I_\perp^{*2}, \tag{23}$$

where $\mathbf{I}_\perp^* = \mathbf{k}\times[\mathbf{I}^*\times\mathbf{k}]$, with $\mathbf{k}$ an arbitrary unit vector and the dot denoting the differentiation d/dt.

In the variation it is necessary to take account of the fact that the tensor of the spectroscopic splitting factor g is a material constant only in the coordinate system that moves and rotates together with the volume element of the medium, and consequently only

$$\delta_2\mathbf{M}^* = \hat{g}\delta_2\mathbf{J}^*. \tag{24}$$

The constancy of the modulus $|\mathbf{I}|$ is taken into account by introducing in the Lagrange function a term $\rho\lambda I^{*2}$, where λ is the Lagrange parameter. Varying with respect to $\mathbf{u}$, $\mathbf{P}$, and $\mathbf{M}$ with constant $\mathbf{A}$, φ, $\mathbf{E}$, and $\mathbf{B}$ (taking into account all the foregoing relations and considerations, carrying out suitable integration by parts with respect to space and time, and recognizing that the variations are equal to zero at the limits of the integration), equating to zero the factors preceding $\delta\mathbf{u}$, $\delta_1\mathbf{m}^*$, and $\delta_1\mathbf{p}^*$, we obtain the following relations:

A. Equation of state for $\mathbf{p}^*$:

$$\mathbf{E}^* = \partial F^*/\partial\mathbf{p}^*. \tag{25}$$

B. Equation of motion for the per-unit mechanical moment I* connected with the magnetization:

$$d\mathbf{I}^*/dt = [\mathbf{I}^*\mathbf{H}_\mathbf{I}^{*\,\mathrm{eff}}], \tag{26}$$

$$\mathbf{H}_\mathbf{I}^{*\,\mathrm{eff}} = \tilde{g}\mathbf{H}_\mathbf{m}^{*\,\mathrm{eff}}. \tag{27}$$

Here $\mathbf{H}_\mathbf{m}^{*\mathrm{eff}}$ and $\mathbf{H}_\mathbf{I}^{*\mathrm{eff}}$ are the effective fields for the magnetic and the associated mechanical mo-

ments, respectively, calculated in the frame with translational motion (~ denotes transposition):

$$\mathbf{H}_{\mathrm{m}}^{*\,\mathrm{eff}} = \mathbf{H}^* - \frac{\partial F^*}{\partial \mathbf{m}^*} + \frac{\partial}{\partial x_j}\frac{\partial F^*}{\partial (\partial \mathbf{m}^*/\partial x_j)}. \tag{28}$$

C. Equation of motion for $\mathbf{u}$:

$$\rho \ddot{u}_i = f_i; \tag{29}$$

$$f_i = \frac{\partial}{\partial x_j}\left(\rho \frac{\partial F^*}{\partial \varepsilon_{ij}}\right) + \frac{1}{c}[\mathbf{jB}]_i + \rho_e E_i + \mathbf{P}\nabla_i \mathbf{E} + \mathbf{M}\nabla_i \mathbf{H} + \frac{1}{c}\rho \frac{d}{dt}([\mathbf{Em}]_i + [\mathbf{pB}]_i) - \frac{1}{2}\mathrm{rot}_i([\mathbf{HM}] + [\mathbf{EP}]) + \frac{1}{2}\mathrm{rot}_i[\mathbf{H}_{\mathrm{I}}^{*\,\mathrm{eff}}\mathbf{J}]. \tag{30}$$

We note that in place of taking the expression for the "kinetic energy" in the form (23), we can use the expression for its variation

$$\delta_1 K^* = -\mathbf{H}_{\mathrm{I}}^{*\,\mathrm{eff}}\delta_1 \mathbf{I}^*, \tag{31}$$

where $\mathbf{H}_{\mathrm{I}}^{*\mathrm{eff}}$ is expressed in terms of the angular velocity of rotation of the vector $\mathbf{I}^*$ (taken with the minus sign), calculated in the translation coordinate system:

$$\mathbf{H}_{\mathrm{I}}^{*\,\mathrm{eff}} = -[\mathbf{I}^*\dot{\mathbf{I}}^*]/I^{*2}. \tag{32}$$

Then, taking (31) into account, we obtain from Hamilton's principle an expression for the effective "field" $\mathbf{H}_{\mathrm{I}}^{*\mathrm{eff}}$ (27). On the other hand, relation (32) can be regarded as an equation of motion for $\mathbf{I}^*$, for under the condition $|\mathbf{I}| = \mathrm{const}$ it is equivalent to Eq. (26).

