# THEORY OF RELAXATION PROCESSES IN A UNIAXIAL ANTIFERRODIELECTRIC

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The relaxation times of antiferromagnetic dielectrics, due to decay of a spin wave into two spin waves (and the reverse processes) and also to interaction between spin waves and phonons, are calculated. The calculations are carried out for two limiting cases of a uniform magnetic field $H_0$: in the absence of any magnetic field and for magnetic fields in which dipole-dipole interaction can be neglected in the spin wave spectrum.

RELAXATION processes in antiferromagnets have been investigated relatively little. These processes, which lead to a broadening of the line of antiferromagnetic resonance and cause an equilibrium value of magnetization and temperature to be established in the system, are the result of interactions of spin waves with one another and with phonons.

The purpose of the present work is to find the temperature dependence of the relaxation time in the interaction of spin waves with one another (with non-conservation of the number of spin waves) and with phonons. Unlike a ferromagnet, where each type of interaction (exchange, relativistic, and others) can be set in correspondence with a definite macroscopic parameter, the establishment of the equilibrium value of which is due to this interaction in a definite region of temperatures (see [1]), in antiferromagnets the relaxation picture is much more complicated.

## 1. DECAY OF A SPIN WAVE INTO TWO WAVES AND INVERSE PROCESSES

The Hamiltonian of the interaction of spin waves with one another contains terms of third and higher order in the spin-wave creation and annihilation operators. It turns out that in addition to the terms of lowest (third) order, it is necessary to take into account in the interaction Hamiltonian also fourth-order terms. This is connected with the fact that the terms have a different origin: the third-order terms are connected with the relativistic dipole interactions of the spins, while those of fourth order are due to magnetic anisotropy and exchange forces. The large value of the exchange interaction constant leads to the need for taking fourth-order terms into account. The average probabilities of the fourth-order processes were calculated previously [2].

Let us consider the third order processes: decay of a spin wave into two and the inverse processes. The Hamiltonian of these processes describes magnetic dipole-dipole interactions of the spin and has in the coordinate representation the form

$$\mathcal{H}^{(3)} = -3\iint \frac{d\mathbf{r}\,d\mathbf{r}'}{R^5}\{(\mathbf{R}\mathbf{M}_1(\mathbf{r}))(\mathbf{R}\mathbf{M}_1(\mathbf{r}')) + 2(\mathbf{R}\mathbf{M}_1(\mathbf{r}))(\mathbf{R}\mathbf{M}_2(\mathbf{r}')) + (\mathbf{R}\mathbf{M}_2(\mathbf{r}))(\mathbf{R}\mathbf{M}_2(\mathbf{r}'))\}, \tag{1}$$

where $\mathbf{M}_1$ and $\mathbf{M}_2$ are the magnetic moments of the first and second sublattices and $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. The oscillations of the moments are assumed to be long-wave ($ak \ll 1$, where k is the wave vector and a the lattice constant).

In the second-quantization representation, expression (1) is written in the form

$$\mathcal{H}^{(3)} = \sum_{\substack{\mathbf{p},\,\mathbf{q},\,\mathbf{f}=\mathbf{p}+\mathbf{q}\\ i,\,k,\,l=1,2}} \Phi^{i;\,kl}_{i;\,\mathbf{pq}}\, a_{\mathbf{f}i} a^{+}_{\mathbf{p}k} a^{+}_{\mathbf{q}l} + \text{c. c.},$$

where the indices i, k, and $l$ number the energy branch of the spin waves and assume two values for an antiferromagnet with two sublattices. The functions $\Phi$ are of the form

$$\begin{aligned}\Phi_{f;\,pq} = {} & \pi\mu V(2\mu M_0)^{1/2}\{\sin 2\theta_f[(u_f^1 + v_f^2)e^{i\varphi_f} \\ & + (v_f^1 + u_f^2)e^{-i\varphi_f}](v_p^{*2}u_q^{*2} \\ & + u_p^{*2}v_q^{*2} - v_p^{*1}u_q^{*1} - u_p^{*1}v_q^{*1}) + \sin 2\theta_p[(u_p^{*1} + v_p^{*2})e^{-i\varphi_p} \\ & + (v_p^{*1} + u_p^{*2})e^{i\varphi_p}](u_q^{*2}u_f^2 + v_q^{*2}v_f^2 - u_q^{*1}u_f^1 - v_q^{*1}v_f^1) \\ & + \sin 2\theta_q[(u_q^{*1} + v_q^{*2})e^{-i\varphi_q} \\ & + (v_q^{*1} + u_q^{*2})e^{i\varphi_q}](u_p^{*2}u_f^2 + v_p^{*2}v_f^2 - u_p^{*1}u_f^1 - v_p^{*1}v_f^1)\}. \end{aligned} \tag{2}$$

Here V—volume of the body, $\mu$—Bohr magneton, $M_0$—equilibrium value of the magnetic moment of the sublattice, $\theta_f$ and $\varphi_f$—polar and azimuthal angles of the wave vector f relative to the easiest magnetization axis. The external constant magnetic field $\mathbf{H}_0$ is directed along the easiest magnetization axis. The values of the amplitudes $u_p \equiv u_{pk}$ and $v_p \equiv v_{pk}$ can be found in the paper by Kaganov and the author [3].

In the general case, the amplitudes u and v have an extremely complicated form. Relatively simple expressions can be obtained for them in two limiting cases: in the absence of a magnetic field and in relatively strong magnetic fields, in which we can neglect the dipole-dipole terms in the spectrum of the spin waves. We shall consider henceforth these two limiting cases with respect to the field $\mathbf{H}_0$.

1. Field $H_0 = 0$. The law of dispersion of spin waves in the absence of a magnetic field is of the form

$$\varepsilon_1 = \sqrt{\varepsilon_0^2 + \Theta_c^2 (ak)^2}\,[1 + (ak)^2/2 + \beta/4\varkappa],$$

$$\varepsilon_2 = \sqrt{\varepsilon_0^2 + \Theta_c^2 (ak)^2}\,[1 + (ak)^2/2 + \beta/4\varkappa + 2\pi\varkappa^{-1}\sin^2\theta], \quad (3)$$

where $\Theta_c$—temperature on the order of the Curie temperature, $\varepsilon_0$—activation energy of the spin wave, $\beta$—magnetic anisotropy constant, and $\kappa \sim \Theta_c/\mu M_0 \gg 1$—constant of homogeneous exchange interaction; $\theta$—angle between the wave vector of the spin wave and the easiest magnetization axis.

