THE MANDELSTAM REPRESENTATION AND THE CONTINUITY THEOREM

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A method is proposed, permitting to determine those points of the Landau surface in which the contributions from Feynman diagrams are holomorphic functions. The Mandelstam representation is proved for the "envelope" diagram, for the Kim diagram, and the "tetrahedron" or "opened envelope" diagram.

1. INTRODUCTION

IT is well-known that all the singularities of contributions from Feynman diagrams are situated on Landau surfaces [1]. In order to prove the Mandelstam representation [2] it is necessary to know which points of the Landau surfaces can be singular points of the diagrams, situated in the "physical" sheet. However, to date, this problem has not been solved. Therefore in proving the Mandelstam representation for concrete diagrams one uses various tricks which do not require the use of the Landau surfaces. There exists only one paper by Tarski [3] in which the Mandelstam representation is proved for a fourth-order diagram by means of an analysis of the character of the singular points of the Landau surfaces.

In recent papers, the present author [4,5] proposed to use the continuity theorem to find those points on the Landau surfaces which are singularities for diagrams situated in the "physical" sheet. It turned out that on the basis of the behavior of the Landau curves in the real domain one can decide whether the Mandelstam representation exists for a given diagram. On the basis of these results the Mandelstam representation is proved in the present paper for certain relatively complicated diagrams. Scalar particles of equal mass will be considered.

Section 2 will be devoted to the problem of finding those points of the Landau surfaces which can be singularities for Feynman diagrams. In Sec. 3 the proof of the Mandelstam representation by means of Eden's method is criticized. Section 4 contains a proof of the Mandelstam representation for diagrams of the "envelope" type, the "truss" diagram (the diagram proposed by Kim in order to prove the Mandelstam representation wrong), and the "tetrahedron" diagram.

2. ANALYTIC PROPERTIES OF DIAGRAMS IN THE COMPLEX DOMAIN

The contribution to a scattering amplitude from Feynman diagrams can be written in the following form

$$F(s, t) = \int_{T_n} \delta\Big(\sum_{i=1}^{n} \alpha_i - 1\Big) \{f(\alpha)s + g(\alpha)t - m^2k(\alpha)\}^{-j} d^{-2}(\alpha)\, d\alpha_1 \ldots d\alpha_n,$$

$$T_n(\alpha \mid \alpha_1 \geqslant 0, \ldots, \alpha_n \geqslant 0), \quad j > 0, \tag{2.1}$$

where s and t are the Mandelstam variables, m is the meson mass, $f(\alpha)$, $g(\alpha)$, $k(\alpha)$, $d(\alpha)$ are real functions of $\alpha = (\alpha_1, \ldots, \alpha_n)$. As is well known [6-8],

$$f(\alpha)\, s + g(\alpha)\, t - m^2 k(\alpha) < 0, \tag{2.2}$$

if $s, t \in B(s, t \mid s < 4m^2,\ t < 4m^2,\ 4m^2 - s - t < 4m^2)$, $\alpha \in T_n$.[1)] (For individual diagrams the domain in which the inequality (2.2) holds may even be larger.) The functions $f(\alpha)$, $g(\alpha)$ are in general of indefinite sign. Utilizing the third Mandelstam variable $u = 4m^2 - s - t$, one can prove the following lemma [5,4]:

Lemma 1. The function $F(s,t)$ can be represented as the sum of functions of three types: $F_1(s,t)$, $F_2(s,u)$, and $F_3(u,t)$; the denominators of the corresponding expressions for $F_1(s,t)$, $F_2(s,u)$, and $F_3(u,t)$ contain positive coefficients in front of s and t, s and u, u and t, respectively, and the denominators do not vanish for $s < 4m^2$, $t < 4m^2$, for $s < 4m^2$, $u < 4m^2$ and for $u < 4m^2$, $t < 4m^2$, for $F_1(s,t)$, $F_2(s,u)$, and $F_3(u,t)$, respectively.

1)The symbol $(\alpha| \ldots)$ $((s,t \mid \ldots))$ denotes the set of points $\alpha = (\alpha_1, \ldots, \alpha_n)$ (or the set of points (s,t)), satisfying the conditions $| \ldots)$.

According to Lemma 1, we may limit ourselves to functions $F(s,t)$ defined by integrals (2.1) such that $f(\alpha) \geq 0$, $g(\alpha) \geq 0$ and with the denominator $f(\alpha)s + g(\alpha)t - m^2k(\alpha) < 0$ in the domain $B_1(s,t \mid s < 4m^2, t < 4m^2)$. The remainder of the present section will refer just to such functions $F(s,t)$. In principle, the properties of the functions $F(s,u)$ and $F(u,t)$ with respect to the variables s, u and u, t do not differ from those of the function $F(s,t)$ with respect to the variables s, t. The function $F(s,t)$ will be considered on its "physical sheet," i.e., for $0 < \arg(s-4m^2), \arg(t-4m^2) < 2\pi$ and for $s,t \in B_1$ the function $F(s,t)$ takes on values determined by the expression (2.1).