We note that the expression for the force (30) can be obtained by using the energy conservation law, as was done by Akhiezer, Bar'yakhtar, and Peletminskiĭ[4]. It is necessary to take into account here the relation

$$\partial \mathcal{F}^*/\partial t = \rho DF^*/Dt - \nabla(\dot{\mathbf{u}}\mathcal{F}^*), \tag{33}$$

relation (22) written in terms of the derivatives D/Dt, relation (9), and Eq. (26).

3. We now consider a second case, which can be extrapolated to arbitrarily low frequencies, i.e., to thermodynamic quasistatic processes for the change in magnetization. In this case it is convenient to express the free energy $\mathbf{F}^{**}$ in the moving and rotating coordinate system. As shown by Landau and Lifshitz[2], such an expression should depend on $\dot{\boldsymbol{\omega}}$ (in the first approximation it should simply contain the gyromagnetic free energy as one of its terms):

$$F^{**} = F^{**}(\mathbf{m}^*, \mathbf{p}^*, \hat{\varepsilon}, \partial \mathbf{m}^*/\partial \mathbf{r}, \dot{\boldsymbol{\omega}}).$$

Since the sum $\mathcal{H} - \mathcal{F} = \rho^{**}(\mathbf{K}^{**} - \mathbf{F}^{**}) - 2\pi \mathbf{M}^{**2}$ should remain relativistically invariant, and since $\mathbf{F}^{**}$ differs from $\mathbf{F}^*$, it is also necessary to choose different values for the per-unit "kinetic energy" $\mathbf{K}^{**}$ in the moving and rotating coordinate system.

Under quasistatic local thermodynamic equilibrium, the rate of change of the magnetization should be equal to zero in the translation-rotation coordinate system. Therefore the "kinetic energy" and its variation should be determined by the rate of change $D\mathbf{I}^*/Dt$ of the mechanical moment $\mathbf{I}^*$ connected with the magnetization, calculated in the translation-rotation coordinate system. We assume, taking into account the remark made in Sec. 2 of the present article, that

$$\delta_1 K^{**} = \delta_2 K^{**} = -\mathbf{H}_{\mathrm{I}}^{**\,\mathrm{eff}}\delta_2 \mathbf{I}^{**}, \tag{34}$$

where $\mathbf{H}_{\mathrm{I}}^{**\mathrm{eff}}$ is some effective "field" for the vector I, calculated in the translation-rotation coordinate system and equal (with a minus sign) to the rate of change of the mechanical moment $\mathbf{I}^*$, calculated in the translation-rotation coordinate system:

$$\mathbf{H}_{\mathrm{I}}^{**\,\mathrm{eff}} = -\left[\mathbf{I}^* \frac{D\mathbf{I}^*}{Dt}\right]\frac{1}{I^{*2}}. \tag{35}$$

Varying the action again, we obtain the following relations:

A. Equations of state for $\mathbf{p}^*$:

$$\mathbf{E}^* = \partial F^{**}/\partial \mathbf{p}^*. \tag{36}$$

B. Expression for the effective field in the translation-rotation frame:

$$\mathbf{H}_{\mathrm{m}}^{**\,\mathrm{eff}} = \mathbf{H}^* - \frac{\partial F^{**}}{\partial \mathbf{m}^*} + \frac{\partial}{\partial x_j}\frac{\partial F^{**}}{\partial (\partial \mathbf{m}^*/\partial x_j)}, \tag{37}$$

connected with $\mathbf{H}_{\mathrm{I}}^{**\mathrm{eff}}$ by the relation (27).

