REGGE POLES IN THE PROBLEM OF THE QUASICLASSICAL POTENTIAL WELL AT ENERGIES BELOW THE BOTTOM OF THE WELL

I. A. FOMIN

Moscow Physico-technical Institute

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It is shown that, for energies below the bottom of the well, poles which are arranged on the real axis in pairs collide and enter the complex plane. The position of the collision points is found and a formula for the behavior of the poles for $E \to -\infty$ is derived.

THE distribution and motion of the Regge poles has been investigated in a paper by Patashinskiĭ, Pokrovskiĭ, and Khalatnikov[1] (PPK) for a quasi-classical square well potential

$$U(r)=\begin{cases}-U_0, & r<a\\ 0, & r>a\end{cases}, \qquad mU_0a^2 \gg 1. \tag{1}$$

That paper was mostly concerned with the physical domain, $-U_0 < E < +\infty$. It is of interest to investigate the region $E < -U_0$ so as to obtain the complete picture of the motion of the poles for the variation of the energy between $-\infty$ and $+\infty$. The motion of the poles for all real energies has already been investigated by Bollini and Giambiagi.[2] However, the method of PPK allows a more detailed discussion.

The equations which determine the poles have the form

$$x_1J_\nu'(x_1)/J_\nu(x_1)=xH_\nu^{(1)\prime}(x)/H_\nu^{(1)}(x),$$
$$x_1^2=2m(E+U_0)a^2, \qquad x^2=2mEa^2, \qquad \nu=l+\tfrac{1}{2}, \tag{2}$$

where J_ν and $H_\nu^{(1)}$ are Bessel and Hankel functions respectively. The roots of this equation were found by the same method as in PPK.

It was shown in the cited paper that, for an energy close to the depth of the potential, there exists a series of poles which are distributed on the real axis close to the negative integers (the "physical" series). Their positions are given by the formula

$$\nu=-n-\frac{1+\sqrt{1-(x/n)^2}}{1-\sqrt{1-(x/n)^2}}\frac{n}{(n!)^2}\left(\frac{x_1}{2}\right)^{2a}. \tag{3}$$

One sees from (3) that at $E = -U_0$ ($x_1 = 0$) the poles coincide with the negative integers, and for $E < -U_0$ the poles move on the real axis from the odd points to the left and from the even points to the right. Equation (3) holds for $|x_1/n| \ll 1$; then $|\nu+n| \ll 1$.

As the energy continues to decrease the poles move pairwise towards each other and collide. We obtain from (2) the following equation which describes the positions of the poles in the vicinity of the collision point:

$$\sin\nu\pi=\frac{1}{2}\frac{y+1}{y-1}\exp\left[2\sqrt{1+\beta^2}\left(\nu+\frac{|x_1|}{\beta}\right)\right];$$
$$y=[(\nu^2-x^2)/(\nu^2-x_1^2)]^{1/2}, \tag{4}$$

where $\beta \cong 0.66$ is the root of the equation $(1+\beta^2)^{1/2} = \coth(1+\beta^2)^{1/2}$.

The position of the poles is given by the abscissae of the crossing points of the sine and the exponential. With increasing $|x_1|$ these points come closer. The points collide when the exponential and the particular crest of the sine wave touch. We thus find the collision point from the condition of a common tangent

$$\tan\nu_c\pi=\pi/2\sqrt{1+\beta^2}, \qquad \nu_c \cong -2n+0.29.$$

Close to ν_c the motion of the poles is given by

$$\nu \cong \nu_c \pm 0.8\sqrt{i(x_1-x_1^c)},$$

where x_1^c is the value of x_1 at which the particular pair of poles collide.

After the collision the poles leave the real axis. They now lie on complex conjugate points. It is therefore sufficient to consider their motion in the upper half-plane only. The poles which are already in the complex domain can be found from the equation

$$\int_\nu^{x_1}\sqrt{1-\frac{\nu^2}{\xi^2}}\,d\xi=\frac{\pi}{4}+n\pi+\tan^{-1}(-iy). \tag{5}$$

At a given energy they lie in the ν-plane close to the line

$$\operatorname{Im}\int_\nu^{x_1}\sqrt{1-\frac{\nu^2}{\xi^2}}\,d\xi=0.$$

This line emerges from x_1 at an angle $2\pi/3$ to the positive direction of the imaginary axis and terminate at the point $\nu = -x_1/i\beta$, having an angle $\approx 125°$ with the positive direction of the real axis.

Besides the considered series of "physical" poles there exists another, "unphysical" series. For negative energies it begins close to the point x and leads up along the imaginary axis. With increasing $|E|$ the points x and x_1 approach each other. When $|x_1|^{4/3} \gg 2mU_0a^2$ the first poles of both series can be described by the common formula:

$$\nu = x - \frac{3^{2/3}}{2} x^{1/3}\left(n\pi + \frac{1}{2i}\ln\left|\frac{4(x-\nu)}{x_1 - x}\right|\right)^{2/3}; \qquad (6)$$

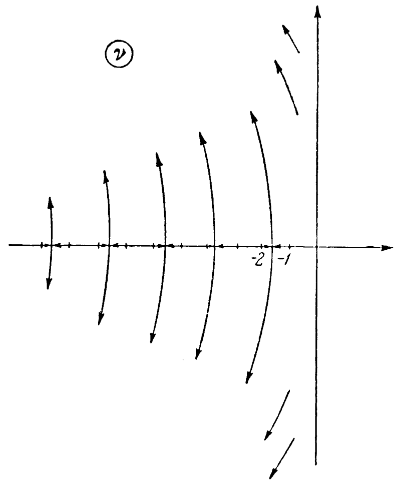
it holds for $|x-\nu|/|x| \ll 1$, $|x-\nu|/|x^{1/3}| \gg 1$.

The poles of the physical series are given by positive n, and of the unphysical series by negative n. Thus both series are related. For $E \to -\infty$ the poles leave for infinity according to

$$\nu = i|x| + \tfrac{1}{2}(i - \sqrt{3})\,(|x|\ln^2|x|/2)^{1/3}.$$

An analogous formula has already been obtained by a different method (see [2]); however, the second term has there an incorrect factor.

The trajectories of the poles are shown in the figure for a change of the energy E from U_0 to $-\infty$.



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[1] Patashinskiĭ, Pokrovskiĭ, and Khalatnikov, JETP **44**, 2062 (1963), Soviet Phys. JETP **17**, 1387 (1963).

[2] C. G. Bollini and J. J. Giambiagi, Nuovo cimento **28**, 356 (1963).

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