Utilizing the inequality (2.2), we can prove a lemma which determines the complex domain in which the function $F(s,t)$ is holomorphic[5,7]:

Lemma 2. The function $F(s,t)$ is holomorphic in the domain D which contains: 1) all points (s,t) with Im s and Im t of the same sign and arbitrary Re s and Re t, 2) all points (s,t) with Im s and Im t of opposite signs, situated in the domain

$$\operatorname{Re} t - 4m^2 - \frac{\operatorname{Im} t}{\operatorname{Im} s}(\operatorname{Re} t - 4m^2) < 0. \tag{2.3}$$

It turns out that the domain (2.3) is a domain of holomorphy[5]. However the domain (2.3) is not the domain of holomorphy of the function $F(s,t)$. All singular points of the function (2.1) (situated not only on the "physical sheet" but on other sheets as well) are determined by the Landau equations[1,9,10] and are situated on the analytic surfaces

$$g_i(s,t) = 0, \quad i = 1, 2, \ldots, l, \tag{2.4}$$

where $g_i(s,t)$ are irreducible polynomials in s and t with real coefficients; the number of surfaces (2.4) depends on the structure of the Feynman diagram.[2)]

Not all points on the Landau surfaces (2.4) are singularities of the function $F(s,t)$ which are situated on the "physical sheet." Indeed, those points of the surfaces (2.4) which are situated within the domain D cannot be singular points. Further, one must take into account the circumstance that the singular points of a function of several complex variables cannot be arbitrarily distributed. Therefore, there arises the problem of determining all points on the Landau surfaces (2.4) which can be singularities on the "physical sheet" of the function $F(s,t)$.

This problem has been solved by the author in the papers[4,5] by use of Bremermann's continuity theorem[11]. Before formulating the basic result we introduce several definitions and notations and formulate the continuity theorem.

Consider a continuous layer of analytic planes $E(c)$ (c is a real number)

$$as + bt = c, \qquad a^2 + b^2 = 1. \tag{2.5}$$

Let there exist on these surfaces some domains $G(c)$ which converge continuously to the domain $G(c_I)$ on the boundary plane $E(c_I)$. Then the holomorphy of the function $F(s,t)$ in all points of the approximating domains $G(c)$ and in a single point of the approximated domain $G(c_I)$ implies the holomorphy of the function $F(s,t)$ in the whole domain $G(c_I)$.

In our case it is necessary to consider only those planes (2.5) for which $a > 0$, $b > 0$, since according to lemma 2 the function $F(s,t)$ is holomorphic for $|\operatorname{Im} s| > 0$, $|\operatorname{Im} t| > 0$ on all planes (2.5) for $a/b < 0$. We separate on the planes (2.5) $(a > 0,\ b > 0)$ the domains $G_1(c)$ $(s,t \mid \operatorname{Im} s > 0, \operatorname{Im} t < 0)$ and $G_2(c)$ $(s,t \mid \operatorname{Im} s < 0, \operatorname{Im} t > 0)$. For $c < c_1(a,b) = 4m^2\ (a+b)$ the domains $G_1(c)$ and $G_2(c)$ are contained in the domain (2.3).

The plane (2.5) has a finite number of intersection points with the surfaces (2.4); among the intersection points there may be both complex and real points. We denote by $c_0(a,b)$ the smallest number determined by the condition that some intersection points which are real for $c = c_0(a,b) \geq c_1(a,b)$ become complex for $c = c_0(a,b)$. (It may also happen that points which are complex for $c < c_0(a,b)$ become real for $c = c_0(a,b)$ and become again complex for $c > c_0(a,b)$. Such a situation arises when there exist isolated singularities.) The possibility that they again become real intersection points as c increases further is not excluded.

Using the continuity theorem, one can prove the validity of the following fundamental theorem:

Theorem. The function $F(s,t)$ is holomorphic in the domains $G_1(c)$ and $G_2(c)$ on the analytic planes $as + bt = c$, $a > 0, b > 0$ for $c \leq c_0(a,b)$.[3)]

Corollary 1. If for $c > c_1(a,b)$ and any $a > 0$, $b > 0$ the real intersection points do not become complex, i.e., the number $c_0(a,b)$ does not exist, the function $F(s,t)$ has only those singularities

[2)]We call Landau surfaces all surfaces formed by singular points of the function F(s,t), and not only those surfaces of singularities for which Landau has given the well known parametric representations[1,10].

[3)]If for some $a > 0$, $b > 0$, $c > c_1(a,b)$ no real intersection points become complex, then we consider that $c_0(a,b) = \infty$.

which are singularities of a function which possesses a Mandelstam representation, i.e., its singularities are situated on the hyperplanes: s real, $s > 4m^2$, t arbitrary; or t real, $t > 4m^2$, s arbitrary. In other words, if the number $c_0(a, b)$ does not exist for any $a > 0$, $b > 0$, then the function $F(s,t)$ admits a Mandelstam representation.

Corollary 2. Singularities of $F(s,t)$ can only be those points of intersection which become complex for $c > c_0(a, b)$ and also those complex intersection points which for a certain $c = c_0'(a, b)$ coincide with complex intersection points which appear for $c > c_0(a, b)$.