If we neglect in the square brackets of (3) the small terms compared with unity, i.e., if we choose a dispersion law in the form $\varepsilon = \sqrt{\varepsilon_0^2 + \Theta_c^2 (ak)^2}$, then the energy and momentum conservation law for the triple processes will not be satisfied. The average probabilities of triple processes were obtained earlier [2] with small dipole-dipole additions taken into account in the dispersion law [the term $2\pi\kappa^{-1}\sin^2\theta$ in (3)]. However, a more detailed analysis shows that the principal role is played by the small addition in the form $(ak)^2$ in the dispersion law.

The decay of a spin wave with quasimomentum $\mathbf{f}$ into two other waves is a threshold process, i.e., the probability of such a process is

$$W_{\mathbf{f}} = \frac{1}{\tau_{\mathbf{f}}} = \begin{cases} 0, & f \leqslant f_{\text{thr}} \\ W_{\mathbf{f}} & f > f_{\text{thr}}, \end{cases}$$

where $f_{\text{thr}} = a^{-1}(2\varepsilon_0/\Theta_c)^{1/2}$—threshold value of the quasi momentum of the decaying spin wave.

Of the six possible processes wherein a spin wave of a given sort decays into two other waves, the matrix elements of the interaction Hamiltonian for four processes are equal to zero at the threshold point ($f = f_{\text{thr}}$) (forbidden transitions). It must be noted that the decay of a spin wave of the first sort into two spin waves of the same sort is forbidden for all values of the wave vector $\mathbf{f}$ of the decaying wave (the matrix elements of the Hamiltonian of such an interaction are in general equal to zero).

At the threshold the matrix elements of the interaction Hamiltonian differ from zero for two processes: the decay of a spin wave of the first sort into two spin waves of the second sort, and the decay of a wave of the second sort into waves of the first and second sorts. For these, the functions $\Phi_{\mathbf{f};\mathbf{pq}}$, defined in (2), take the form

$$\Phi_{\mathbf{f};\,\mathbf{pq}}^{1;\,22} = -\frac{\pi\mu}{4}\left(\frac{2\mu M_0}{\Theta_c V}\right)^{1/2}(\varepsilon_{f1}\varepsilon_{p2}\varepsilon_{q2})^{-1/2}$$
$$\times \{\sin 2\theta_p e^{-i\varphi_p}\,[1 + e^{2i(\varphi_f - \varphi_q)}]\varepsilon_{p2}(\varepsilon_{q2} + \varepsilon_{f1})$$
$$+ \sin 2\theta_q e^{-i\varphi_q}\,[1 + e^{2i(\varphi_f - \varphi_p)}]\,\varepsilon_{q2}(\varepsilon_{p2} + \varepsilon_{f1})\},$$

$$\Phi_{\mathbf{f};\,\mathbf{pq}}^{2;\,12} = -\frac{\pi\mu}{4}\left(\frac{2\mu M_0}{\Theta_c V}\right)^{1/2}(\varepsilon_{f2}\varepsilon_{p1}\varepsilon_{q2})^{-1/2}\{\sin 2\theta_q e^{-i\varphi_q}\,[1 + e^{2i(\varphi_f - \varphi_p)}]$$
$$\times\,\varepsilon_{q2}(\varepsilon_{p1} + \varepsilon_{f2})$$
$$+ \sin 2\theta_f e^{i\varphi_f}(e^{-2i\varphi_p} + e^{-2i\varphi_q})\,\varepsilon_{f2}(\varepsilon_{q2} - \varepsilon_{p1})\}.$$

These processes turn out to be the most probable, i.e., the lifetime of the spin wave of the first sort is determined by the process of its decay into two waves of the second sort, and for the spin wave of the second sort the most probable decay is into waves of different sorts.

The probabilities of these processes near the threshold ($f \sim f_{\text{thr}}$) and away from it ($f \gg f_{\text{thr}}$) will be

$$W_{\mathbf{f}}^{1;\,22} \sim W_{\mathbf{f}}^{2;\,12}$$
$$= \frac{1}{\tau_{\mathbf{f}}} \sim \begin{cases} \dfrac{\Theta_c}{\hbar}\left(\dfrac{\mu M_0}{\Theta_c}\right)^2 \sin^2 2\theta_f\,(af_{\text{thr}})^{5/2}\,\text{cth}\,\dfrac{\Theta_c a f_{\text{thr}}}{4T}(af - af_{\text{thr}})^{1/2}, & f \sim f_{\text{thr}}, \\ \dfrac{T}{\hbar}\left(\dfrac{\mu M_0}{\Theta_c}\right)^2 \sin^2 2\theta_f\,(af)^2 \ln\dfrac{f}{f_{\text{thr}}}, & f \gg f_{\text{thr}}. \end{cases} \quad (4)^*$$

Averaging these probabilities over the wave vector $\mathbf{f}$ we obtain for the average probabilities of triple processes

$$\overline{W}^{(3)} = \sum_{\mathbf{f}}\frac{n_{\mathbf{f}}}{\tau_{\mathbf{f}}} \Big/ \sum_{\mathbf{f}} n_{\mathbf{f}}$$

the following expressions:

$$\overline{W}^{(3)} \sim \frac{\Theta_c}{\hbar}\left(\frac{\mu M_0}{\Theta_c}\right)^2\left(\frac{\varepsilon_0}{\Theta_c}\right)^{9/4}\left(\frac{\Theta_c}{T}\right)^{3/2}\exp\left[\frac{-(2\varepsilon_0\Theta_c)^{1/2}}{T}\right],$$
$$T \ll (2\varepsilon_0\Theta_c)^{1/2}; \quad (5a)$$

*cth = coth

$$\overline{W}^{(3)} \sim \frac{\Theta_c}{\hbar}\left(\frac{\mu M_0}{\Theta_c}\right)^2\left(\frac{T}{\Theta_c}\right)^3 \ln\frac{T}{(2\varepsilon_0\Theta_c)^{1/2}}, \quad (2\varepsilon_0\Theta_c)^{1/2} \ll T \ll \Theta_c. \tag{5b}$$

2. Field $H_0 \neq 0$. In this case in fields $\varepsilon_0\pi M_0/\Theta_c < \mu H_0 \ll \varepsilon_0$ for $T \ll T_1 = [(\mu H_0\Theta_c)^2 - (\pi\mu M_0\varepsilon_0)^2]^{1/2}(\pi\mu M_0)^{-1}$ we can neglect the dipole dipole terms in the general dispersion law (see [4]). Then the law of spin wave dispersion will be of the form

$$\varepsilon_{1,2} = \sqrt{\varepsilon_0^2 + \Theta_c^2(ak)^2} \mp \mu H_0. \tag{6}$$

The triple processes are also threshold processes in this case.

