SPECTRAL REPRESENTATIONS IN THE QUASIOPTICAL APPROACH

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The analytic properties of the scattering amplitude as a function of energy and momentum transfer are investigated in the quasioptical approach, i.e., in the framework of a potential-scattering problem with a complex potential and an equation which implies the relativistic two-particle unitarity condition. The analysis is carried out with the aid of a Fredholm series. It is shown that such an amplitude does not possess a Mandelstam representation but admits one-dimensional dispersion relations in each variable, with the other variable held fixed within a certain range.

1. INTRODUCTION

IT is the aim of the present paper to investigate the analytic properties of the scattering amplitude in the quasioptical approach [1]. Already in the work of Charap and Fubini [2] and later in the work of Chew and Frautschi [3] the idea had been expressed that the concept of potential might have a certain sense in some problems of quantum field theory. The "quasipotential" description of the scattering problem in quantum field theory is especially interesting, since Regge [4] has developed the technique of complex angular momenta for potential scattering; it is particularly in quantum field theory, that the most advantageous aspects of this technique may become apparent, owing to the presence of crossing symmetry.

The Schrödinger equation with a potential has been well studied. However the potentials obtained before [2,3] have not served as a basis for a complete description, since it had already been clear at that time [5,6] that in order to obtain the results of quantum field theory it was not only necessary to select potentials of a special type, but also to modify in a certain way the Schrödinger equation itself. Such a modified equation has been found in the papers of Logunov and Tavkhelidze [1] devoted to the quasioptical approach.

In the quasioptical approach one exploits the advantage of the two-particle problem, which can be formulated in mechanical (particle) and not in the field-theoretical form. The equation and the form of the potential are selected in such a manner that the amplitude has the same structure as the corresponding field-theoretical amplitude. It is obvious that in this case the potential must depend on the energy in such a manner that it becomes complex starting from a certain value – the threshold of the inelastic processes. The imaginary part of the potential has a simple physical interpretation: it reflects the influence of inelastic processes on the elastic channel. This implies the definiteness of the sign of the imaginary part $\operatorname{Im} V(s+i\varepsilon, t) > 0$ for $s > s_0$, or the absorptive character of the total potential. The condition that the potential be absorptive implies a spectral type of dependence of the potential on s [5]. Finally, the potential is chosen to be "spectral" also in t, i.e., to be a superposition of Yukawa potentials. This guarantees, as will be seen, the existence of dispersion relations in t for the scattering amplitude in a certain energy domain. Thus a certain correspondence with quantum field theory is obtained.

Thus, finally, the potential of the quasioptical approach has the form

$$V(s, t) = \frac{1}{\pi}\int_{\mu^2}^{\infty} d\nu \frac{\sigma(s, \nu)}{\nu - t}, \qquad (1.1)$$

where, in turn,

$$\sigma(s, \nu) = \frac{1}{\pi}\int_{s_0}^{\infty} ds' \frac{\operatorname{Im}\sigma(s', \nu)}{s' - s}.$$

The form of the equation for the scattering amplitude will be determined from the condition that, in the region where the potential is real, the unitarity condition for our amplitude coincide exactly with the two-particle unitarity condition in quantum field theory:

$$A^+(p, p') - A(p, p') = 2\pi i \int A^+(p, p'')\,\delta\,[(p^2 - p''^2)\sqrt{p^2+m^2}]\,A(p'', p')\,dp'' = \frac{2\pi i}{E^2}\int A^+(p, p'')\times\delta\,(\sqrt{p^2+m^2} - \sqrt{p''^2+m^2})\,A(p'', p')\,dp''. \qquad (1.2)$$

Selecting the amplitude in the form

$$A = V + V\{(E^2 - p^2 - m^2)\sqrt{p^2 + m^2} + i\varepsilon - V\}^{-1} V, \qquad (1.3)$$

we get, in the region where the potential is real,

$$A^+ - A = 2A^+ i\varepsilon\{[(E^2 - p^2 - m^2)\sqrt{p^2+m^2}]^2 + \varepsilon^2\}^{-1} A.$$

Observing that

$$i\varepsilon\{[(E^2 - p^2 - m^2)\sqrt{p^2+m^2}]^2 + \varepsilon^2\}^{-1} = i\pi\delta[(E^2 - p^2 - m^2)\sqrt{p^2+m^2}]$$

and rearranging the delta function by the standard procedure to the form $\delta(\sqrt{p^2+m^2} - \sqrt{p'^2 - m^2})$, we obtain the unitarity condition precisely in the form (1.2).

Thus the selection of the amplitude in the form (1.3) allows us to satisfy the principal correspondence condition. On the other hand, the same equation is the sought-for Lippman-Schwinger equation for the scattering amplitude.

This equation and the corresponding equation for the wave function

$$(E^2 - p^2 - m^2)\sqrt{p^2+m^2}\,\psi(\mathbf{p}) - \int V(E, \mathbf{p} - \mathbf{p}')\,\psi(\mathbf{p}')\,d\mathbf{p}' = 0 \qquad (1.4)$$

were the starting point of a series of papers [1,7,8] devoted to the quasioptical approach, in which the analytic properties of this equation and its connections with the Mandelstam representation in quantum field theory and with the Regge asymptotic behavior have been investigated, mainly within the framework of perturbation theory.

Among these papers the one closest to the present one in method is the paper by Arbuzov [7], where the analytic properties of partial wave amplitudes in the s-plane are investigated. In the present paper we investigate the analytic properties of the full scattering amplitude in the s and t variables by means of the Fredholm method.

2. THE STARTING EQUATIONS AND GREEN'S FUNCTIONS

The equation for the wave function

$$(E^2 - p^2 - m^2)\sqrt{p^2+m^2}\,\psi(\mathbf{p}) - \int V(E, \mathbf{p} - \mathbf{p}')\,\psi(\mathbf{p}')\,d^3\mathbf{p}' = 0 \qquad (2.1)$$

differs from the usual Klein-Gordon equation by the presence of the kinematic factor $\sqrt{p^2 + m^2}$.

The Green's function for this equation has the form

$$G_0(p) = \{[(E + i\varepsilon)^2 - p^2 - m^2]\sqrt{p^2 + m^2}\}^{-1}, \qquad (2.2)$$

and therefore the corresponding Lippman-Schwinger equation for the scattering amplitude will be

$$f(\mathbf{p}_\alpha, \mathbf{p}_\beta) = V(E, \mathbf{p}_\alpha - \mathbf{p}_\beta) + \int \frac{d\mathbf{p}}{(2\pi)^3} V(\mathbf{p}_\alpha, \mathbf{p}) \frac{f(\mathbf{p}, \mathbf{p}_\beta)}{[(E + i\varepsilon)^2 - p^2 - m^2]\sqrt{p^2 + m^2}}, \qquad (2.3)$$

where $\mathbf{p}_\alpha$ and $\mathbf{p}_\beta$ are the momenta of the initial and final states.