In order to determine the intersection points of the surface (2.4) with the surface (2.5) one must find the roots of the equation

$$g_i(s, -(a/b)\, s + c/b) = 0. \qquad (2.6)$$

In order that some intersection points which are real for $c = c_0(a, b)$ become complex for $c > c_0(a, b)$ it is necessary that some roots of (2.6), which are real for $c = c_0(a, b)$, become complex for $c > c_0(a, b)$ (due to the fact that the coefficients of the polynomials $g_i(s, t)$ are real there will be an even number of pairwise complex conjugate roots).

There exists a simple but rather lengthy criterion[12] which permits, on the basis of the coefficients of Eq. (2.6), i.e., on the basis of the coefficients of the polynomials $g_i(s,t)$, to determine the number of pairs of complex roots of the equation (2.6) and thus answers the question whether a number $c_0(a, b)$ exists. However, if the algebraic curve $g_i(s,t) = 0$ has no singularities in the domain $s \geq 4m^2$ or $t \geq 4m^2$ (the real section of the surface (2.4) is an algebraic curve which we will call the Landau curve) one can indicate a simple geometrical condition which allows one to establish the existence or non-existence of the number $c_0(a, b)$ on the basis of the behavior of the algebraic curve only in the domain $s > 4m^2$ or $t > 4m^2$. More precisely, we have the lemma:

Lemma 3. A necessary and sufficient condition for the appearance of complex intersection points for $c > c_0(a, b) > c_1(a, b)$, points which for $c \leq c_0(a, b)$ had been real, is that in the domain $s > 4m^2$ or $t > 4m^2$ there exist at least one portion on one of the branches of the algebraic curve $g_i(s,t) = 0$ with its convex side directed upward and for which the straight lines $as + bt = c_0(a, b)$ are tangents. If such a portion with upward convexity does not exist for the curve $g_i(s,t) = 0$ in the domain $s > 4m^2$ or $t > 4m^2$, then the number $c_0(a, b)$ does not exist and in this case the function $F(s,t)$ possesses a Mandelstam representation.

We remark here on the following circumstance. The Landau curve $g(s,t) = 0$ consists of a certain number of branches. In some cases it is known exactly which branch of this curve can be singular in the "physical sheet." Let this branch be described by the equation $t = t(s)$. It is also known that complex singularities of the contribution $F(s,t)$ on the "physical sheet" can be situated only on the surface $t = t(s)$. In order to solve the problem of the existence of a Mandelstam representation in this case, it is not necessary to analyze the behavior of all the branches of the Landau curve $g(s,t) = 0$, but it is sufficient to show that for the branch $t = t(s)$ the number $c_0(a, b)$ does not exist. If, for instance, the branch $t = t(s)$ has its convexity directed downward and has no singularities, the number $c_0(a, b)$ does not exist.

3. A CRITICISM OF THE PROOF OF THE MANDELSTAM REPRESENTATION BY MEANS OF EDEN'S METHOD

From the reasoning above it follows that the function $F(s,t)$ may have complex singularities only if certain roots of the equation $g_i(s, -(a/b)s + c/b) = 0$, which are real for $c = c_0(a, b) \geq c_1(a, b)$ become complex for $c > c_0(a, b)$ (a and b are arbitrary numbers such that $a > 0$, $b > 0$, $a^2 + b^2 = 1$). This result which we have obtained by means of the strengthened Bremermann continuity theorem may produce certain misunderstandings, the essence of which reduces to the following.

On the basis of their proof of the Mandelstam representation Eden et al.[13] have reached the conclusion that the function $F(s,t)$ can have complex singularities only in the case when the Landau curves have isolated singularities in the domain $s > 4m^2$, $t > 4m^2$. This result seems more restrictive than the one given above. Indeed, instead of investigating the intersection points of the curves $g_i(s,t) = 0$ with all straight lines $as + bt = c$ ($a > 0$, $b > 0$, $a^2 + b^2 = 1$, $c > c_1(a, b)$), one proposes to find only the isolated real singularities of the Landau curves in the domain $s > 4$, $t > 4$. We show that this result of Eden et al. is insufficiently founded.

Eden's mistake consists in an incorrect application of the continuity theorem (to say nothing of other false assertions). As we have seen above, in order to apply the continuity theorem it is necessary that there exist a layer of analytic planes $E(c)$ and on these planes there should be separated domains $G(c)$, which converge continuously to the domain $G(c_I)$ on the boundary plane $E(c_I)$. Eden has used that version of the continuity theorem, according to which the holomorphy of the function $F(s,t)$ in all points of the approximating domains $G(c)$ and their boundaries and the holomorphy on the boundary of the approximated domain $G(c_I)$ (and not just in one single point of the domain $G(c_I)$, as demanded by Bremermann's theorem) imply the holomorphy of the function $F(s,t)$ in the whole domain $G(c_I)$.