At sufficiently high temperatures ($T \gg T_0$, see below), there will be two out of the six possible processes of the decay of a spin wave into two, which are the most probable and have a decaying wave quasimomentum with threshold value $f_{thr} = 3\varepsilon_0^2/2a\Theta_c\mu H_0$. One is the decay of the spin wave of the first sort into spin waves of the same sort, while the other is the decay of the spin wave of the second sort into waves of different sorts. The average probabilities of these processes are of the same order of magnitude

$$\overline{W}^{1;11} \sim \overline{W}^{2;12} = 1/\tau$$

and have the form

$$\frac{1}{\tau} \sim \frac{\Theta_c}{\hbar}\left(\frac{\mu M_0}{\Theta_c}\right)^2\left(\frac{T_0}{\Theta_c}\right)^3\left(\frac{T_0}{T}\right)^{3/2} e^{-T_0/T}, \quad T \ll T_0; \tag{7a}$$

$$\frac{1}{\tau} \sim \frac{\Theta_c}{\hbar}\left(\frac{\mu M_0}{\Theta_c}\right)^2\left(\frac{T}{\Theta_c}\right)^3 \ln\frac{T}{T_0}, \quad T_0 \ll T \ll T_1 \approx \frac{H_0\Theta_c}{\pi M_0}, \tag{7b}$$

where $T_0 = 3\varepsilon_0^2/2\mu H_0$, and the field $H_0$ satisfies the conditions

$$\varepsilon_0(3\pi\mu M_0/2\Theta_c)^{1/2} \ll \mu H_0 \ll \varepsilon_0.$$

At a temperature $T \ll T_0$, the most probable will be the decay of a spin wave of the second sort into two waves of the first sort, since the threshold value of the quasimomentum of the decaying wave of these processes is ⅓ as large as the threshold value of the corresponding quasimomentum for the processes considered above.

In Fig. 1, the solid curve shows the dependence $T_1(H_0)$ while the dashed curve shows the dependence $T_0(H_0)$. The temperature and field region under consideration lies below the curve $T_1(H_0)$ and to the left of the vertical straight line $\mu H_0 = \varepsilon_0$[1]. The shaded portions of this region are

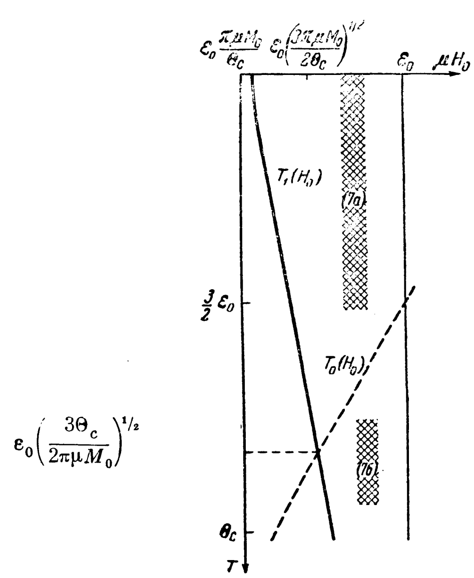


FIG. 1

those in which it is possible to obtain limiting expressions (7a) and (7b) for the relaxation times.

A comparison of the obtained formulas (5a), (5b) and (7a), (7b) with the average probabilities of the fourth-order processes, calculated in [2] [2]), shows that in the entire temperature region $T \ll \Theta_c$ the probability of the decay of one spin wave into two is much smaller than the probability of the scattering of a spin wave by a spin wave due to exchange and anisotropic interactions. Unlike a ferromagnet, where at sufficiently low temperatures the exchange interactions become insignificant and cede the main role in the establishment of equilibrium to the triple (relativistic) processes, in an antiferromagnet even at low temperatures the probability of triple processes remains considerably smaller than the probabilities of exchange and anisotropic interactions, owing to the presence of a large threshold energy. (The probability of the anisotropic and exchange processes is of the order of $\exp(-2\varepsilon_0/T)$, while that of the processes of decay of spin waves into two is $\overline{W}^{(3)} \sim \exp[-(2\varepsilon_0\Theta_c)^{1/2}/T]$, or else $\overline{W}^{(3)} \sim \exp(-T_0/T)$ [see (5) and (7)], with $T_0$ and $(2\varepsilon_0\Theta_c)^{1/2} \gg \varepsilon_0$.)

## 2. INTERACTION OF SPIN WAVES WITH PHONONS

The Hamiltonian of the interactions between spin waves and phonons can be constructed in

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1) We note that in Fig. 1 the scale is not uniform for the the sake of convenience. Actually, if drawn to scale, the disregarded region between the curve $T_1(H_0)$ and the temperature axis would turn out to be very narrow ($T_1 \sim \Theta_c$ for $\mu H_0 \sim \pi\mu M_0$).