We also need the Green's function in the coordinate representation, which can be obtained easily by utilizing the spectral representation of the square root:

$$\frac{1}{\sqrt{p^2+m^2}} = \frac{1}{\pi}\int_0^\infty \frac{d\xi}{\sqrt{\xi}\,(\xi + p^2 + m^2)}. \qquad (2.4)$$

We obtain

$$G_0(x, y) = \frac{-1}{4\pi^2|\mathbf{x} - \mathbf{y}|}\int_0^\infty \frac{d\xi}{\sqrt{\xi}}\left\{\frac{e^{ik|\mathbf{x}-\mathbf{y}|} - e^{-\sqrt{\xi+m^2}|\mathbf{x}-\mathbf{y}|}}{k^2 + m^2 + \xi}\right\}. \qquad (2.5)$$

In establishing the analytic properties of the scattering amplitude corresponding to Eq. (2.1) we shall use the resolvent representations obtained by means of the total Green's function

$$G = \{[(E + i\varepsilon)^2 - p^2 - m^2]\sqrt{p^2 + m^2} - V(E, \mathbf{p} - \mathbf{p}')\}^{-1}. \qquad (2.6)$$

We shall use two such representations:

$$f(\mathbf{p}_\alpha, \mathbf{p}_\beta) = V(E, \mathbf{p}_\alpha - \mathbf{p}_\beta) + \int \frac{d\mathbf{p}}{(2\pi)^3} V(E, \mathbf{p}_\alpha - \mathbf{p})\, G(\mathbf{p}_\alpha, \mathbf{p})\, V(E, \mathbf{p} - \mathbf{p}_\beta), \qquad (2.7)$$

$$f(\mathbf{p}_\alpha, \mathbf{p}_\beta) = V(E, \mathbf{p}_\alpha - \mathbf{p}_\beta) - \frac{1}{4\pi}\int d\mathbf{r}\,d\mathbf{r}' e^{i(\mathbf{p}_\alpha\mathbf{r} - \mathbf{p}_\beta\mathbf{r}')} V(\mathbf{r})\, V(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}', k),$$

$$p_\alpha^2 = p_\beta^2 = k^2 = s = E^2 - m^2, \qquad (2.8)$$

where the second representation is easily obtained from the first by means of a Fourier transform. Having in mind the application of the Fredholm method, we also rewrite the resolvent representation in the form

$$f(\mathbf{p}_\alpha, \mathbf{p}_\beta) = V(E, \mathbf{p}_\alpha - \mathbf{p}_\beta) + \int \frac{d\mathbf{p}'}{(2\pi)^3} \frac{N(\mathbf{p}_\alpha, \mathbf{p}', k)}{D(k)} V(\mathbf{p}' - \mathbf{p}_\beta, E) \qquad (2.9)$$

and

$$f(\mathbf{p}_\alpha, \mathbf{p}_\beta) = V(E, \mathbf{p}_\alpha - \mathbf{p}_\beta) - \frac{1}{4\pi}\int d\mathbf{r}\,d\mathbf{r}' e^{i(\mathbf{p}_\alpha\mathbf{r} - \mathbf{p}_\beta\mathbf{r}')} \frac{N(\mathbf{r}, \mathbf{r}'; k)}{D(k)} V(\mathbf{r}'). \qquad (2.10)$$

Here N and D are the Fredholm numerator and denominator in the respective representations.

3. ANALYTIC PROPERTIES IN THE ENERGY VARIABLE

In order to establish the analytic properties of the scattering amplitude in the energy variable k^2 we start from the form (2.10).

Since the quasioptical potential is spectral with respect to k^2, we must consider the second term in the right-hand side of (2.10)[1)]:

$$F(\mathbf{p}_\alpha, \mathbf{p}_\beta) = \frac{-1}{4\pi}\int d\mathbf{r}d\mathbf{r}' e^{i(\mathbf{p}_\alpha \mathbf{r} - \mathbf{p}_\beta \mathbf{r}')} \frac{N(\mathbf{r}, \mathbf{r}'; k)}{D(k)} V(\mathbf{r}'). \quad (3.1)$$

In order to establish its analytic properties it is necessary to investigate the analytic properties of the numerator $N(\mathbf{x}, \mathbf{y}; k)$ and of the Fredholm denominator $D(k)$ and also the convergence properties of the integral. $N(\mathbf{x}, \mathbf{y}; k)$ and $D(k)$ are expressed by the well known series:

$$N(\mathbf{x}, \mathbf{y}; k) = K(\mathbf{x}, \mathbf{y}) + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2\pi)^{3n}} \frac{1}{n!} \int d\mathbf{x}_1 \ldots \int d\mathbf{x}_n N^{(n)}(k; \mathbf{x}, \mathbf{y}; \mathbf{x}_1 \ldots \mathbf{x}_n), \quad (3.2)$$

$$D(k) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2\pi)^{3n}} \frac{1}{n!} \int d\mathbf{x}_1 \ldots \int d\mathbf{x}_n D^{(n)}(k; \mathbf{x}_1 \ldots \mathbf{x}_n), \quad (3.3)$$

where

$$K(\mathbf{x}, \mathbf{y}) = G_0(\mathbf{x}, \mathbf{y}) V(y), \quad (3.4)$$

and $N^{(n)}(k; \mathbf{x}, \mathbf{y}; \mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $D^{(n)}(k; \mathbf{x}_1 \ldots \mathbf{x}_n)$ are the Fredholm determinants, expressed in the standard way in terms of the kernel $K(\mathbf{x}, \mathbf{y})$.

We note that since the Green's function $G_0(\mathbf{x}, \mathbf{y})$ has no singularity in the point where its arguments coincide [this can be seen from the representation (2.5)] there is no need to use the iterated equation, as has been done by Khuri [9], or to replace the diagonal elements of the determinant by zeros, following the example of Jost and Pais [10].

We now estimate the integrand in Eq. (3.1). It is obvious that it will essentially be determined by the estimate of the Green's function

$$G(r) = G_1 + G_2 = \frac{-1}{4\pi} \frac{e^{ik|\mathbf{r}|}}{|\mathbf{r}|\sqrt{k^2+m^2}} + \frac{1}{4\pi^2|\mathbf{r}|} \int_0^\infty \frac{d\xi}{\sqrt{\xi}} \frac{\exp\{-\sqrt{\xi+m^2}|\mathbf{r}|\}}{k^2+m^2+\xi}. \quad (3.5)$$

1)In reality, our potential V is always a function of the parameter k^2 with a cut along the real k^2 axis. Wherever this is inessential, we will omit this dependence, in order to simplify the notation.