Eden applies the continuity theorem to the domains 1) $\operatorname{Im} s > 0$ and 2) $\operatorname{Im} s < 0$, situated on the analytic planes $t = c$. It follows from lemma 2 that the function $F(s,t)$ is holomorphic in these domains for $c < 4m^2$. For $c = 4m^2$ all points of the plane $t = c$ are singularities of the function $F(s,t)$ even if the Mandelstam representation is true, and the continuity theorem gives nothing for $c > 4m^2$. Therefore Eden chooses the number c as follows: $c = \operatorname{Re} c \pm i\epsilon$, $\epsilon > 0$, applies the continuity theorem to the domains 1) $\operatorname{Im} s > 0$ and 2) $\operatorname{Im} s < 0$ and arrives at the already mentioned conclusion on the validity of the Mandelstam representation, if the curve (2.4) has no isolated points. (Even disregarding the fact that not all conditions which are necessary for the applicability of the continuity theorem are satisfied, by applying the continuity theorem in an accurate and rigorous manner he could only arrive at the conclusion that the Mandelstam representation is valid under the condition that no real roots of the equation $g_i(s,t) = 0$, for $t > 4m^2$, should become complex.)

The error in Eden's proof consists in the following: his assertion that for $\operatorname{Re} c < 4m^2$, $\operatorname{Im} c = +\epsilon$ the function $F(s,t)$ is holomorphic in all points of the domains 1) $\operatorname{Im} s > 0$ and 2) $\operatorname{Im} s < 0$, as required by the continuity theorem, cannot be considered as having a rigorous foundation. Eden takes this fact as obvious, as if its validity would follow from property (2.2) of the denominator in the integrand of (2.1). Using the property (2.2) of the denominator Eden asserts that in the domain $s < 4m^2$, $t > 4m^2$ or $s > 4m^2$, $t < 4m^2$ there are no continuous branches of the Landau curves and no isolated singularities, and from this he derives the holomorphy of the function $F(s,t)$ in the required domain. As follows from subsequent work of Eden and collaborators, they admit now that the problem of the existence of isolated singularities is not solved even for the case of scattering of particles of equal mass. A concrete example of a diagram (cf. Sec. 4, Fig. 1c) shows that there are portions of branches of Landau curves in the domains $s < 4m^2$, $t > 4m^2$ or $s > 4m^2$, $t < 4m^2$.

Thus Eden's assertion that the function $F(s,t)$ is holomorphic for $\operatorname{Re} c < 4m^2$, $\operatorname{Im} c = \pm\epsilon$ in all points of the domains $\operatorname{Im} s > 0$ and $\operatorname{Im} s < 0$ cannot be considered as proved, since it is based on incorrect facts. It remains to be shown that the domain D does not contain all points of the domains $\operatorname{Im} s > 0$ and $\operatorname{Im} s < 0$, with $\operatorname{Re} c < 4m^2$, $\operatorname{Im} c = \pm\epsilon$, either. Indeed, the domain contains: for $\operatorname{Im} t = +\epsilon$ all points of the domain $\operatorname{Im} s > 0$ and for $\operatorname{Im} t = -\epsilon$ —all points of the domain $\operatorname{Im} s < 0$ and only those points of the domain $\operatorname{Im} s < 0$, for $\operatorname{Im} t = +\epsilon$ and of the domain $\operatorname{Im} s > 0$ for $\operatorname{Im} t = -\epsilon$, which also satisfy the inequality (2.3). Therefore one is not allowed to apply the continuity theorem for $|\epsilon| > 0$ in the form which has been used by Eden, and his conclusions are not proved.

4. PROOF OF THE MANDELSTAM REPRESENTATION FOR CERTAIN FEYNMAN DIAGRAMS

A. Let us consider the "envelope" diagram, represented in Fig. 1a. It is obvious that for the "envelope" the domain B_1 has the form $B_1(s,t \mid s < 9m^2,\ t < 9m^2)$ and the functions $f(\alpha) \geq 0$ and $g(\alpha) \geq 0$. For the sake of convenience we will set $m = 1$ in this section.

The Landau curve corresponding to the proper singularities of this diagram has been computed by Kolkunov, Okun', and Rudik [14] and has the following parametric representation:

$$\begin{aligned} t &= 1 + 2\,[1 + \cos(\alpha + \pi/3)]^2 - 2\,(1 \\ &\quad + \cos\alpha)\ [1 + \cos(\alpha + \pi/3)]\ \sin(\alpha + \pi/3)/\sin\alpha, \\ s &= 1 + 2\,(1 + \cos\alpha)^2 - 2\,(1 + \cos\alpha)\ [1 \\ &\quad + \cos(\alpha + \pi/3)]\sin\alpha/\sin(\alpha + \pi/3). \end{aligned} \tag{4.1}$$

It is easy to see that the Landau curves for the reduced diagrams corresponding to the "envelope" degenerate into the straight lines $s = 9$ and $t = 9$.