2) In [2] are calculated the average probabilities of the fourth-order processes at high temperatures $T \gg \epsilon_0$. At low temperatures ($T \ll \epsilon_0$) these probabilities are of the order of $\exp(-2\epsilon_0/T)$.

analogy with the phenomenological Hamiltonian of such interactions in a ferrodielectric[1], by expanding the energy density of the antiferromagnet in powers of the deformation tensor $\xi_{ik}$, which describes the lattice vibrations. At low temperatures, it is possible to confine oneself in the Hamiltonian of the spin-phonon interactions to terms which are linear in $\xi_{ik}$ and quadratic in the deviations of the magnetic moments of the sublattices from their equilibrium values

$$\mathcal{H}^{s.ph} = \int \Big\{ \varkappa \mathbf{M}_1\mathbf{M}_2\xi_{ii} + \Lambda\xi_{ik}\left(\frac{\partial \mathbf{M}_1}{\partial x^i}\frac{\partial \mathbf{M}_1}{\partial x^k} + \frac{\partial \mathbf{M}_2}{\partial x^i}\frac{\partial \mathbf{M}_2}{\partial x^k}\right)$$

$$+ 2\Lambda_{12}\xi_{ik}\frac{\partial \mathbf{M}_1}{\partial x^i}\frac{\partial \mathbf{M}_2}{\partial x^k} + \Delta\xi\quad\left(\frac{\partial \mathbf{M}_1}{\partial x^i}\right)^2 + \Delta\xi_{kk}\left(\frac{\partial \mathbf{M}_2}{\partial x^i}\right)^2$$

$$+ 2\Delta_{12}\xi_{kk}\frac{\partial \mathbf{M}_1}{\partial x^i}\frac{\partial \mathbf{M}_2}{\partial x^i} + \gamma\xi_{ik}(M_i^1M_k^1 + M_i^2M_k^2)$$

$$+ 2\gamma_{12}\xi_{ik}M_i^1M_k^2\Big\}\, d\mathbf{r},$$

$$\xi_{ik} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x^k} + \frac{\partial \xi_k}{\partial x^i}\right) \tag{8}$$

($\xi$—displacement vector).

All the terms in the Hamiltonian (8), except for the last two, describe the magnetostriction energy of the crystal for homogeneous and inhomogeneous magnetization due to the exchange forces, $\kappa$—constant of homogeneous exchange interaction; $\Lambda$, $\Lambda_{12}$, $\Delta$, and $\Delta_{12}$ are the inhomogeneous exchange constants, and it is convenient to introduce the following notation: $\Lambda = \lambda a^2/4$ and $\Delta = \delta a^2/4$, where $\lambda$, $\delta \sim \kappa \sim \Theta_C/\mu M_0 \gg 1$. The last two terms in the Hamiltonian (8) with coefficients $\gamma$ and $\gamma_{12}$ represent part of the magnetostriction energy of the body, due to small relativistic (spin-orbit) interactions.

We expand the operators of the moment in the Hamiltonian (8) in terms of the spin wave creation and annihilation operators ($a^+$, $a$), and we expand the displacement operator $\boldsymbol{\xi}$ in terms of the phonon creation and annihilation operators ($b^+$, $b$). We have

$$\boldsymbol{\xi} = \sum_{\mathbf{f},s}\sqrt{\frac{\hbar}{2\rho V\omega_{\mathbf{f}s}}}\,\mathbf{P}_{\mathbf{f}s}\left(b_{\mathbf{f}s}(t)\,e^{i\mathbf{f}\mathbf{r}} + b_{\mathbf{f}s}^{+}(t)\,e^{-i\mathbf{f}\mathbf{r}}\right),$$

where $b_{fs}(t) = b_{fs}\exp(-i\omega_{fs}t)$, $\rho$ is the density of the matter, $\mathbf{P}_{fs}$ is the unit vector of the s-th polarization, and $\mathbf{f}$ and $\omega_{fs}$ are the wave vector and oscillation frequency of the phonon with polarization s. For longitudinally polarized sound $\omega_{fl} = c_l f$, while for transversely polarized sound $\omega_{ft} = c_t f$ ($c_l$ and $c_t$ are the velocities of the longitudinally and transversely polarized sound, respectively, with $c_l > c_t$).

After expanding the moments and displacements in terms of the spin wave and the phonon creation and annihilation operators the interaction Hamiltonian (8) is written in the form

$$\mathcal{H}^{s.ph} = \sum_{\substack{\mathbf{p},\mathbf{q},\mathbf{f},s\\ i,k=1,2}}\left(\Psi^{is;k}_{\mathbf{pf};\mathbf{q}}a^{+}_{\mathbf{q}k}a_{\mathbf{p}i}b_{\mathbf{f}s}\delta(\mathbf{p}+\mathbf{f}-\mathbf{q})\right.$$

$$\left. + \Psi^{s,ik}_{\mathbf{f};\mathbf{pq}}a^{+}_{\mathbf{p}i}a^{+}_{\mathbf{q}k}b_{\mathbf{f}s}\delta(\mathbf{p}+\mathbf{q}-\mathbf{f})\right) + \text{c.c.} \tag{9}$$

The first term in (9) and the corresponding complex conjugate term describe processes of absorption (or emission) of a phonon by a spin wave. The second term describes the decay of the phonon into two spin waves, and the corresponding complex-conjugate term describes the conversion of two spin waves into a phonon.

### A. Absorption (Emission) of a Phonon by a Spin Wave

The processes of interaction of spin waves with phonons will exhibit different behaviors, depending on the magnitude of the constant external field $\mathbf{H}_0$ in which the antiferromagnet is situated.

We consider the case when $H_0 \neq 0$. As before, we consider sufficiently strong magnetic fields ($H_0 > \varepsilon_0\pi M_0/\Theta_C$), in which the spin-wave dispersion is described by formula (6).

The processes of absorption (emission) of a phonon by a spin wave correspond to the following energy and momentum conservation laws:

$$\varepsilon_i(\mathbf{p}) + \hbar c_s f - \varepsilon_k(\mathbf{q}) = 0, \qquad \mathbf{p} + \mathbf{f} - \mathbf{q} = 0.$$

An investigation of these conservation laws with the spin-wave dispersion law (6) shows that only the absorption of a phonon by a spin wave of the first sort, with formation of waves of the second sort (or the inverse processes) can occur both when $\Theta_C \geq \Theta_d$, and when $\Theta_C < \Theta_d$ (where $\Theta_d = \hbar c/a$ is a temperature on the order of the Debye temperature). It was shown in[5] that this process is a threshold process with respect to the phonon frequency, and the threshold value of this frequency is $\omega_{thr} \sim 2gH_0/(1 + c'\Theta_C/\Theta_d)$ ($c' = \text{const} \sim 1$). It is obtained at relatively high values of the wave vectors of the spin wave ($ak \sim 1$), where the phenomenological analysis is not applicable.