C. Equation of motion for $\mathbf{u}$ (29), where

$$f_i = \frac{\partial}{\partial x_j}\left(\rho \frac{\partial F^{**}}{\partial \varepsilon_{ij}}\right) + \frac{1}{c}[\mathbf{jB}]_i + \rho_e E_i + \mathbf{P}\nabla_i \mathbf{E} + \mathbf{M}\nabla_i \mathbf{H} + \frac{1}{c}\rho \frac{d}{dt}([\mathbf{Em}]_i + [\mathbf{pB}]_i) - \frac{1}{2}\mathrm{rot}_i([\mathbf{HM}] + [\mathbf{EP}]) + \frac{1}{2}\mathrm{rot}_i\left(\rho \frac{d}{dt}\frac{\partial F^{**}}{\partial \dot{\boldsymbol{\omega}}}\right). \tag{38}$$

Relation (35) gives the equation of motion for the magnetization, which for $D\mathbf{I}^*/Dt \to 0$ goes into the equation of state

$$\mathbf{H}_{\mathrm{I}}^{**\,\mathrm{eff}} = 0 \text{ or } \mathbf{H}^* = \frac{\partial F^{**}}{\partial \mathbf{m}^*} - \frac{\partial}{\partial x_j}\frac{\partial F^{**}}{\partial (\partial \mathbf{m}^*/\partial x_j)}. \tag{39}$$

The equation of motion (35) can be regarded here as a particular case of a more general equation of motion, which takes into account absorption and the diffusion processes, and which is discussed in [11]:

$$(\mathbf{H}_{\mathrm{I}}^{**\,\mathrm{eff}})_i = \rho\left\{\left[\frac{D\mathbf{I}^*}{Dt}\mathbf{J}^*\right]_i \frac{1}{J^{*2}} + \beta_{ij}\frac{DI_j^*}{Dt} + D_{ijkl}\frac{\partial^2}{\partial x_k \partial x_l}\frac{DI_j^*}{Dt}\right\}. \tag{40}$$

4. Let us discuss the results obtained. In all cases we get for the dielectric polarization the equation of state (25) or (36). For the magnetization, in the case when $|\mathbf{I}| = \text{const}$, we have the equation of motion (26), and for the case when the modulus of the magnetization can vary, the equation of motion (35) [or (40)], which for $DI^*/Dt \to 0$ goes over into the equation of state (39). Recognizing that F** depends on $\dot{\boldsymbol{\omega}}$, we see that the equation of state (39) defines also the Barnet effect.

The force (30) and (38), which acts on the volume element, can be expressed, using Maxwell's equations (5) and (6) and the equation of motion (26) for I, in terms of the stresses and the rate of change of I:

$$f_i = \partial(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)})/\partial x_j + f_i^{(1)} - f_i^{(2)}, \tag{41}$$

where $\mathbf{f}^{(2)}$ is the variation of the momentum per unit volume of the electromagnetic field:

$$\mathbf{f}^{(2)} = \frac{1}{4\pi c}\frac{\partial}{\partial t}[\mathbf{DB}] - \rho\frac{d}{dt}([\mathbf{Em}] + [\mathbf{pB}]), \tag{42}$$

$$\sigma_{ij}^{(2)} = \frac{1}{8\pi}[D_iE_j + D_jE_i + B_iH_j + B_jH_i - \delta_{ij}(E^2 + H^2)]. \tag{43}$$

For the case $|\mathbf{I}| = \text{const}$

$$\sigma_{ij}^{(1)} = \rho\partial F^*/\partial\varepsilon_{ij}, \tag{44}$$

$$\mathbf{f}^{(1)} = -\tfrac{1}{2}\operatorname{rot}(\rho\dot{\mathbf{I}}). \tag{45}$$

For the case $|\mathbf{I}| \neq \text{const}$

$$\sigma_{ij}^{(1)} = \rho\partial F^{**}/\partial\varepsilon_{ij}, \tag{46}$$

$$\mathbf{f}^{(1)} = \tfrac{1}{2}\operatorname{rot}\left(\rho\frac{d}{dt}\frac{\partial F^{**}}{\partial\dot{\boldsymbol{\omega}}}\right). \tag{47}$$

Expression (41) for the force contains the hitherto disregarded term $\mathbf{f}^{(1)}$. The remaining terms coincide accurate to a small term $(\nabla\dot{\mathbf{u}})(\mathbf{E}\times\mathbf{M} + \mathbf{P}\times\mathbf{B})$ with Eq. (56.17) of [2]. This can be readily obtained by changing over in F from per-unit quantities to densities of these quantities, and by taking account of the fact that the thermodynamic potential used in [2] is a function of E and B, whereas in (44) or (46) we deal with F* or F**, which are functions of p and m.