We estimate the second term for real k

$$|G_2(r)| \leqslant e^{-m|\mathbf{r}|}[4\pi|\mathbf{r}|\sqrt{k^2+m^2}]^{-1}.$$

From this we obtain an estimate for $\operatorname{Im} k > 0$:

$$|G(r)| \leqslant e^{-\omega|\mathbf{r}|}[4\pi|\mathbf{r}|\sqrt{k^2+m^2}]^{-1},$$

where

$$\omega = \min\{\operatorname{Im} k, m\},$$

from which we obtain an estimate for the fundamental kernel $K(\mathbf{x}, \mathbf{y})$:

$$|K(\mathbf{x}, \mathbf{y})| \leqslant \frac{e^{-\omega|\mathbf{x}-\mathbf{y}|}}{\sqrt{k^2+m^2}} \frac{|V(y)|}{4\pi|\mathbf{x}-\mathbf{y}|}.$$

These estimates allow us to majorize the Fredholm numerator $N(k; \mathbf{x}, \mathbf{y})$. Indeed, the general term of the determinant has the majorant

$$|N^n(k; \mathbf{x}, \mathbf{y}; \mathbf{x}_1 \ldots \mathbf{x}_n)| \leqslant \frac{V(y)}{(k^2+m^2)^{(n+1)/2}} \cdot \frac{V(x_1)}{|\mathbf{x}-\mathbf{x}_1|} \cdot \frac{V(x_2)}{|\mathbf{x}_1-\mathbf{x}_2|} \cdots \frac{V(x_n)}{|\mathbf{x}_{n-1}-\mathbf{x}_n| \cdot |\mathbf{x}_n-\mathbf{y}|} \exp\{-\omega(|\mathbf{x}-\mathbf{x}_1| + |\mathbf{x}_1-\mathbf{x}_2| + \ldots + |\mathbf{x}_{n-1}-\mathbf{x}_n| + |\mathbf{x}_n-\mathbf{y}|)\}.$$

Applying successively the triangle inequality to the exponent and noting that

$$\int \frac{V(x_i)}{|\mathbf{x}_i-\mathbf{x}_{i+1}|} d\mathbf{x}_i \leqslant A < \infty,$$

$$\int \frac{V(x_k)}{|\mathbf{x}_{k-1}-\mathbf{x}_k| \cdot |\mathbf{x}_k-\mathbf{x}_{k+1}|} d\mathbf{x}_k \leqslant N < \infty,$$

one obtains directly the following inequality for the general term of the Fredholm numerator

$$\left|\int d\mathbf{x}_1 \ldots d\mathbf{x}_n N^{(n)}(k; \mathbf{x}, \mathbf{y}; \mathbf{x}_1 \ldots \mathbf{x}_n)\right| \leqslant M \frac{e^{-\omega|\mathbf{x}-\mathbf{y}|}}{(k^2+m^2)^{(n+1)/2}} V(y), \quad (3.6)$$

where $M = NA^{n-1}$.

From here it already follows directly that the series (3.2) converges uniformly with respect to k and represents an analytic function of k in the whole k plane with purely kinematical cuts along the imaginary axis $[\pm im, \pm i\infty]$ and with a cut along the real axis due to the cut in the potential V.

For the total $N(k; \mathbf{x}, \mathbf{y})$ we have

$$|N(k, \mathbf{x}, \mathbf{y})| \leqslant C \frac{e^{-\omega|\mathbf{x}-\mathbf{y}|}}{\sqrt{k^2+m^2}} V(y). \quad (3.7)$$

The analyticity and convergence properties of the series (3.2) and (3.3) are completely identical. Therefore the Fredholm denominator $D(k)$ exists and is an analytic function of k in the whole k plane with cuts along the imaginary and real axes.

Returning to the scattering amplitude $f(p_\alpha, p_\beta)$, we rewrite Eq. (3.1) in the form

$$F(\mathbf{p}_\alpha, \mathbf{p}_\beta) = \frac{-1}{4\pi}\int d\mathbf{x}\, d\mathbf{y} \exp\left\{i\left[\sqrt{s-\tau^2/4}\ \mathbf{n}(\mathbf{x}-\mathbf{y}) + \boldsymbol{\tau}\frac{\mathbf{x}+\mathbf{y}}{2}\right]\right\} V(x)\frac{N(k;\mathbf{x},\mathbf{y})}{D(k)}, \tag{3.8}$$

where

$$s = p_\alpha^2 = p_\beta^2 = k^2, \quad \boldsymbol{\tau} = \mathbf{p}_\alpha - \mathbf{p}_\beta,$$

$$\mathbf{n} = (\mathbf{p}_\alpha + \mathbf{p}_\beta)/|\mathbf{p}_\alpha + \mathbf{p}_\beta|.$$

Hence, using the estimates obtained above, we obtain

$$|F(\mathbf{p}_\alpha, \mathbf{p}_\beta)| \leqslant \Big| \frac{C}{4\pi\sqrt{k^2+m^2}}\frac{1}{D(k)} \times \int d\mathbf{x}\, d\mathbf{y} \exp\left[i\sqrt{s-\frac{\tau^2}{4}}\ \mathbf{n}(\mathbf{x}-\mathbf{y})\right] \times V(x)V(y)e^{-\omega|\mathbf{x}-\mathbf{y}|}\Big|. \tag{3.9}$$

We are interested in the behavior of $F(p_\alpha, p_\beta)$ for physical values of $\boldsymbol{\tau}$ on the real k axis, and also in the upper half-plane Im k > 0.

On the real axis

$$|F(\mathbf{p}_\alpha, \mathbf{p}_\beta)| \leqslant \Big| \frac{C}{4\pi\sqrt{k^2+m^2}}\frac{1}{D(k)}\int \exp\left\{\frac{|\boldsymbol{\tau}|}{2}|\mathbf{x}-\mathbf{y}|\right\} \times V(x)V(y)\, d\mathbf{x}\, d\mathbf{y}\Big|$$

and, consequently (due to the definition of the potential) F will be an analytic function of k on the real axis with the cuts $|k| > k_0$ with the usual restriction $|\tau| \leq 2\mu$ imposed on the momentum transfer.

