NONLINEAR EFFECTS LIMITING THE AMPLIFICATION OF SOUND IN PIEZO-ELECTRICS

V. L. GUREVICH and B. D. LAĬKHTMAN

Semiconductor Institute, U.S.S.R. Academy of Sciences

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Nonlinear effects limiting the amplification of sound in piezoelectric semiconductors in a stationary electric field are considered. It is demonstrated that nonlinearities of electronic origin set in as a rule long before the nonlinearities of the crystal elastic properties. Nonlinear effects result in stationary waves which move in the crystal without amplification or damping. An expression is obtained for the amplitude of the waves as a function of the stationary electric field strength. The damping or growth of waves that differ little from the stationary ones is studied. Stability of stationary waves against small changes of the amplitude is investigated.

1. INTRODUCTION

WHEN a sound wave propagates in a piezoelectric, an alternating inhomogeneous electric field is produced, proportional to the strain. In the presence of conduction electrons [1)] this field produces an alternating current, which causes both a specific electronic absorption of sound and a renormalization of its velocity. Such an absorption was considered theoretically by Hutson and White[1] and by one of the authors[2], and was also observed in experiment[3].

As shown earlier[1,2], the corresponding sound absorption coefficient Γ_e, characterizing the spatial attenuation of the square of the sound-wave amplitude, is equal to

$$\Gamma_e = \frac{4\pi\beta^2}{\varepsilon\rho w^2}\frac{\omega}{w_g}\frac{\omega\tau_M}{(1+q^2/\varkappa^2)^2+(\omega\tau_M)^2}. \quad (1.1)$$

Here β, σ, ϵ, and ρ are respectively the piezoelectric coefficient, the conductivity, the dielectric constant, and the density; κ is the reciprocal Debye-Huckel radius, $\tau_M = \epsilon/4\pi\sigma$—Maxwellian relaxation time, ω and q—frequency and wave vector of the sound, $w = \omega/q$, and $w_g = \delta\omega/\delta q$ are the phase and group velocities of sound.

The dependence of Γ_e on ω and on q can be explained as follows. The alternating electric field produces an alternating conduction-current density j. This current depends on the coordinates and consequently causes a redistribution of the free charges. The redistributed charges produce in turn an additional electric field. This field has a direction opposite that of the initial piezoelectric field and consequently leads to a decrease in the conduction current. τ_M is the characteristic time during which the redistribution of the free charges occurs. Under static deformation the charges are redistributed in such a way that their field cancels out completely the piezoelectric field so that the conduction current becomes equal to zero.

If the deformation varies with the frequency $\omega \ll 1/\tau_M$, there is no chance for total compensation to set in, and a small difference field proportional to ω remains. Consequently, $j \sim \omega$ and the coefficient Γ_e is determined by the mean square of the current density and is proportional to ω^2.

In the opposite limiting case, when $\omega \gg 1/\tau_M$, the space-charge field does not have any time at all to become produced during one period of the sound. Therefore the proportionality coefficient between j and the electric field turns out to be independent of the frequency. The coefficient Γ_e is also independent of the frequency. The term $(\omega\tau_M)^2$ in the denominator of (1.1), which determines the temporal dispersion of Γ_e, is just the one ensuring transition from one limiting case to the other.

Another mechanism which also contributes to the decrease in the absorption is diffusion. The diffusion current is proportional to the gradient of the electron concentration and is directed opposite to this gradient. For long waves, when the concentration gradient is small, the diffusion current is

[1)]We are considering a semiconductor with carriers of the same polarity; for concreteness we shall assume them to be electrons.

also small. It becomes comparable with the conduction current when $q \sim \kappa$, and when $q \gg \kappa$ it almost completely cancels out the latter. This circumstance is reflected by the term $(1 + q^2/\kappa^2)^2$ in the denominator of (1.1), which determines the spatial dispersion of the absorption coefficient.

The equation for the phase velocity of sound in a piezoelectric dielectric, $w = \omega/q$, is of the form

$$w_1^2 = w_0^2 + \delta w^2 = c / \rho + 4\pi\beta^2 / \varepsilon\rho, \qquad (1.2)$$

where the first term w_0^2 is connected with the short-range forces of ordinary elasticity theory, and the second δw^2 is due to the additional quasi-elastic forces connected with the piezoelectric field.

In piezoelectric semiconductors we obtain in place of (1.2) the following expression for δw^2:

$$\delta w^2 = \frac{4\pi\beta^2}{\varepsilon\rho} \frac{(q^2/\varkappa^2)(1 + q^2/\varkappa^2) + (\omega\tau_M)^2}{(1 + q^2/\varkappa^2)^2 + (\omega\tau_M)^2}. \qquad (1.3)$$

If $\omega\tau_M$ is small then, as noted above, the field of the free charges compensates the piezoelectric field or, in other words, the free charges screen the piezoelectric field. Because of this there appears in the expression for the piezoelectric addition δw_0^2 a factor $q^2/(q^2 + \kappa^2)$, which takes the screening effect into account. At large $\omega\tau_M$ the screening does not have time to become established, and when $\omega\tau_M$ increases the expression for δw^2 tends to the value which this quantity has in dielectrics.