Let us concentrate our attention on the qualita-

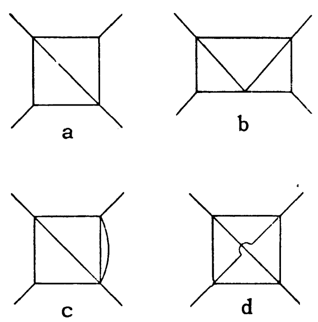

FIG. 1

tive behavior of the curve (4.1). Since the right-hand sides of the equalities (4.1) are periodic functions of α with the period 2π, we consider the expressions (4.1) for $0 \leq \alpha \leq 2\pi$. From this expression it follows that the Landau curve of our diagram has the following asymptotes (we give not only the asymptotes but also the values to which they correspond):

$$\begin{aligned}
&1)\ \alpha = +0,\ t = -\infty,\ s = 9-0;\\
&2)\ \alpha + \pi/3 = 2\pi - 0,\ t = 9-0,\ s = -\infty;\\
&3)\ \alpha + \pi/3 = 2\pi + 0,\ t = 9+0,\ s = +\infty;\\
&4)\ \alpha = 2\pi - 0,\ t = +\infty,\ s = 9+0.
\end{aligned} \tag{4.2}$$

From Eq. (4.1) it also follows that $t(\alpha) < 9$ and $s(\alpha) < 9$ for $0 < \alpha < \frac{5}{3}\pi$. An elementary computation shows that $dt/d\alpha$ and $ds/d\alpha$ can be expressed in the following form:

$$\begin{aligned}
\frac{dt}{d\alpha} &= -2\sin^{-2}\alpha\Big\{(1+\cos\alpha)\sin\alpha\Big[\cos\Big(\alpha+\frac{\pi}{3}\Big) + \cos 2\Big(\alpha+\frac{\pi}{3}\Big)\Big]\\
&\quad - (\cos\alpha + \cos 2\alpha)\sin\Big(\alpha+\frac{\pi}{3}\Big)\Big[1+\cos\Big(\alpha+\frac{\pi}{3}\Big)\Big]\Big\},\\
\frac{ds}{d\alpha} &= -2\sin^{-2}\Big(\alpha+\frac{\pi}{3}\Big)\Big\{\Big[1+\cos\Big(\alpha+\frac{\pi}{3}\Big)\Big]\sin\Big(\alpha+\frac{\pi}{3}\Big)\\
&\quad \times(\cos\alpha + \cos 2\alpha)\\
&\quad - \Big[\cos\Big(\alpha+\frac{\pi}{3}\Big) + \cos 2\Big(\alpha+\frac{\pi}{3}\Big)\Big]\sin\alpha(1+\cos\alpha)\Big\}.
\end{aligned} \tag{4.3}$$

This implies that for $\frac{5}{3}\pi < \alpha < 2\pi$ the function $dt/d\alpha > 0$ and the function $ds/d\alpha < 0$.

From (4.3) we obtain

$$dt/ds = -\sin^2(\alpha + \pi/3)/\sin^2\alpha, \tag{4.4}$$

i.e., the curve (4.1) has everywhere a negative slope. Taking into account Eqs. (4.2) we come to the conclusion that the curve (4.1) has two branches: one in the domain $s > 9,\ t > 9$ ($\frac{5}{3}\pi < \alpha < 2\pi$), and the other in the domain $s < 9,\ t < 9$ ($0 < \alpha < \frac{5}{3}\pi$). On the basis of Eq. (4.4) we obtain

$$\begin{aligned}
\frac{d^2t}{ds^2} &= \frac{1}{s'_\alpha}\frac{d}{d\alpha}\left(\frac{t'_\alpha}{s'_\alpha}\right)\\
&= \frac{1}{s'_\alpha}\,\frac{\sin 2\alpha \sin^2(\alpha+\pi/3) - \sin 2(\alpha+\pi/3)\sin^2\alpha}{\sin^4\alpha},
\end{aligned}$$

i.e., $d^2t/ds^2 > 0$ for $\frac{5}{3}\pi < \alpha < 2\pi$. Thus the branch which is situated in the domain $s > 9,\ t > 9$ has its convex side downward.

Since $dt/d\alpha$ and $ds/d\alpha$ do not vanish anywhere for $\frac{5}{3}\pi < \alpha < 2\pi$, the branch which is situated in the domain $s > 9,\ t > 9$ has no singularities; it is easy to show that isolated singularities are also absent [13].

Thus, according to lemma 3, the number $c_0(a, b)$ does not exist for the curve (4.1) and the contribution of the "envelope" diagram admits the Mandelstam representation

$$F(s, t) = \int_9^\infty\int_9^\infty \frac{\rho(s', t')}{(s'-s)(t'-t)}\,ds'dt'.$$

B. Let us consider the "truss" diagram of Fig. 1b. The interest in this diagram has been aroused by a recent paper by Kim, in which it is claimed that the contribution from this "truss" diagram does not possess a Mandelstam representation.

The Landau curves corresponding to the proper singularities of this diagram have been computed by Liu Yi-ch'en and Todorov [15] and have the following parametric representation

$$\begin{aligned}
\frac{1}{3}s &= 3 + \frac{2}{\lambda}\,\frac{1}{\lambda^2+3\lambda+3},\\
\frac{1}{3}t &= \frac{(\lambda+2)^3}{(\lambda^2+3\lambda+3)^2}(4\lambda^2+9\lambda+6).
\end{aligned} \tag{4.5}$$

The "truss" diagram has only one reduced diagram, for which the singularity curves depend simultaneously on s and t and which is shown in Fig. 1c. The proper singularities of this diagram are situated on the curve [15]

$$t = 16s(s-4)/(s-1)(s-9). \tag{4.6}$$

For the diagram represented in Fig. 1c, as for the "envelope" diagram, all singularities of the reduced diagrams do not depend on s and t simultaneously. For the diagrams in Figs. 1b and c the domain B_1 has the form $B_1(s, t \mid s < 9,\ t < 16)$, and the functions $f(\alpha) \geq 0$ and $g(\alpha) \geq 0$.