We go over now to an analysis of the complex plane. We observe that

$$V(x)V(y) \leqslant \text{const}\, e^{-\mu(x+y)} \leqslant \text{const}\, e^{-\mu|\mathbf{x}-\mathbf{y}|},$$

and also

$$\operatorname{Im}\sqrt{s-\tau^2/4} \xrightarrow[|k|\to\infty]{} \operatorname{Im} k \quad \text{for} \quad \operatorname{Im} k > 0.$$

Consequently, we see from Eq. (3.9), that the amplitude $F(p_\alpha, p_\beta)$ is an analytic function in the strip[2)]

$$|\operatorname{Im} k| \leqslant (m+\mu). \tag{3.10}$$

Remembering that the total scattering amplitude is f = V + F and that the potential V(s, t) is by assumption an analytic function of k with a cut in the k plane ($|k| > k_0$, Im k = 0) we conclude that the amplitude f(k, t) is an analytic function of k in the strip $|\operatorname{Im} k| \leq (m+\mu)$ with cuts along the real axis:

$$\operatorname{Im} k = 0, \quad |k| > \tau; \quad \operatorname{Im} k = 0, \quad |k| > k_0, \tag{3.11}$$

which have their origin in the branch points of the integral (3.8) and the branch point of the potential, respectively.

Further, we succeed in extending the initial analyticity domain by means of the well-known procedure (cf., e.g.,[11]) which utilizes the possibility of continuing Yukawa potentials into the complex domain of z (Re z > 0). In its usual form this technique starts out from the equation for the wave function

$$(L_z(k) + V_z)\psi(k, z) = 0 \tag{3.12}$$

along a fixed ray $z = \rho\exp(i\theta)$ $(-\pi/2 < \theta < \pi/2)$; it is easy to see that in this case the explicit form of the operator L_z implies that the equation becomes

$$(L_\rho(k') + V'_\rho)\psi(k', \rho) = 0 \tag{3.12'}$$

where $k' = k\exp(i\theta)$, $V'_\rho = V(\rho\exp(i\theta))\exp(2i\theta)$.

Obviously the new potential possesses all the properties that V does. Therefore, if the solution of Eq. (3.12) is analytic in a certain domain with respect to k, the solution of (3.12′) will be analytic in the same domain with respect to k′. In the k plane this analyticity domain will differ from the analyticity domain of the original solution. It is clear now, that if the solution (3.12), after being analytically continued with respect to z onto the same ray $z = \rho\exp i\theta$, coincides with the solution of the (3.12′), then it will be analytic in the k plane in the union of analyticity domains of the two solutions in this plane. Varying the angle θ in its admissible limits, we find that the total analyticity domain Γ of the initial solution is obtained from the initial analyticity domain Γ_0 by adding to the latter all domains which are obtained from Γ by means of the transformation $k \to k\exp i\theta$ with arbitrary $\theta \in (-\pi/2, \pi/2)$.

In order to extend the initially obtained domain (the strip $|\operatorname{Im} k| \leq (m+\mu)$ with cuts), we carry out the same reasoning for our particular case. First of all, it is obvious, that the integral form of the operator L which has been used until now, does not allow us to use a reasoning in which z and k have to be free variables.

We therefore start from the differential equation for the wave function

[2)]The scattering amplitude will also have the well-known kinematic cut [im, i ∞] due to the occurrence of the factor $(k^2+m^2)^{-1/2}$ in front of the integral (3.9). This cut can be easily eliminated and we will not mention it any more in what follows.

$$(\mathcal{L}_z(k,m)+V_z(k^2))\psi(m,k,z)$$

$$\equiv[(k^2+\Delta)\sqrt{\Delta+m^2}+V_z(k^2)]\psi(m,k,z)=0. \qquad (3.13)$$

Transforming this equation to the ray $z = \rho \exp i\theta$, we obtain

$$(\mathcal{L}_\rho(k', m') + V'_\rho(k^2))\psi = 0, \qquad (3.13')$$

where $\mathcal{L}$ is the same operator and

$$m' = me^{i\theta}, \quad V'_\rho(k^2) = V(k^2, \rho e^{i\theta})e^{3i\theta}, \quad k' = ke^{i\theta}. \quad (3.14)$$

The difference from the usual case consists, first, in the fact that one carries out a supplementary substitution of the parameter m and, second, the potential does not participate in the substitution $k \to k \exp i\theta$, depending only on the original variable k^2. Obviously, the first point is inessential, since the substitution $m \to m \exp i\theta$ does not change the form of the operator $\mathcal{L}$ and the only thing which has to be checked is that there appear no new singularities due to this substitution.

One can establish this, for instance, considering the Born expansion of the solution, in which the m dependence enters only via the Green's function $\{(E^2 - p^2 - m^2)(p^2 + m^2)^{1/2}\}^{-1}$. It is obvious that under the substitution $m \to m \exp i\theta$ with $|\theta| < \pi/2$ there appear no new singularities.

Let us consider the second point of difference. It boils down to the fact that by conserving the form of the operator $\mathcal{L}_\rho$ which we must do to prove that the solutions of Eqs. (3.13) and (3.13′) coincide, we cannot tell at first glance whether this operator depends on the parameter k or k′. We therefore use the following reasoning. Considering (3.13) and (3.13′) with their connecting substitution (3.14) we can prove the identity of their solutions in the usual manner, since in the form (3.14) the potentials differ from each other in exactly the same manner as in the usual reasoning. Thus the identity of the solutions (3.13) and (3.13′) is proved.

It remains to be shown that Eq. (3.13′) really depends only on k′ and therefore its solution will be analytic in k′ in the analyticity domain of the initial solution with respect to k. We use for this purpose the analyticity of the potential

$$V(k^2, x) = \frac{1}{\pi}\int_{k_0^2}^{\infty} ds' \frac{v(s', x)}{s' - k^2}.$$

This equation shows that V can be easily transformed to a form in which it will depend on k'^2:

$$V(k'^2, x) = \frac{1}{\pi}\int_{k_0^2 e^{2i\theta}}^{\infty} ds' \frac{v(s', x)}{s' - k'^2}.$$

This proves not only that the solutions coincide, but also that the new solution possesses the required analyticity domain with respect to k′.

By the same token, the analyticity domain of the wave function in the k-plane has been extended to the union of the analyticity domains of both solutions in this plane. Comparing the Fredholm equations (2.3) and (2.4) we conclude that the scattering amplitude will possess the same analyticity domain.