Akhiezer et al.[4] obtained a somewhat different expression for the force, since they actually assumed

$$\frac{\partial\mathcal{F}}{\partial t} = \frac{\partial\mathcal{F}}{\partial\mathbf{M}}\frac{\partial\mathbf{M}}{\partial t} + \frac{\partial\mathcal{F}}{\partial\varepsilon_{ij}}\frac{\partial\varepsilon_{ij}}{\partial t} + \cdots$$

in place of (33), causing for them to obtain an additional term which in our notation has the form $M\nabla_i H_m^{*\text{eff}}$.

Let us show, finally, that the presence of a term $\mathbf{f}^{(1)}$ in the force does not contradict the angular momentum conservation law, and that in the case when $\dot{\boldsymbol{\omega}}$ and $\dot{\mathbf{I}}$ are homogeneous it gives rise to the Einstein–de Haas effect.

Let us assume first, for example, that

$$\rho\ddot{u}_i = \partial\sigma_{ij}/\partial x_j - \tfrac{1}{2}\operatorname{rot}_i(\rho\dot{\mathbf{I}}). \tag{48}$$

We take the vector product of both halves with r and integrate over the volume. Going over then to a surface integral and recognizing that $\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} = \sigma_{ji}$ we obtain

$$\int\rho([\mathbf{r}\ddot{\mathbf{u}}]_i + \dot{I}_i)\,dv = \oint\varepsilon_{ijk}x_j\sigma_{kl}dS_l + \frac{1}{2}\oint\rho[\mathbf{r}[\dot{\mathbf{I}}d\mathbf{S}]]_i. \tag{49}$$

For the case when $|\mathbf{I}| = \text{const}$, we obtain in place of $\dot{\mathbf{I}}$ in (49) the term $-\mathrm{d}(\partial F^{**}/\partial\dot{\boldsymbol{\omega}})/\mathrm{dt}$. In the left half of (49) we have in both cases the rate of change of the total angular momentum. It is expressed only in terms of the integral over the surface, which is necessary in order to satisfy the angular momentum conservation law.

If $\dot{\mathbf{I}}$ is homogeneous inside a bounded body on the surface of which there are no acting forces, then, recognizing that $\dot{\mathbf{I}} = 0$ on the outside of the surface bounding the body, we obtain an expression for the Einstein–de Haas effect

$$\int\rho[\mathbf{r}\ddot{\mathbf{u}}]\,dv = -\rho V\dot{\mathbf{I}}$$

or

$$\int\rho[\mathbf{r}\ddot{\mathbf{u}}]\,dv = \rho V\frac{\partial F^{**}}{\partial\dot{\boldsymbol{\omega}}},$$

where ρV is the mass of the body.

We note that (44) or (46) are equations of state. In the dynamic mode, for periodic processes in the presence of a constant magnetic field or magnetization, it is necessary to use in place of these equations the dynamic equations of state[12], in which account is taken of the fact, for example, that the tensor of the elastic moduli can have antisymmetrical terms, which are odd functions of the polarizing magnetic field or the polarization magnetization.

It is also necessary to take into account the singularities of the media to which the obtained system of equations is applied. For example, in good conductors, according to [2], it is meaningless to include in (6) the term $c^{-1}\partial D/\partial t$.

5. Let us investigate by way of an example the possible oscillations of an isotropic ferrodielectric medium magnetically-polarized (along the z axis) in some homogeneous constant magnetic field H_0.

We shall consider ultrasonic frequencies that are much lower than the frequency of ferromagnetic resonance. In this case we can use the equation of motion (29), Maxwell's equations (6) in

which we neglect the conduction and displacement currents, the equation of state (39), the expression for the force (41) in which we neglect the terms $\mathbf{f}^{(2)}$ and $\sigma_{ij}^{(2)}$, relation (47), and also the equation of state (46). We shall likewise disregard the density variation. In the free energy we shall take account of the elastic, magnetic, magnetoelastic, and gyromagnetic free energies:

$$\mathcal{F}^{**} = \tfrac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \tfrac{1}{2}\gamma_{mn}\mu_m\mu_n + h_{m,ij}\mu_m\varepsilon_{ij} - \lambda_{mn}\mu_m\dot{\omega}_n, \tag{50}$$

where $\mu = \mathbf{M} - \mathbf{M}_0$ is the deviation of the magnetization $\mathbf{M}$ from its value $\mathbf{M}_0$ in the absence of oscillations.