Let us investigate the curve (4.5). It is easy to see that it has the following asymptotes:

$$\begin{aligned}
&1)\ \lambda = +0,\ s = +\infty,\ t = 16+0;\\
&2)\ \lambda = -0,\ s = -\infty,\ t = 16-0;\\
&3)\ \lambda = +\infty,\ s = 9+0,\ t = +\infty;\\
&4)\ \lambda = -\infty,\ s = 9-0,\ t = -\infty.
\end{aligned} \tag{4.7}$$

From Eqs. (4.5) we obtain

$$\begin{aligned}
\frac{ds}{d\lambda} &= -3\,\frac{2\lambda^4+10\lambda^3+27\lambda^2+36\lambda+18}{\lambda^2(\lambda^2+3\lambda+3)^2},\\
\frac{dt}{d\lambda} &= 3\,\frac{(\lambda+2)^2(4\lambda^4+20\lambda^3+54\lambda^2+72\lambda+36)}{(\lambda^2+3\lambda+3)^3}.
\end{aligned} \tag{4.8}$$

On the basis of Eqs. (4.8) we see that $ds/d\lambda < 0$ and $dt/d\lambda > 0$ for $\lambda > 0$, and from Eqs. (4.8) it follows that

$$dt/ds = -2\lambda^2(\lambda+2)^2/(\lambda^2+3\lambda+3),$$

i.e., $dt/ds < 0$ for all λ.

Thus, the curve (4.5) consists of two branches, one in the domain $s > 9$, $t > 16$ and the other in the domain $s < 9$, $t < 16$, both having negative slopes.

For d^2t/ds^2 we obtain the following formula:

$$\frac{d^2t}{ds^2} = -2\frac{1}{s'_\lambda}\frac{\lambda(\lambda+2)[(4\lambda+4)(\lambda^2+3\lambda+3)-(2\lambda+3)(\lambda^2+2\lambda)]}{(\lambda^2+3\lambda+3)^2}, \tag{4.9}$$

which implies that $d^2t/ds^2 > 0$ in the domain $s > 9$, $t > 16$. Thus the branch of the curve (4.5) which is situated in the domain $s > 9$, $t > 16$ has its convex side pointing downwards and thus has no singularities.

Since for real positive λ, $ds/d\lambda \neq 0$, the implicit function theorem shows that for $\lambda > 0$, $s > 9$ the first equation (4.5) can be represented in the form $\lambda = \lambda(s)$. Substituting $\lambda = \lambda(s)$ into the second equation (4.5) we obtain the equation of the analytic surface $t = t(\lambda(s))$.

In the paper by Liu and Todorov[15] it has been shown that the singular curve is on the "physical sheet" for real positive λ and consequently the singularities from the contribution on the "physical" sheet can be situated only on the surface $t = t(\lambda(s))$. Therefore we can restrict ourselves to a consideration of that branch of the Landau curve (4.5) which corresponds to real positive λ. Obviously, the number $c_0(a, b)$ does not exist for this branch.

Let us now consider the curve (4.6). It has the asymptotes $s = 9$, $s = 1$. We obtain the following expressions for the derivatives:

$$\frac{dt}{ds} = -96\frac{s^2-3s+6}{(s-1)^2(s-9)^2},$$

$$\frac{d^2t}{ds^2} = 96\frac{2s^3-9s^2+36s-93}{(s-1)^3(s-9)^2}. \tag{4.10}$$

It follows from (4.10) that $dt/ds < 0$ for all s, therefore the curve (4.6) has no singularities.

Equation (4.6) implies that the inequality $16 < t(s) < \infty$ holds for $9 < s < +\infty$, the inequality $-\infty < t(s) < \infty$ for $1 < s < 9$, and the inequality $t(s) < 16$ for $s < 1$. The curve (4.6) has three branches, one in the domain $s > 9$, $t > 16$, the second in the domain $1 < s < 9$, and the third in the domain $s < 1$, $t < 16$. For $s > 9$ we have $d^2t/ds^2 > 0$, i.e., the branch situated in the domain $s > 9$, $t > 16$ is convex downward.

From (4.6) it follows that $t(s) > 16$ for $1 < s < 3/2$, and from Eq. (4.10) it follows that $d^2t/ds^2 > 0$ for $1 < s < 3/2$. Therefore the portion of the branch of the curve (4.6) which is situated outside the domain $s < 9$, $t < 16$ has a negative slope and is convex downward. One should note that the curve (4.6) has the asymptote $s = 1$ which has nothing to do with normal thresholds, and for $1 < s < 3/2$ its branch leaves the domain $t < 16$ (cf. Sec. 3 in this connection). According to lemma 3 the curve (4.6) also does not have a number $c_0(a, b)$.