Summing the analyticity domains corresponding to all values of θ in the interval $(-\pi/2, \pi/2)$ we reach the conclusion that the total analyticity domain Γ of the scattering amplitude will consist of all points of the form $k = k_0 \exp i\theta$, where k_0 belongs to the initial analyticity domain Γ_0, i.e., the strip $|\operatorname{Im} k| \le (m + \mu)$ with cuts. This will be the whole complex k plane with the following cuts:

$$\operatorname{Re} k = 0, \qquad |\operatorname{Im} k| \geqslant (m+\mu);$$

along the imaginary axis

$$\operatorname{Im} k = 0, \quad |\operatorname{Re} k| \geqslant \tau/2; \qquad \operatorname{Im} k = 0, \quad |\operatorname{Re} k| \geqslant k_0. \qquad (3.15)$$

We note that in obtaining the domain Γ from Γ_0 we naturally find branch points on the real axis in the plane $k' = k \exp i\theta$. However, if we first consider small values of θ, it always turns out that upon returning to the k plane the new branch point will be situated in a region where the analyticity had already been proved by a direct method. This means that in reality this branch point does not actually exist. Going over gradually to larger values of θ, up to $\theta = \pi/2 - \varepsilon$, we can see each time that the new branch point will be situated in a region where the analyticity is already proved, and that therefore these branch points do not actually exist.

To conclude the analysis it remains to consider the spectrum of the eigenvalues. This is easily achieved by means of a modification of the reasoning which is usually employed[5]. Consider the quantity $W = (p^2 + m^2)^{-1/2} V$. Equation (2.1), when expressed in terms of W, formally coincides with the ordinary Schrödinger equation. Therefore it is easy to establish the relation

$$\int d\mathbf{x}\,|\psi(\mathbf{x},s)|^2(\operatorname{Im} W(x,s) - \operatorname{Im} s) = 0. \qquad (3.16)$$

The condition that the potential V be absorptive

$$\operatorname{sign} \operatorname{Im} V = -\operatorname{sign} \operatorname{Im} s \qquad (3.17)$$

immediately implies a similar condition for W. Therefore, as in the usual case, Eq. (3.16) can be satisfied only for $\operatorname{Im} s = 0$, i.e. the eigenvalue spectrum for s is localized about the real axis.

As shown above, the positive semiaxis is occupied by the continuous spectrum, and the discrete spectrum, if it exists, is consequently located at negative real values of s.

Thus the foregoing investigation of analytic properties of the scattering amplitude, allows us to write down a dispersion relation for this amplitude with respect to the variable s:

$$f(s,t) = f_B(s,t) + \frac{1}{\pi}\int_{-\infty}^{-(m+\mu)^2} \frac{ds'}{s'-s}\,\mathrm{Im}\, f(s',t) + \frac{1}{\pi}\int_0^\infty \frac{ds'}{s'-s}\,\mathrm{Im}\, f(s',t) + \sum_i \frac{\Gamma_i}{s-s_i}, \tag{3.18}$$

which is valid for $t < -4\mu^2$, and possibly requiring several subtractions. We have not investigated this latter point.

4. THE ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDE WITH RESPECT TO THE MOMENTUM TRANSFER

The starting point of our investigation will now be the resolvent representation (2.9), where

$$N(k;\mathbf{p}_\alpha,\mathbf{p}) = K(\mathbf{p}_\alpha,\mathbf{p}) + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2\pi)^{3n}}\frac{1}{n!}\int d\mathbf{p}_1 .. \int d\mathbf{p}_n N^{(n)}(k;\mathbf{p}_\alpha,\mathbf{p};\mathbf{p}_1\ldots\mathbf{p}_n) \tag{4.1}$$

and the determinant is

$$N^{(n)}(k;\mathbf{p}_\alpha,\mathbf{p};\mathbf{p}_1\ldots\mathbf{p}_n) = \begin{vmatrix} K(\mathbf{p}_\alpha,\mathbf{p}) & K(\mathbf{p}_\alpha,\mathbf{p}_1) & K(\mathbf{p}_\alpha,\mathbf{p}_n) \\ K(\mathbf{p}_1,\mathbf{p}) & 0 & K(\mathbf{p}_1,\mathbf{p}_n) \\ \cdots & \cdots & \cdots \\ K(\mathbf{p}_n,\mathbf{p}) & K(\mathbf{p}_n,\mathbf{p}_1) & 0 \end{vmatrix}. \tag{4.2}$$

The denominator $D(k)$ has obviously nothing to do with the analytic properties with respect to t and we will not consider it here.

In order to establish the analytic properties we are interested in, we have to prove the uniform convergence of the Fredholm series for the scattering amplitude and also find the analyticity domain in the t-plane for each term of this series.

To solve the problem of convergence of the Fredholm series one usually makes use of the Lehmann representation (cf. [12]). However, a weaker result is sufficient for the proof, and we will establish this result. Substituting

$$N(k;\mathbf{p}_\alpha,\mathbf{p})/D(k) = \int d\mathbf{p}_1 V(\mathbf{p}_\alpha-\mathbf{p}_1)\,G(\mathbf{p}_1.\,\mathbf{p};\,k)$$

in (2.10) we obtain

$$f(\mathbf{p}_\alpha,\mathbf{p}_\beta) - V(E,\mathbf{p}_\alpha-\mathbf{p}_\beta) = F(\mathbf{p}_\alpha,\mathbf{p}_\beta) = \int d\mu_1 d\mu_2 \sigma(s,\mu_1)\,\sigma(s,\mu_2)\int d\mathbf{p}_1 d\mathbf{p}_2\,[\mu_1^2+(\mathbf{p}_\alpha-\mathbf{p}_1)^2]^{-1} \times G(s;\mathbf{p}_1\mathbf{p}_2)\,[\mu_2^2+(\mathbf{p}_2-\mathbf{p}_\beta)^2]^{-1}.$$

Introducing the Lehmann variables [12] we rewrite $F(\mathbf{p}_\alpha, \mathbf{p}_\beta)$ in the form

$$F(\mathbf{p}_\alpha,\mathbf{p}_\beta) = \iint d\mu_1 d\mu_2 \sigma(s,\mu_1)\,\sigma(s,\mu_2) \times \iint_{\lambda_0}^{\infty} d\lambda_1 d\lambda_2 \int_0^{2\pi} d\chi \int_0^{2\pi} d\alpha_1 \frac{W(\lambda_1,\lambda_2,\chi;s)}{[\lambda_1-\cos(\theta-\alpha_1)]\,[\lambda_2-\cos(\alpha_1-\chi)]}.$$