It can be assumed that the elastic modulus tensor c_{ijkl} and the gyromagnetic ratio tensor λ_{mn} are practically independent of the polarization magnetization $\mathbf{M}_0$. They have therefore the same form in a magnetically polarized isotropic medium as in a non-polarized isotropic medium, i.e., they have nonvanishing components of the type

$$c_{1111} = c_{2222} = c_{3333}, \quad c_{1122} = c_{2211} = \ldots = c_{3311},$$
$$c_{1212} = c_{2112} = \ldots = c_{1313} = {}^1/_2\,(c_{1111} - c_{1212});$$
$$\lambda_{ij} = \delta_{ij}\lambda. \tag{51}$$

On the other hand, the tensors of the magnetostriction constant $h_{m,ij}$ and the values of the reciprocal susceptibility γ_{mn} depend essentially on the polarization magnetization and have the following non-vanishing components [13]

$$h_{3,33}; \quad h_{3,11} = h_{3,22}; \quad h_{1,13} = h_{1,31} = h_{2,23} = h_{2,32};$$
$$\gamma_{11} = \gamma_{22}, \ \gamma_{33}. \tag{52}$$

The components of the tensor γ_{mn} are even, while those of the tensor $h_{m,ij}$ are odd functions of the polarization magnetization.

We shall consider excitations of the plane-wave type, i.e., we shall assume that all quantities vary like $\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})]$, and assume that the inhomogeneities of the magnetization of the medium, due to the domain structure, have characteristic dimensions which are much smaller than the excitation wavelength.

We consider first a ferromagnet in a demagnetized state, when $h_{m,ij} = 0$ and $\gamma_{mn} = \delta_{mn}\gamma$. Solving simultaneously the aforementioned equations with allowance for (50), we find that the following can propagate in the ferromagnet in this case: a longitudinal elastic wave with velocity

$$v_l^0 = (c_{1111}/\rho)^{1/2} \tag{53}$$

and two degenerate transverse elastic waves accompanied by magnetization oscillations, the dispersion of which, generally speaking, is not linear:

$$\omega = v_t^0 k\,(1 + \lambda^2 k^2/4\rho\gamma)^{-1/2}, \tag{54}$$

$$v_t^0 = (c_{1212}/\rho)^{1/2}. \tag{55}$$

Recognizing that in order of magnitude $\lambda \sim 10^{-7}$ Oe-sec and $\gamma \sim 10^{-2}$, we conclude that the nonlinearity can come into play at frequencies of the order of 10^{10} sec^{-1}, when the equation of state (39) can no longer be used. We shall therefore neglect the terms quadratic in λ.

However, the change in the magnetic moment, occurring in a plane perpendicular to the plane of polarization of the transverse elastic wave, may turn out to be appreciable. The amplitude of the magnetization oscillations is equal to

$$\mu_x^0 = -i\frac{\lambda\omega}{2\gamma}\left(\frac{\partial u_y}{\partial z}\right)^0, \qquad \mu_y^0 = i\frac{\lambda\omega}{2\gamma}\left(\frac{\partial u_x}{\partial z}\right)^0. \tag{56}$$

These magnetization oscillations can be detected at relatively low frequencies (for $\omega \sim 10^6$ sec^{-1} we have $\mu \sim 10\ \partial u/\partial z$ Gauss), for example by measuring the emf induced when μ is changed. This effect uncovers another possibility of measuring the gyromagnetic ratio.

We now consider a ferromagnet which is magnetically polarized (in the z direction). Let first the elastic waves propagate along the polarization magnetization. In this case the following can propagate: a longitudinal wave with propagation velocity

$$v_l = (c'_{1111}/\rho)^{1/2}, \tag{57}$$

where

$$c'_{1111} = c_{1111}\,[1 - h_{3,33}^2/c_{1111}\,(\gamma_{33} + 4\pi)], \tag{58}$$

and two circularly polarized transverse waves, the frequencies of which are determined from the equation

$$\rho\omega^2 - c_{1212}k^2 + (k^2/\gamma_{11})\,(h_{1,31} \pm \lambda\omega/2)^2 = 0. \tag{59}$$