Thus for the curves (4.5) and (4.6) the number $c_0(a, b)$ does not exist and the "truss" diagram possesses the Mandelstam representation

$$F(s, t) = \int_9^\infty ds' \int_{16}^\infty dt' \frac{\rho(s', t')}{(s'-s)(t'-t)}.$$

At the same time we have proved that the diagram in Fig. 1,c admits a Mandelstam representation.

C. Let us consider the diagram of the "opened envelope" or "tetrahedron" type, represented in Fig. 1d. For this diagram the domain B has the form $B(s, t \mid s < 16, t < 16, u < 16)$ and the functions $f(\alpha)$ and $g(\alpha)$ are of indefinite sign. The diagram of Fig. 1d is of very great interest, since the indefiniteness of the sign of the functions $f(\alpha)$ and $g(\alpha)$ implies that it can be represented on the basis of lemma 1 in the form of three functions $F_1(s, t)$, $F_2(s, u)$, and $F_3(u, t)$ and that it will possess a Mandelstam representation with three terms.

The Landau curve corresponding to the proper singularities of this diagram has been computed in the paper by Kolkunov, Okun', and Rudik[14] as well as in the paper of Lugunov, Todorov, and Chernikov[10]. We will use the results of the latter work. The Landau curve can be represented in the form

$$(s/16)^{1/3} + (t/16)^{1/3} + (u/16)^{1/3} = 1. \tag{4.11}$$

The singularities of all the reduced diagrams are situated on the surfaces $s = 16$, $t = 16$, $u = 16$. We will prove the Mandelstam representation for the function $F_1(s, t)$. Due to the symmetry of the "opened envelope" diagram we have the relation $F_1(s, u) = F_2(s, u)$, $F_1(u, t) = F_3(u, t)$, therefore the Mandelstam representation is valid for $F_2(s, u)$ and $F_3(u, t)$ if it is valid for $F_1(s, t)$. The singularities of the function $F_1(s, t)$ are determined by the same Landau equations as the singularities of the functions $F(s, t)$, i.e., the singularities of the functions $F_1(s, t)$ are also situated on the surface (4.11).

We start the discussion of the curve (4.11). For the sake of convenience we also make use of the parametric representation of this curve[10]:

$$\frac{1}{2}s(\lambda) = \left(\frac{1/2-\lambda}{1/4-\lambda}\right)^3,$$

$$t_{\mp}^{(\lambda)} = \left\{\frac{\lambda \mp 1/2\,\lambda^{-1/2}[(1/2-\lambda)(1-2\lambda-4\lambda^2)]^{1/2}}{2(\lambda-1/4)}\right\}^3. \tag{4.12}$$

Since we are not interested in the behavior of the curve (4.11) in the region $s < 16$, $t < 16$, we only investigate its behavior for $s > 16$, $t > 16$. Owing to the symmetry of (4.11) with respect to s and t, it is sufficient to investigate the curve (4.11) for $s > 16$.

It follows from (4.12) that $s > 16$ only for $0 < \lambda < 1/4$. Indeed, from (4.12) it follows that

$$\frac{ds}{d\lambda} = \frac{3}{2}\left(\frac{1/2-\lambda}{1/4-\lambda}\right)^2 \frac{1}{(1/4-\lambda)^2}, \tag{4.13}$$

i.e., $ds/d\lambda > 0$. Besides, we have

$$s(\lambda)|_{\lambda=\pm\infty} = 2, \qquad s(\lambda)|_{\lambda=0} = 16, \qquad s(\lambda)|_{\lambda=1/4\mp 0} = \pm\infty.$$

Thus, $16 < s(\lambda) < \infty$ for $0 < \lambda < 1/4$.

We show that for $0 < \lambda < 1/4$ the variable $t_-(\lambda)$ is inside the interval $16 < t_-(\lambda) < \infty$. Indeed, we have

$$t_-(\lambda)|_{\lambda=+0} = +\infty, \qquad t_-(\lambda)|_{\lambda=1/4-0} = 16+0.$$

From Eq. (4.11) we obtain

$$\frac{dt}{ds} = -\frac{1-(4-s-t)^{2/3}/s^{2/3}}{1-(4-s-t)^{2/3}/t^{2/3}}, \tag{4.14}$$

therefore the slope of the curve (4.11) is negative in the domain $s > 16$, $t > 16$.

Instead of considering the function $t(\lambda)$ for $0 < \lambda < 1/4$ we can consider the function $t(s) = t(\lambda(s))$ for $16 < s < \infty$. If for a certain $s > 16$, the function $t(s)$ would take on values which are smaller than 16, then, due to the fact that $t(s) \to 16 + 0$ as $s \to \infty$, the quantity ds/dt would take on positive values in the region $t > 16$, $s > 16$. Thus we have $16 < t(s) < \infty$ for $16 < s < \infty$ or $16 < t(\lambda) < \infty$ for $0 < \lambda < 1/4$. Thus the curve (4.11) has one branch in the domain $s > 16$, $t > 16$, with the asymptotes $s = 16$ and $t = 16$ and with negative slope.