Taking the integral with respect to $d\alpha$, and introducing a delta function, we obtain

$$F(\mathbf{p}_\alpha,\mathbf{p}_\beta) = \iint d\mu_1 d\mu_2 \sigma(s,\mu_1)\,\sigma(s,\mu_2) \times \iint_{\lambda_0}^{\infty} d\lambda_1 d\lambda_2 \int_0^{2\pi} d\chi \int dy\, \frac{\delta\left[y-(\lambda_1\lambda_2+\sqrt{(\lambda_1^2-1)(\lambda_2^2-1)})\right]}{y-\cos(\theta-\chi)} \times W(\lambda_1,\lambda_2,\chi,y;s)\left(\frac{\lambda_1}{\sqrt{\lambda_1^2-1}}+\frac{\lambda_2}{\sqrt{\lambda_2^2-1}}\right)$$

$$= \iint_{\lambda_0}^{\infty} d\lambda_1 d\lambda_2 \int dy \int_0^{2\pi} d\chi\, \frac{\psi(\lambda_1,\lambda_2,y,\chi;s)}{y-\cos(\theta-\chi)}. \tag{4.3}$$

Here W is the complete Green's function G, multiplied by an algebraic function of λ_1 and λ_2 and integrated with respect to μ_1 and μ_2 with appropriate weight functions. The Green's function G can be represented by the Fredholm series, the terms of which are (apart from inessential factors) the Fourier transforms of the terms of the corresponding series in the x-representation. As has been shown in Sec. 3, this latter series converges uniformly for physical values of the variables t and k. Blankenbecler et al.[12] have proved that the convergence for such values of t and k, translated into Lehmann variables, means convergence within the whole range of integration in (4.3).

The remainder of the proof coincides entirely with the one described in the Appendix of [12]. Substituting into (4.3) the series expansion of ψ, which converges uniformly in the range of integration, it takes an elementary ε-δ reasoning to show that the corresponding series for F is also uniformly convergent, at least for all those values of t which are not singular for any one of its terms.

Thus the representation (4.3) is sufficient in order to establish the convergence of the Fredholm series. We note that the Lehmann representation is obtained from (4.4) by changing the order of integrations with respect to dy and also $d\lambda_1$ and $d\lambda_2$. This latter step (in distinction from the change of order of the integration involving dy and the integration with respect to the weight functions of the representation of the potential) is legitimate, in our case, not for arbitrary values of s, as indicated by the example of the second Born approximation [14], where there is no analyticity inside the Lehmann ellipse. A completely analogous instance where a change in the order of integration is not allowed will be encountered below, and will be discussed there in more detail.

We go over to the problem of the analytic properties of the individual terms of the series for the scattering amplitude. A term with arbitrary n in (4.1) will in turn be a sum of terms which are obtained by expanding the determinant $N^{(n)}(k; p, p; p_1, \ldots, p_n)$ and contain different numbers $l \leq n$ of factors $K(p_i, p_{i+1})$. Therefore the general term in the series (4.1) will have the form

$$N_l^{(n)}(k; \mathbf{p}_\alpha, \mathbf{p}) = \int \prod_{j=1}^{l} \frac{d\mathbf{p}_j}{(p_j^2 - k^2 - i\varepsilon)\sqrt{p_j^2 + m^2}} \frac{\Phi_l^{(n)}(k)}{p^2 - k^2 - i\varepsilon} V(\mathbf{p}_\alpha - \mathbf{p}_1) \ldots V(\mathbf{p}_l - \mathbf{p}), \tag{4.4}$$

where $l \leq n$ and the function $\Phi_l^{(n)}(k)$ is the result of integrating with respect to the remaining momenta $p_{l+1} \ldots p_n$. Substituting (4.4) into (2.10) (and replacing p there by p_{l+1}), separating the integration with respect to dp_j^2 by means of a delta function, and utilizing the spectral representation (1.1) for the potentials, we obtain the following representation for an arbitrary term in the scattering amplitude:

$$F_l^{(n)}(s, t) = \frac{\Phi_l^{(n)}(k)}{(2\pi)^3 D(k)} \int \prod_{j=1}^{l+1} \frac{\sigma(u_j, \mu_j)\, d\mu_j\, du_j \sigma(u_{l+1}, \mu_{l+2})\, d\mu_{l+2}}{\sqrt{u_j^2 + m^2}\,(u_j - s - i\varepsilon)} \times \int d\mathbf{p}_1 \ldots d\mathbf{p}_{l+1} \times \frac{\delta(p_1^2 - u_1)\,\delta(p_2^2 - u_2) \ldots \delta(p_{l+1}^2 - u_{l+1})}{[\mu_1^2 + (\mathbf{p}_\alpha - \mathbf{p}_1)^2][\mu_2^2 + (\mathbf{p}_1 - \mathbf{p}_2)^2] \ldots [\mu_{l+2}^2 + (\mathbf{p}_{l+1} - \mathbf{p}_\beta)^2]}. \tag{4.5}$$

There is no difficulty in establishing the analytic properties with respect to t of the internal integral $\int dp_1 \ldots dp_{l+1}$, by making use of the following fundamental formula

$$\int d\mathbf{p}_1 \frac{\delta(p_1^2 - u_1)}{[\mu_1^2 + (\mathbf{p}_\alpha - \mathbf{p}_1)^2][\mu_2^2 + (\mathbf{p}_1 - \mathbf{p}_2)^2]} = \frac{\pi}{2} \int \frac{dt_1}{t_1 + (\mathbf{p}_\alpha - \mathbf{p}_2)^2} \frac{\theta(\sqrt{t_1} - \mu_1 - \mu_2)\,\theta(\Delta)}{\sqrt{\Delta}}, \tag{4.6}$$

where

$$\Delta = -\begin{vmatrix} s & {}^1/_2(\mu_1^2 + s + u_1) & {}^1/_2(t_1 + s + u_2) \\ {}^1/_2(\mu_1^2 + s + u_1) & u_1 & {}^1/_2(\mu_2^2 + u_1 + u_2) \\ {}^1/_2(t_1 + s + u_2) & {}^1/_2(\mu_2^2 + u_1 + u_2) & u_2 \end{vmatrix} = \Delta(\mu_1^2\mu_2^2 \,|\, su_1u_2 \,|\, t_1),$$

for $p_\alpha^2 = s$, $p_1^2 = u_1$, $p_2^2 = u_2$.

It is immediately clear that successive application of this formula permits one to obtain a spectral representation for the whole integral $\int dp_1 \ldots dp_{l+1}$ also. Let us however go through this procedure in more detail, since for what follows it is important to know the concrete form of the spectral function of the representation so obtained.