All the remaining processes can take place only if the Curie temperature ($\Theta_C$) exceeds the Debye temperature ($\Theta_d$). A comparison of the values of the functions $\Psi^{i;s;k}_{pf;q}$ for all possible processes in the considered limiting case with respect to the field shows that of all the processes involving absorption (emission) of a phonon by spin waves, the most probable are the following two: absorption (emission) of a phonon by a spin wave of the first sort with formation of a wave of the same sort, and absorption (emission) of a phonon by a spin wave of the second sort with formation of a wave of the second sort, i.e., it is most probable that the spin wave, after absorbing (or emitting) a phonon, does not change its sort (its dispersion law).

The functions $\Psi$ for these processes will be the following:

$$\Psi^{1s;1}_{\mathbf{pf;q}} = \Psi^{2s;2}_{\mathbf{pf;q}} = \tfrac{1}{2} i(\mu M_0/\Theta_c)(\hbar/2\rho V\omega_{\mathbf{f}s})^{1/2}$$
$$\times \{[\varepsilon_0^2 + \Theta_c^2(ap)^2][\varepsilon_0^2 + \Theta_c^2(aq)^2]\}^{-1/4}$$
$$\times \{\varkappa(\mathbf{fP}_{\mathbf{f}s})\sqrt{\varepsilon_0^2+\Theta_c^2(ap)^2}\sqrt{\varepsilon_0^2+\Theta_c^2(aq)^2}$$
$$+ (a\Theta_c)^2[(\delta-\delta_{12})(\mathbf{f}P_{\mathbf{f}s})(\mathbf{pq})$$
$$+\tfrac{1}{2}(\lambda-\lambda_{12})[(\mathbf{P}_{\mathbf{f}s}\mathbf{p})(\mathbf{fq}) + (\mathbf{P}_{\mathbf{f}s}\mathbf{q})(\mathbf{fp})]]$$
$$+ 2\Theta_c^2(\gamma-\gamma_{12})(\mathbf{fP}_{\mathbf{f}s} - 3f_zP^z_{\mathbf{f}s})\}. \qquad (10)$$

In the derivation of (10) it was assumed that, for constant $\gamma$ and $\gamma_{12}$, the following condition is satisfied

$$|\gamma-\gamma_{12}| \gg |\gamma+\gamma_{12}|(T/\Theta_c)^2.$$

Let us consider first the longitudinally polarized sound. The average probability that the spin wave will absorb a longitudinally polarized phonon is

$$\overline{W}^l_s = \frac{1}{\overline{\tau}^l_s} = \sum \frac{n^0_{\mathbf{p}}}{\tau} \Big/ \sum_{\mathbf{p}} n^0_{\mathbf{p}}$$
$$= \frac{2\pi}{\hbar} \sum_{\substack{\mathbf{p},\mathbf{q}\\ \mathbf{f}=\mathbf{q}-\mathbf{p}}} |\Psi_{\mathbf{pf;q}}|^2 [(N^0_{\mathbf{f}l}+1)n^0_{\mathbf{q}}$$
$$+ N^0_{\mathbf{f}l}n^0_{\mathbf{p}}]\,\delta(\varepsilon_{\mathbf{p}} + \hbar\omega_{\mathbf{f}l} - \varepsilon_{\mathbf{q}}) \Big/ \sum_{\mathbf{p}} n^0_{\mathbf{p}},$$

where $n^0_{\mathbf{p}}$ and $N^0_{\mathbf{f}l}$ are the equilibrium Bose distribution functions of the spin waves and phonons.

With the aid of (10) we obtain:

1) for $\varepsilon_0\pi\mu M_0/\Theta_c < \mu H_0 \leqslant \varepsilon_0\sqrt{2}\,\pi\mu M_0/\Theta_c \quad (T_1 \lesssim \varepsilon_0)$

$$\overline{W}^l_s \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}\left(\frac{\Theta_c}{\Theta_l}\right)^2\left[\varkappa^2\left(\frac{\varepsilon_0}{\Theta_c}\right)^4 + \frac{16}{5}(\gamma-\gamma_{12})^2\right]\frac{T}{\Theta_c}e^{-\varepsilon_0/T},$$
$$T \ll T_1, \quad \Theta_c \gg \Theta_l; \qquad (11a)$$

2) for $\varepsilon_0\sqrt{2}\,\pi\mu M_0/\Theta_c \ll \mu H_0 \ll \varepsilon_0 \quad (T_1 \gg \varepsilon_0)$

$$\overline{W}^l_s \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}\left(\frac{\Theta_c}{\Theta_l}\right)^2\left[\varkappa^2\left(\frac{\varepsilon_0}{\Theta_c}\right)^4 + \frac{16}{5}(\gamma-\gamma_{12})^2\right]\frac{T}{\Theta_c}e^{-\varepsilon_0/T},$$

$T \ll \varepsilon_0$,

$$\overline{W}^l_s \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}(\varkappa+\delta-\delta_{12})^2\frac{T}{\Theta_c}\left[\left(\frac{T}{\Theta_c}\right)^4 + \frac{(\gamma-\gamma_{12})^2}{105}\right],$$

$$\varepsilon_0 \ll T \ll T_1, \qquad (11b)$$

where $\Theta_l = \hbar c_l/a$ is of the order of the Debye temperature for longitudinally polarized sound.

If we assume that the small relativistic constants $\gamma$ and $\gamma_{12}$ satisfy the conditions

$$|\gamma+\gamma_{12}|(T/\Theta_c)^2 \ll |\gamma-\gamma_{12}| \ll (T/\Theta_c)^2,$$

then the second terms in the square brackets of (11) can be neglected and the formulas for the average probabilities simplify to

$$\overline{W}^l_s \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}\varkappa^2\left(\frac{\Theta_c}{\Theta_l}\right)^2\left(\frac{\varepsilon_0}{\Theta_c}\right)^4\frac{T}{\Theta_c}e^{-\varepsilon_0/T}, \text{ regions I, II,} \qquad (12a)$$

$$\overline{W}^l_s \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}(\varkappa+\delta-\delta_{12})^2\left(\frac{T}{\Theta_c}\right)^5, \quad \text{region III} \qquad (12b)$$

($\Theta_c \gg \Theta_l$), where regions I, II, and III denote the following regions of temperatures and fields (see Fig. 2):

Region I: $\varepsilon_0\pi\mu M_0/\Theta_c < \mu H_0 \leqslant \varepsilon_0\sqrt{2}\,\pi\mu M_0/\Theta_c$, $T \ll T_1$;

Region II: $\varepsilon_0\sqrt{2}\,\pi\mu M_0/\Theta_c \ll \mu H_0 \ll \varepsilon_0$, $T \ll \varepsilon_0$;

Region III: $\varepsilon_0\sqrt{2}\,\pi\mu M_0/\Theta_c \ll \mu H_0 \ll \varepsilon_0$, $\varepsilon_0 \ll T \ll T_1$.