For the investigation of the second case $t = t_+(\lambda)$ it is more convenient to use the (s, u)-plane, since in this case $u(\lambda) = t_-(\lambda)$. As in the first case, we come to the conclusion that the second branch of the curve (4.11) is entirely contained inside the domain $s > 16$, $u > 16$ and has the asymptotes $s = 16$ and $u = 16$. Obviously, there are no singularities in the domain $s > 16$.

Let us now investigate the slope of the curve (4.11) situated in the domain $s > 16$, $s+t < -12$ $(u > 16)$. It follows from Eq. (4.14) that $dt/ds > 0$ in the domain $s > 16$, $2s+t < 4$, and $ds/dt < 0$ in the domain $s > 16$, $2s+t > 4$. For d^2t/ds^2 we obtain the expression

$$\frac{d^2t}{ds^2} = \frac{2}{3}\left\{\frac{[t^{-2/3}-(4-s-t)^{-2/3}]\,[s^{-5/3}+(4-s-t)^{-5/3}\,dt/ds]}{[t^{-2/3}-(4-s-t)^{-2/3}]^2} - \frac{[s^{-2/3}-(4-s-t)^{-2/3}]\,[(4-s-t)^{-5/3}+t^{-5/3}\,dt/ds+(4-s-t)^{-5/3}\,dt/ds]}{[t^{-2/3}-(4-s-t)^{-2/3}]^2}\right\}. \tag{4.15}$$

It follows from Eq. (4.15) that $d^2t/ds^2 > 0$ in the domain $s > 16$, $t > 16$, i.e., the branch which is situated in the domain $s > 16$, $t > 16$ is convex downward and has negative slope. In the domain $s > 16$, $2s+t < 4$, the quantity $d^2t/ds^2 < 0$, and in the domain $s > 16$, $2s+t > 4$, $d^2t/ds^2 < 0$. Thus the second branch, situated in the domain $s > 16$, $s+t < -12$ has in the domain $s > 16$, $2s+t < 4$ a positive slope and the convexity directed upward, and in the domain $s > 16$, $2s+t > 4$ it has a negative slope and the convexity also directed upward.

The domain $t > 16$ contains, besides the already investigated branch in the domain $s > 16$, $t > 16$, another branch in the domain $t > 16$, $s+t < -12$. For $2t+s < 4$ its slope is positive, $dt/ds > 0$, and for $2t+s > 4$ its slope is negative, $dt/ds < 0$; besides $d^2t/ds^2 > 0$ for $t > 16$, $s+2t > 4$ and $d^2t/ds^2 < 0$ for $t > 16$, $s+2t > 4$. Thus for $2t+s < 4$, this branch is convex downward and for $2t+s > 4$ it is convex upward.

The qualitative behavior of the curve (4.11) in the domain $s > 16$, $t > 16$ is shown in Fig. 2. In this case, too, there is no number $c_0(a, b) \geq c_1(a, b) = 16(a+b)$. Indeed, although there are portions of branches of the curve which are convex upward and have positive slope, those are situated in the region $s+t < -12$, and it is easy to see that the

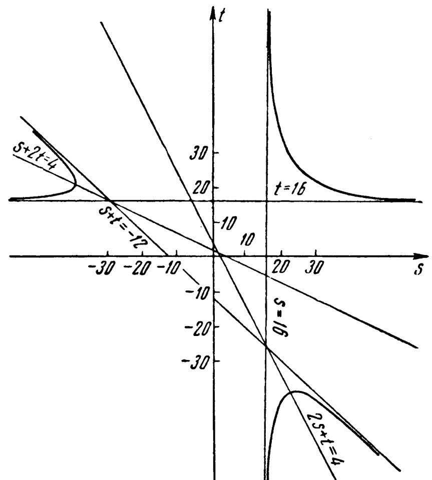

FIG. 2

distance $c(a, b)$ between the tangents $as + bt = c(a, b)$ and the origin of the coordinates will always be smaller than $c_1(a, b)$.

Thus, the number $c_0(a, b)$ does not exist and the function $F_1(s, t)$ possesses the Mandelstam representation

$$F_1(s, t) = \int_{16}^{\infty} ds' \int_{16}^{\infty} dt' \frac{\rho(s', t')}{(s' - s)(t' - t)}.$$

5. CONCLUSION

It is obvious that the method developed above can be used to prove the Mandelstam representation for many diagrams. It is sufficient to obtain the Landau curves and to analyze these curves qualitatively. Difficulties can appear only in the case when the Landau curves corresponding to the singularities of the reduced diagrams have asymptotes situated higher than the curves which correspond to the proper singularities of the given diagram. Such a situation arises when one considers ladder diagrams. But these difficulties can be overcome, and we hope to expose the corresponding results in a subsequent paper.

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Note added in proof (January 16, 1964). In the proof of the Mandelstam representation in Sec. 4 we have in fact used the assumption that on the "physical sheet" all singularities are exhausted by those surfaces which admit the Landau parametric representations[1]. It can be shown that this assumption is satisfied for the diagrams which we have considered.

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