Thus,

$$\int d\mathbf{p}_1 \ldots d\mathbf{p}_{l+1} \times \frac{\delta(p_1^2 - u_1)\,\delta(p_2^2 - u_2) \ldots \delta(p_{l+1}^2 - u_{l+1})}{[\mu_1^2 + (\mathbf{p}_\alpha - \mathbf{p}_1)^2][\mu_2^2 + (\mathbf{p}_1 - \mathbf{p}_2)^2] \ldots [\mu_{l+2}^2 + (\mathbf{p}_{l+1} - \mathbf{p}_\beta)^2]}$$
$$= \frac{\pi}{2} \int_{(\mu_1+\mu_2)^2}^{\infty} dt_1 \frac{\theta(\Delta_1)}{\sqrt{\Delta(\mu_1^2\mu_2^2 \,|\, su_1u_2 \,|\, t_1)}} \times \int d\mathbf{p}_2 \frac{\delta(p_2^2 - u_2)}{[t_1 + (\mathbf{p}_\alpha - \mathbf{p}_2)^2][\mu_3^2 + (\mathbf{p}_2 - \mathbf{p}_3)^2]} \times \int d\mathbf{p}_3 \ldots d\mathbf{p}_{l+1} \frac{\delta(p_3^2 - u_3) \ldots \delta(p_{l+1}^2 - u_{l+1})}{[\mu_4^2 + (\mathbf{p}_3 - \mathbf{p}_4)^2] \ldots [\mu_{l+2}^2 + (\mathbf{p}_{l+1} - \mathbf{p}_\beta)^2]}$$
$$= \ldots = \left(\frac{\pi}{2}\right)^{l+1} \int_{(\mu_1+\mu_2)^2}^{\infty} dt_1 \frac{\theta(\Delta_1)}{\sqrt{\Delta(\mu_1^2\mu_2^2 \,|\, su_1u_2 \,|\, t_1)}} \times \int_{(\sqrt{t_1}+\mu_3)^2}^{\infty} dt_2 \frac{\theta(\Delta_2)}{\sqrt{\Delta(t_1\mu_3^2 | su_2u_3 | t_2)}} \times \ldots \times \ldots \times \int_{(\sqrt{t_{k-1}}+\mu_{k+1})^2}^{\infty} dt_k \frac{\theta(\Delta_k)}{\sqrt{\Delta(t_{k-1}\mu_{k+1} \,|\, su_k u_{k+1} \,|\, t_k)}} \times \ldots \times \ldots \times \int_{(\sqrt{t_l}+\mu_{l+2})^2}^{\infty} \frac{dt_{l+1}}{t_{l+1} + (\mathbf{p}_\alpha - \mathbf{p}_\beta)^2} \frac{\theta(\Delta_{l+1})}{\sqrt{\Delta(t_l\mu_{l+2}^2 \,|\, su_{l+1}s \,|\, t_{l+1})}} \tag{4.7}$$

and we obtain, obviously, the well known result

$$\int d\mathbf{p}_1 \ldots d\mathbf{p}_{l+1} \frac{\delta(p_1^2 - u_1) \ldots \delta(p_{l+1}^2 - u_{l+1})}{[\mu_1^2 + (\mathbf{p}_\alpha - \mathbf{p}_1)^2] \ldots [\mu_{l+2}^2 + (\mathbf{p}_{l+1} - \mathbf{p}_\beta)^2]} = \int_{t_{0l}^{(n)}}^{\infty} \frac{dt_{l+1}}{t_{l+1} - t} \rho_l^{(n)}(t_{l+1}; s; u_1 \ldots u_{l+1}; \mu_1, \ldots \mu_{l+2}), \tag{4.8}$$

where

$$t_{0l}^{(n)} \geqslant (\mu_1 + \mu_2 + \ldots + \mu_{l+2})^2. \tag{4.9}$$

In order to deduce from this the analytic properties of the whole term $F_l^{(n)}(s, t)$ it is necessary to substitute (4.8) in (4.5):

$$F_l^{(n)}(s,t)=\frac{\Phi_l^{(n)}(k)}{(2\pi)^3 D(k)}\int\prod_{j=1}^{l+1}\frac{\sigma(u_j,\mu_j)\,d\mu_j du_j\sigma(u_{l+1},\mu_{l+2})\,d\mu_{l+2}}{\sqrt{u_j+m^2}\,(u_j-s-i\varepsilon)}$$
$$\times\int_{t_{0l}^{(n)}}^{\infty}\frac{dt_{l+1}}{t_{l+1}-t}\rho_l^{(n)}(t_{l+1};\,s;\,u_1\ldots u_{l+1};\,\mu_1\ldots\mu_{l+2}). \tag{4.10}$$

and to interchange the order of integration in the expression so obtained, putting the spectral integral

$$\int_{(\mu_1+\ldots+\mu_{l+2})^2}^{\infty}\frac{dt_{l+1}}{t_{l+1}-t}$$

at the beginning of the complete expression for $F_l^{(n)}(s, t)$.

Thus, the possibility of changing the order of integration in (4.10) is a necessary and sufficient condition for obtaining the spectral representation for an arbitrary term in the scattering amplitude. In turn, such an interchange is admissible if all integrals obtained in this process of changing the integration limits are convergent.

It is easy to see from (4.5) and (4.8) that upon successively interchanging the integrations with respect to dt_{l+1} with all the integrations in (4.5), except with the integration with respect to du_{l+1}, no integrals with a new structure, other than those present in the preceding order of integration will appear, and therefore no new convergence conditions will be imposed. The interchange of the integration with respect to dt_{l+1} with the one with respect to du_{l+1} produces the integral

$$J(s,t_{l+1})=\int du_{l+1}\,\theta(\Delta)\,[\sqrt{u_{l+1}+m^2}(u_{l+1}-s)]^{-1}$$
$$\times[\sqrt{\Delta(\mu^2,\mu^2\,|\,su_{l+1}s\,|\,t_{l+1})}]^{-1}, \tag{4.11}$$

for which one has to investigate the convergence conditions.

Applying the formula

$$\frac{1}{\sqrt{u_{l+1}+m^2}}=\frac{1}{\pi}\int_0^{\infty}\frac{d\xi}{\sqrt{\xi}}\frac{1}{\xi+u_{l+1}+m^2},$$

we obtain (cf. [14]):

$$J(s,t_{l+1})=\frac{1}{\pi}\int_0^{\infty}\frac{d\xi}{\sqrt{\xi}}\frac{1}{s+m^2+\xi}$$
$$\times\int_{u_-}^{u_+}\frac{du}{u+m^2+\xi}\frac{1}{\sqrt{t_{l+1}(u-u_-)(u_+-u)}}$$
$$+\frac{1}{\sqrt{s+m^2}}\int_{u_-}^{u_+}\frac{du}{u-s}\frac{1}{\sqrt{t_{l+1}(u-u_-)(u_+-u)}}=J_1+J_2, \tag{4.12}$$

where the integration limits

$$u_\pm=u_\pm(s,t_{l+1})=\tfrac{1}{2}\left(t_{l+1}+2s-2\mu^2\right.$$
$$\left.\pm\sqrt{(4s+t_{l+1})(t_{l+1}-4\mu^2)}\right) \tag{4.13}$$

are the roots of the determinant Δ.