We shall henceforth neglect in the interaction Hamiltonian the terms with the small magnetostriction constants $\gamma$ and $\gamma_{12}$.

As was already mentioned, these processes occur only when $\Theta_c > \Theta_l$. When $\Theta_c = \Theta_l$, these processes are forbidden by the energy and momentum conservation laws. Let us present an expression for the average probability of these processes near the value $\Theta_c = \Theta_l$ (as $\Theta_c \to \Theta_l$ from above):

$$\overline{W}^l_s \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}(\varkappa+\delta-\delta_{12}+\lambda-\lambda_{12})^2\left(\frac{\varepsilon_0}{\Theta_c}\right)^4$$
$$\times\frac{T}{\Theta_c}\left(\frac{\Theta_l}{\Theta_c-\Theta_l}\right)^2\exp\left(-\frac{\varepsilon_0}{T}\sqrt{\frac{\Theta_l}{2(\Theta_c-\Theta_l)}}\right).$$

It is seen from this expression that when $\Theta_c = \Theta_l$ the probability $\overline{W}^l_s$ vanishes. But since the dispersion law (6) is approximate, we shall observe actually at the point $\Theta_c = \Theta_l$ a sharp decrease in the average probability (although the probability does not vanish).

One can also speak of an average phonon lifetime relative to the emission (absorption) phonon by a spin wave. A quantity that is the reciprocal of the average phonon lifetime is defined by the expression

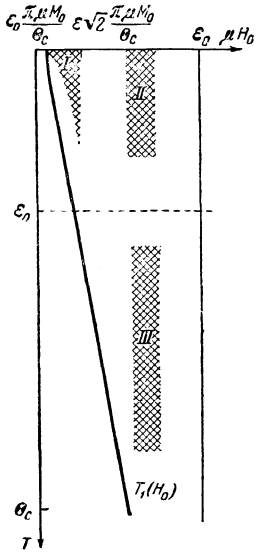


FIG. 2

$$\frac{1}{\bar{\tau}_{\Phi}^{l}} = \overline{W}_{\text{ph}}^{l} = \sum_{\mathbf{f}} \frac{N_{\mathbf{f}l}^{0}}{\tau_{\mathbf{f}}^{l}} \Big/ \sum_{\mathbf{f}} N_{\mathbf{f}l}^{0} = \frac{4\pi}{\hbar} \sum_{\substack{\mathbf{p},\mathbf{q} \\ (\mathbf{f}=\mathbf{q}-\mathbf{p})}} |\Psi_{\mathbf{pf};\mathbf{q}}|^2$$

$$\times\ N_{\mathbf{f}l}^{0} (n_{\mathbf{p}}^{0} - n_{\mathbf{q}}^{0})\, \delta (\varepsilon_{\mathbf{p}} + \hbar\omega_{\mathbf{f}l} - \varepsilon_{\mathbf{q}}) \Big/ \sum_{\mathbf{f}} N_{\mathbf{f}l}^{0}.$$

After the calculations we obtain

$$\overline{W}_{\text{ph}}^{l} \sim \left(\frac{\Theta_l}{\Theta_c}\right)^3 \overline{W}_s^l, \text{ regions I, II;} \qquad (13)$$
$$\overline{W}_{\text{ph}}^{l} \sim \left(\frac{\Theta_l}{\Theta_c}\right)^4 \overline{W}_s^l, \text{ region III,}$$

where $\overline{W}_s^l$ are defined by (12). It is seen from (13) that in all three temperature and field regions $\overline{W}_{\text{ph}}^l \ll \overline{W}$

For transversely polarized phonons, the average probabilities of the processes under consideration are of the form

$$\overline{W}_s^t \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_t^2} (\delta - \delta_{12})^2 \left(\frac{\varepsilon_0}{\Theta_c}\right)^3 \left(\frac{T}{\Theta_c}\right)^2 e^{-\varepsilon_0/T}, \text{regions I, II,}$$
$$\overline{W}_s^t \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_t^2} (\delta - \delta_{12})^2 \left(\frac{\Theta_t}{\Theta_c}\right)^2 \left(\frac{T}{\Theta_c}\right)^5, \quad \text{region III,} \quad (14)$$
$$\overline{W}_{\text{ph}}^t \sim \left(\frac{\Theta_t}{\Theta_c}\right)^3 \overline{W}_s^t, \text{ regions I, II,}$$
$$\overline{W}_{\text{ph}}^t \sim \left(\frac{\Theta_t}{\Theta_c}\right)^4 \overline{W}_s^t, \text{ region III.} \qquad (15)$$

It is seen from (15) that for transverse phonons, like for longitudinal ones, $\bar{\tau}_s \ll \bar{\tau}_{\text{ph}}$. In addition, a comparison of formulas (12) and (14) shows that in all three regions of temperatures and fields absorption (emission) of a longitudinal phonon is more probable than that of a transverse one.

2. We now consider the case $H_0 = 0$. In the investigation of the interaction of spin waves with phonons in the absence of a magnetic field, the dispersion law can be regarded to be the same for the two sorts of spin waves

$$\varepsilon_1 \approx \varepsilon_2 = \sqrt{\varepsilon_0^2 + \Theta_c^2 (ak)^2}.$$

Four processes of phonon absorption (emission) by a spin wave are possible[3)], as shown in Fig. 3

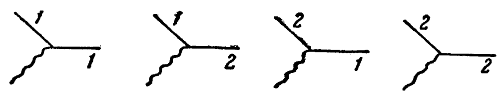


FIG. 3

For all these processes, the conservation laws are satisfied only if $\Theta_c > \Theta_d$. Their average probabilities are equal and coincide in order of magnitude with the previously obtained formulas (12)—(15) for the probabilities in the presence of a magnetic field. However, regions I—III are determined in this case by the following inequalities:

Regions I, II: $H_0 = 0$; $T \ll \varepsilon_0$;
Region III: $H_0 = 0$, $\varepsilon_0 \ll T \ll \Theta_c$.