It follows from (59) that

$$\omega_\pm = v_t\,(1 \mp \lambda h_{1,31}k/v_t\gamma_{11})\,k, \tag{60}$$

$$v_t = (c'_{1212}/\rho)^{1/2}; \qquad c'_{1212} = c_{1212}\,(1 - h_{1,31}^2/c_{1212}\gamma_{11}). \tag{61}$$

The more appreciable dispersion already observed here for the left-hand and right-hand polarized elastic waves should lead to the rotation of the plane polarization of the transverse elastic wave. The angle of rotation of the plane of polarization per unit length, as follows from (59), is equal to

$$\varphi = \tfrac{1}{2}\,(k_+ - k_-) = \lambda h_{1,31}\omega^2/2\rho v_t^3\gamma_{11}. \tag{62}$$

Putting $\gamma \sim 10^{-2}$, $h_{1,31} \sim 10^4$ Oe, $\rho \sim 10$ g/cm^3, and $v_t \sim 10^5$ cm/sec we find that when ω is of the order of 10^7 sec^{-1} we can expect a rotation of the order of $\varphi \sim 10^{-3}$ rad/cm.

The magnetization oscillations accompanying the longitudinal wave should have an amplitude

$$\mu_z^0 = -\frac{h_{3,33}}{\gamma_{33}+4\pi}\left(\frac{\partial u_z}{\partial z}\right)^0, \quad (63)$$

while those accompanying the transverse circularly-polarized wave should have an amplitude

$$\mu_x^0 \pm i\mu_y^0 = -\frac{h_{1,31} \pm \lambda\omega/2}{\gamma_{11}}\left[\left(\frac{\partial u_x}{\partial z}\right)^0 \pm i\left(\frac{\partial u_y}{\partial z}\right)^0\right]. \quad (64)$$

Finally, let us consider the propagation of elastic waves in a direction perpendicular to the polarization magnetization (for concreteness, in the x direction). In this case three elastic waves can propagate: one linearly polarized and two with elliptic polarizations, the very weak ellipticity of which we can neglect.

The rate of propagation of the longitudinal wave is

$$v_l = (c''_{1111}/\rho)^{1/2}, \quad (65)$$

$$c''_{1111} = c_{1111}(1 - h^2_{3,11}/c_{1111}\gamma_{33}). \quad (66)$$

It should be accompanied by magnetization oscillations with amplitude

$$\mu_z^0 = -\frac{h_{3,11}}{\gamma_{33}}\left(\frac{\partial u_x}{\partial x}\right)^0. \quad (67)$$

The elastic transverse wave, polarized in a direction perpendicular to the polarization magnetization, should propagate with a velocity

$$v_t^{\perp} = v_t^0 \quad (68)$$

and is accompanied by magnetization oscillations with amplitude

$$\mu_z^0 = i\frac{\lambda\omega}{2\gamma_{33}}\left(\frac{\partial u_y}{\partial x}\right)^0. \quad (69)$$

The elastic transverse wave polarized in the direction parallel to the polarization magnetization should propagate with velocity

$$v_t^{\parallel} = (c''_{1212}/\rho)^{1/2}, \quad (70)$$

$$c''_{1212} = c_{1212}(1 - h^2_{1,31}/c_{1212}(\gamma_{11} + 4\pi)) \quad (71)$$

and is accompanied by magnetization oscillations in the plane perpendicular to the polarization magnetization with amplitudes

$$\mu_y^0 = -i\frac{\lambda\omega}{2\gamma_{11}}\left(\frac{\partial u_z}{\partial x}\right)^0, \quad (72)$$

$$\mu_x^0 = -\frac{h_{1,31}}{\gamma_{11}+4\pi}\left(\frac{\partial u_z}{\partial x}\right)^0. \quad (73)$$

Thus, even in a polycrystalline isotropic ferromagnet we can expect the following in the case of propagation of ultrasound:

1) Rotation of the plane of polarization of the ultrasound, due to the simultaneous action of gyromagnetism and magnetostriction.

2) Appearance of forced magnetization oscillations accompanying the elastic waves due to both magnetostriction and gyromagnetism.

3) Anisotropy of the velocity of sound and dependence of the velocity of sound on the magnetic state of the ferromagnet.

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31