The convergence of the integral $J_1(s, t_{l+1})$ is violated only when $u_+ + m^2 = 0$ when the integral has an end-point singularity; for $u_- < m^2 + \xi < u_+$ the singularity of J_1 can be removed by deforming the contour of integration. Similarly the integral J_2 is not convergent only for $u_\pm - s = 0$.

Thus, conditions have been obtained for the impossibility of interchanging the integration limits in (4.10), so that the spectral representation (4.8) does not give any information about the analytic properties of $F_l^{(n)}(s, t)$ in the t variable, namely, for:

$$u_\pm+m^2=0,\qquad u_\pm-s=0. \tag{4.14}$$

These conditions obviously determine those values of the fixed variable s, for which we do not succeed in establishing dispersion relations in t for the amplitude $F_l^{(n)}(s, t)$. According to Eq. (4.13) the values of s which satisfy the first condition occupy in the s-plane the vertical line Re s $= -(m^2 + \mu^2)$ and the s values which satisfy the second condition occupy the positive real semi-axis.

We note that for values of s which satisfy the second condition, the dispersion relation with respect to t exists in fact. Indeed, these values occupy only the right half of the real axis and in the remainder of the half-plane Re $s > -(m^2 + \mu^2)$, including the real interval $-(m^2 + \mu^2) < s < 0$, the dispersion relation in t exists.

Therefore above and below the semiaxis Im s = 0, Re s > 0, $F_l^{(n)}(s, t)$ is that branch of the analytic function of s and t which can be represented in the form

$$F_l^{(n)}(s,t)=\frac{1}{\pi}\int_{(l+1)^2\mu^2}^{\infty}\frac{dt'}{t'-t}\varphi_l^{(n)}(s,t'). \tag{4.15}$$

The discontinuity of this function across the indicated semi-axis is zero, as can be easily seen. Therefore we can define the function $F_l^{(n)}(s, t)$ by continuity on the real positive s semi-axis by means of the same spectral formula (4.15), i.e., the dispersion relation in t remains valid also for values of s which satisfy the second condition of (4.14).

Obviously, such a reasoning cannot be applied to the first condition, since the values of s which are determined by that condition separate completely the half-planes Re $s > -(m^2 + \mu^2)$ and Re $s < -(m^2 + \mu^2)$. Therefore, defining first the scattering amplitude for physical values of s, we can continue the dispersion relation in t only up to the line $\mathrm{Re}\ s = -(m^2 + \mu^2)$ but not to the left of it. In other words, since the interchange of integration limits is valid to the left as well as to the right of the line $\mathrm{Re}\ s = -(m^2 + \mu^2)$, we can obtain spectral representations for $F_l^{(n)}(s, t)$ on both sides of this line. However we can never prove that these two expressions are the same branch of an analytic function.

Thus the first restriction of (4.14) cannot be eliminated and therefore in the quasipotential approach we cannot prove dispersion relations in t for arbitrary s, but only for s values situated in the half-plane $\mathrm{Re}\ s > -(m^2 + \mu^2)$. The appearance of this restriction is not a peculiarity of the method employed, but seems to be a genuine property of the quasipotential scattering amplitude, since an actual calculation of the second Born approximation, carried out in [14] without using the spectral formula (4.6), indicates that for $\mathrm{Re}\ s \leq -(m^2 + \mu^2)$ the amplitude does indeed have anomalous singularities in the t plane.

We note further that in ordinary Schrödinger theory such singularities do not appear, for then Eq. (4.11) does not contain the factor $1/(u_{l+1} + m^2)^{1/2}$ and only the second condition of (4.14) remains, which is a restriction which can be easily eliminated.

By virtue of the proved uniform convergence of the Fredholm series, the full scattering amplitude will also possess all the analytic properties that have been obtained. We write down the derived dispersion relation

$$f(s, t) = f_B(s, t) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{dt'}{t' - t} \mathrm{Im}\, f(s, t'). \qquad (4.16)$$

This relation is valid for $\mathrm{Re}\ s > -(m^2 + \mu^2)$. The number of subtractions which is necessary in (4.16) is determined by the position of the pole which is farthest to the right in the l-plane for a given energy.

5. CONCLUSION

The fundamental equation (1.4) of the quasipotential approach, which we have investigated in the present paper, has been obtained as a result of an attempt to unify the potential description of the scattering of elementary particles with the two-particle unitarity condition of quantum field theory. This inevitably (Sec. 1) introduces into the equations a kinematic factor $1/(p^2 + m^2)^{1/2}$.

When this factor is absent we have the ordinary Schrödinger equation, which admits of a Mandelstam representation for the scattering amplitude. It is easy to see that in our case such a representation does not exist. Indeed, just because of the presence of the factor $1/(p^2 + m^2)^{1/2}$ [cf. the text following Eq. (4.14)], the spectral representation in t is not valid for all values of s; on the other hand the dispersion relation in s has likewise been proved only under certain restrictions on t. The ensemble of these two dispersion relations does not allow us to derive from them the double spectral representation.

Thus, the two-particle unitarity condition of quantum field theory in the framework of the quasipotential approach has turned out to be incompatible with the requirements of maximal analyticity in the form of the existence of a Mandelstam representation. The analytic properties in the quasipotential approach are more reminiscent of the situation encountered in simple dispersion relations in the framework of quantum field theory, where it is also possible to obtain rigorous proofs only when the free variable is suitably restricted.

In connection with the technique of complex angular momenta it will be very interesting to investigate to what extent crossing relations can be introduced into the quasipotential approach, as one does in quantum field theory. In this connection we note that the appearance of the left hand cut in our case in the s dispersion relations has nothing in common with crossing properties, since it has been obtained in a one-channel theory, without any account of crossed channels. In distinction from the right hand cut, this cut has no direct physical significance in terms of intermediate scattering states.

The study of problems of crossing symmetry and the closely connected problems of analytic properties of partial wave amplitudes in the l plane seems to be most important for a further investigation of the quasipotential approach.

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