B. Decay of a Phonon Into Two Spin Waves (Or Merging of Two Spin Waves Into a Phonon)

Three types of decay of a phonon into two spin waves (and the inverse process) are possible, depending on the sort of the participating spin waves (see Fig. 4). As will be shown in what follows, all these are threshold processes both in the presence of a magnetic field and when $H_0 = 0$.

1. We consider the case $H_0 \neq 0$. An investigation of the laws of energy and momentum conservation with dispersion law (6) for the spin waves shows that all three possible processes in which a phonon decays into two spin waves are threshold processes with respect to the magnitude of the momentum of the decaying phonon. The threshold value of the phonon momentum $f_{\text{thr}}$ depends on the type of the process. In addition, in turns out that the processes of the decay of a phonon into two spin waves of the second sort or into spin

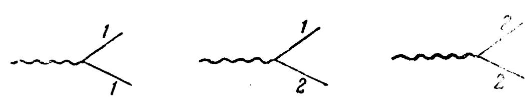


FIG. 4

[3)]The wavy line denotes a phonon and the straight line a spin wave. The numbers indicate the sort of the spin wave.

waves of different sorts occur only if $\Theta_c < \Theta_d$ [4], and the decay of a phonon into spin waves of the first sort occurs both when $\Theta_c \geq \Theta_d$ and when $\Theta_c < \Theta_d$.

Near the threshold, the probability W of the decay of a phonon with energy $\hbar\omega$ is of the order of $(\omega - \omega_{thr})^{1/2}$. Let us write out the threshold values of the phonon energy $\hbar\omega_{thr} = \hbar c f_{thr}$ for three posible processes.

1) For the process of the decay of a phonon into spin waves of the second sort

$$\hbar\omega_{thr} = \frac{2}{1-\alpha^2}\left(\mu H_0 + \sqrt{(\alpha\mu H_0)^2 + \varepsilon_0^2(1-\alpha^2)}\right), \quad \omega > \omega_{thr},$$

where $\alpha = \Theta_c/\Theta_d < 1$.

2) For the decay of a phonon into spin waves of different sorts

$$\hbar\omega_{thr} = 2\varepsilon_0/\sqrt{1-\alpha^2}, \quad \omega > \omega_{thr}, \quad \alpha < 1. \quad (16)$$

3) In the case when the phonon decays into spin waves of the first sort, there are three possibilities, depending on the value of the parameter $\alpha = \Theta_c/\Theta_d$. We have

a) For $\alpha < 1$

$$\hbar\omega_{thr} = \frac{2}{1-\alpha^2}\left(\sqrt{(\alpha\mu H_0)^2 + \varepsilon_0^2(1-\alpha^2)} - \mu H_0\right), \quad \omega > \omega_{thr};$$

b) for $\alpha = 1$

$$\hbar\omega_{thr} = \varepsilon_0^2/\mu H_0, \quad \omega > \omega_{thr};$$

c) for $\alpha > 1$, $\varepsilon_0\sqrt{1-1/\alpha^2} < \mu H_0 \ll \varepsilon_0$,

$$\hbar\omega_{min} = 2(\alpha^2-1)^{-1}\left(\mu H_0 - \sqrt{(\alpha\mu H_0)^2 + \varepsilon_0^2(1-\alpha^2)}\right),$$
$$\hbar\omega_{max} = 2(\alpha^2-1)^{-1}\left(\mu H_0 + \sqrt{(\alpha\mu H_0)^2 + \varepsilon_0^2(1-\alpha^2)}\right).$$
$$\omega_{min} \leqslant \omega \leqslant \omega_{max}.$$

We see therefore that for the last process in the case $\Theta_c > \Theta_d$ the energy of the phonon has in the indicated fields not only a minimum value (as for the two other processes), but also a maximum value.

The comparison of the values of the functions $\Psi$ [see (9)] shows that of all three processes, the most probable is the decay of a phonon into two spin waves of different sorts, and the inverse process. For this process we have calculated the average probabilities.

The average lifetimes of a phonon decaying into two spin waves of different sorts has the following form: for a longitudinally-polarized phonon ($\Theta_c \ll \Theta_l$)

$$\frac{1}{\bar{\tau}^l_{ph}} \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}\varkappa^2\left(\frac{\varepsilon_0}{\Theta_c}\right)^{13/2}\left(\frac{\Theta_c}{T}\right)^{3/2} e^{-2\varepsilon_0/T}, \text{ regions I, II,}$$
$$\frac{1}{\bar{\tau}^l_{ph}} \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}(\varkappa - \delta + \delta_{12})^2\left(\frac{T}{\Theta_c}\right)^5, \text{ region III.} \quad (17)$$

For a transversely polarized phonon ($\Theta_c \ll \Theta_t$)

$$\frac{1}{\bar{\tau}^t_{ph}} \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_t^2}(\delta - \delta_{12})^2\left(\frac{\varepsilon_0}{\Theta_c}\right)^{9/2}\left(\frac{T}{\Theta_c}\right)^{1/2} e^{-2\varepsilon_0/T}, \text{ regions I, II,}$$
$$\frac{1}{\bar{\tau}^t_{ph}} \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_t^2}(\delta - \delta_{12})^2\left(\frac{T}{\Theta_c}\right)^5, \text{ region III.} \quad (18)$$

In formulas (17) and (18) regions I, II, and III are defined as the following regions of temperatures and fields:

Region I: $\varepsilon_0\pi\mu M_0/\Theta_c < \mu H_0 < \sqrt{5}\,\varepsilon_0\mu M_0/\Theta_c$, $T \ll T_1$,

Region II: $\varepsilon_0\sqrt{5}\pi\mu M_0/\Theta_c \ll \mu H_0 \ll \varepsilon_0$, $T \ll 2\varepsilon_0$,

Region III: $\varepsilon_0\sqrt{5}\pi\mu M_0/\Theta_c \ll \mu H_0 \ll \varepsilon_0$, $2\varepsilon_0 \ll T \ll T_1$.

The reciprocal of the average lifetime of the spin wave relative to the process under consideration will be (for both longitudinal and transverse phonons):

$$1/\bar{\tau}_s \sim (\Theta_c/\Theta_d)^3\bar{\tau}_{ph}^{-1} \quad (\Theta_c \ll \Theta_d), \quad (19)$$

where $1/\bar{\tau}_{ph}$ are given by (17) and (18). It is seen from (19) that $\bar{\tau}_{ph} \gg \bar{\tau}_s$.

If we are interested in the decay of a phonon with arbitrary polarization into two spin waves, then the average probability of such a process is determined by the expression ($\Theta_c \ll \Theta_d$, $\gamma \approx 0.1$):

$$\overline{W}_{ph} \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}\varkappa^2\left(\frac{\varepsilon_0}{\Theta_c}\right)^{13/2}\left(\frac{\Theta_c}{T}\right)^{3/2} e^{-2\varepsilon_0/T}, \text{ regions I, II;}$$
$$\overline{W}_{ph} \sim \frac{(\mu M_0)^2}{\hbar\rho a^3 c_l^2}\left[\frac{(\varkappa - \delta + \delta_{12})^2}{c_l^2} + \gamma(\delta - \delta_{12})^2\right]\left(\frac{T}{\Theta_c}\right)^5, \text{ region III} \quad (20)$$

2. Let us proceed to the case $H_0 = 0$. If there is no external magnetic field then, as follows from the conservation laws, the processes wherein a phonon decays into two spin waves (and the inverse processes) are also threshold processes with threshold (minimum) value of phonon energy:

$$\hbar\omega_{thr} = 2\varepsilon_0/\sqrt{1 - \Theta_c^2/\Theta_d^2}.$$

Near the threshold ($\omega \to \omega_{thr}$) the probability of decay of a phonon with a given frequency $\omega$ into two spin waves will have the following dependence on $(\omega - \omega_{thr})$:

$$W_{f;11} \sim W_{f;22} \sim (\omega - \omega_{thr})^{1/2}, \quad W_{f;12} \sim (\omega - \omega_{thr})^{3/2}. \quad (21)$$

The dependence on $(\omega - \omega_{thr})$ for the probability

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[4] When $\Theta_c \geq \Theta_d$ these processes are forbidden by the conservation laws if the dispersion law (6) is used.

of the decay of the phonon into spin waves of different sorts is connected with the fact that this process is forbidden at the threshold $\omega = \omega_{thr}$ (the matrix elements of the interaction Hamiltonian are equal to zero).

The processes of decay of a phonon into spin waves of one sort are the most probable. Their average probabilities are the same order of magnitude as the probabilities in the presence of a magnetic field, given by formulas (17)—(20). In this case regions I and II must be defined as those where $H_0 = 0$ and $T \ll 2\mathcal{E}_0$, while region III is where $H_0 = 0$, $2\mathcal{E}_0 \ll T \ll \Theta_C$.

A comparison of the obtained formulas (12)—(19) and (20) shows that a spin wave at all temperatures will most probably absorb (or emit) a longitudinally polarized phonon. In the presence of a magnetic field ($H_0 \neq 0$) the spin wave does not change its dispersion law. For phonons at low temperatures ($T \ll \mathcal{E}_0$), the most probable processes are the emission (absorption) of longitudinally polarized phonons by a spin wave.

At high temperatures ($2\mathcal{E}_0 \ll T \ll \Theta_C$), for both longitudinal and transverse phonons, the most probable is the decay into two spin waves (of the same sort, if there is no magnetic field and of different sorts in the presence of a field).

## CONCLUSION

The results obtained enable us to draw some conclusions concerning the relaxation picture in an antiferrodielectric, a picture which is much more complicated and much more entangled than for ferromagnets.

In an antiferromagnet, when account is taken of the anisotropy energy in the interaction Hamiltonian, the establishment of the equilibrium magnetization is described by two coupled equations of motion for the quantities $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ and $\mathbf{L} = \mathbf{M}_1 - \mathbf{M}_2$. The exchange part of the interaction Hamiltonian commutes with all the projections of the density of the magnetic moment of the system **M**, but does not commute with **L**. Since **L** enters in the right half of the equation of motion for **M**, the exchange forces participate through it directly in the establishment of the equilibrium magnetic moment of the body. Thus, unlike ferromagnets and antiferromagnets, the exchange interactions influence the establishment of the equilibrium magnetic moment and can lead to a broadening of the resonant line. Genkin and Faĭn[6] obtained the antiferromagnetic resonance line width for the interaction of homogeneous precession with spin waves due to exchange forces[5)].

The Hamiltonian of the anisotropic and relativistic interactions does not commute with the total angular momentum of the system, i.e., these interactions, as well as the exchange interactions, participate in the establishment of the equilibrium magnetization. The relativistic interactions turn out to be much weaker than all other interactions in the spin-wave system at both high and low temperatures, owing to the presence of the large energy threshold. As to the interactions between the spin wave and the phonons, as can be seen from formulas (5), (7), (12), (19), etc., the spin-phonon interaction probabilities can exceed the probabilities of third-order spin-wave interactions with one another.

Of all the possible spin-spin and spin-phonon interactions at high temperatures ($\mathcal{E}_0 \ll T \ll \Theta_C$), the decisive ones will be the inhomogeneous exchange interactions, which have the greatest probability.

In conclusion I take this opportunity to express gratitude to M. I. Kaganov for help with the work and for valuable advice, and also to V. M. Tsukernik for useful discussions.

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[5)] Our estimates of the average probabilities of such processes lead to a considerably lower line width:

$$\overline{W} \sim \Theta_c \hbar^{-1} (\varepsilon_0/\Theta_c)^5 (T/\Theta_c)^2 e^{-\varepsilon_0/T} \quad (T \ll \varepsilon_0).$$

From this we see, incidentally, that the exchange forces make a contribution to the line width only if the anisotropy of $\epsilon_0$ is taken into account in the energy spectrum ($\overline{W} = 0$ when $\epsilon_0 = 0$).

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