Hutson et al.[4] investigated the absorption of sound in piezoelectrics in the presence of a constant electric field U_0. It was observed that in this case

$$\Gamma_e = \frac{4\pi\beta^2}{\varepsilon\rho w^2} \frac{\omega}{w_g} \frac{(\omega - qV)\tau_M}{(1 + q^2/\varkappa^2)^2 + (\omega - qV)^2\tau_M^2}, \qquad (1.4)$$

where V is the drift velocity of the conduction electrons in the field E_0. When $qV > \omega$, this expression reverses sign, that is, electronic amplification of sound takes place.

The denominators in (1.1) and (1.4) are quite similar. They differ only in the fact that in (1.4) ω is replaced by $\omega - qV$, which is the frequency of sound in the coordinate system that moves with velocity V together with the electron stream. For the piezoelectric addition to the sound velocity we obtain an analogous expression

$$\delta w^2 = \frac{4\pi\beta^2}{\varepsilon\rho} \frac{(q^2/\varkappa^2)(1 + q^2/\varkappa^2) + (\omega - qV)^2\tau_M^2}{(1 + q^2/\varkappa^2)^2 + (\omega - qV)^2\tau_M^2}. \qquad (1.5)$$

The total absorption coefficient Γ is the sum of the electronic part Γ_e and of the lattice part Γ_l. When this sum is negative, sound is amplified. These are the main results of the linear theory.

With increasing amplitude of the amplified wave, its velocity and its amplification coefficient begin to become amplitude-dependent, that is, nonlinear effects appear. If the amplification coefficient decreases with increasing amplitude and at some value of the amplitude it vanishes, then this value corresponds to a stationary wave which propagates in the crystal without being amplified or damped 2). Its amplitude and phase velocity depend on E_0. The purpose of the present paper is to study the role of the nonlinear effects in the propagation of stationary waves and, in particular, to determine this dependence 3). We assume here that the constant electric field is independent of the amplitude of the sound wave, something which is readily realized in experiment.

The nonlinearity can be due both to nonlinear effects of the elasticity theory and to effects of purely electronic origin. The former begin to assume a role when the deformations become comparable with unity, and the latter, as will be shown below, at much lower deformations. There should therefore exist a sound-wave amplitude interval in which the former are insignificant and the latter already assume a role. Only this region will be considered, so that the equations of elasticity theory will be assumed to be linear. We have confined ourselves to an analysis of the simplest possibility, when the current density is a linear function of the electron concentration and of the electron field.

We shall analyze qualitatively a case when the lattice absorption of sound is negligibly small. It can be shown that in the nonlinear theory the sign of Γ_e coincides with the sign of the difference $w - V$. Let us see how this difference behaves with the increasing amplitude of the sound. W_0^2, which is a quantity not of piezoelectric origin, should remain unchanged. In order to visualize the behavior of δw^2, we must understand what happens to the screening effects at large sound amplitudes.

Since the conduction electron concentration is relatively small in a semiconductor, with the in-

2)We shall see below that there exists also another possibility, when the gain increases with increasing amplitude in a definite interval of amplitude values. In this case the stationary waves are either missing or unstable.

3)For the particular case of a stationary wave of small amplitude in a medium with negligibly small lattice absorption, these questions were considered in a paper by one of the authors [5].

creasing wave amplitude the displacement of the free charges becomes difficult, and in the limiting case of large amplitudes they all cluster in regions where the potential has a minimum, and move together with the wave. In other words, the nonlinear effects make it more difficult to screen the piezoelectric fields by the free charges. Therefore when the amplitude changes from infinitesimally small to very large values, the quantity δw^2 increases from the value given by (1.4) to the value $4\pi\beta^2/\epsilon\rho$ which is characteristic of a dielectric in which there are no screening effects at all.

Therefore, if the drift velocity V is in the interval between w_h, which is the velocity of sound given by the linear theory, and w_1 we can point to the following mechanism whereby the amplification of the sound is limited. Specifically, the amplitude of the sound wave will increase until the difference V − w vanishes, after which further amplification ceases. In Sec. 3 of the present paper we obtain an exact solution of this problem for a stationary wave of finite amplitude; the solution confirms these arguments.[4)]

An account of the lattice absorption makes this picture more complicated. For a stationary wave to exist it is necessary that the total absorption coefficient be equal to zero. To this end the electronic absorption coefficient should be negative, that is, the drift velocity V should exceed the velocity of sound w. As can be seen from formula (1.4) in the linear theory $|\Gamma_e|$ goes through a maximum as a function of V. If Γ_l exceeds this maximum value, then the amplification of the sound in the linear theory is completely impossible. In the converse case there are two velocity values V_1 and V_2 ($V_1 < V_2$) at which a stationary linear mode is realized.

It can be shown that when $4\pi\beta^2/\epsilon\rho w^2 \ll 1$, $\eta\omega/\rho w^2 \ll 1$, and when $V_1 < V < V_2$, there exists a stationary mode with finite amplitude. We investigated by the iteration method the stationary modes of finite amplitude, which are close to one of two linear stationary modes. We also investigated the stability of the corresponding waves against small changes in amplitude. It turns out that at sufficiently large viscosity they are stable, and at low viscosity they are stable if V is close to V_1 and unstable if V is close to V_2. We note that the question of stability of the mode with stationary wave against small perturbations with a different frequency or with a different direction of propagation remains open and is worthy of special analysis.

[4)]The first to point out the existence of nonlinear effects connected with the redistribution of the electrons upon propagation of a sound wave was Hutson[6]. He proposed that in the case of generation of sound, that is, during the growth of sound oscillations amplified from thermal noise, the strong scattering of different oscillations by one another, due to these nonlinear interactions, can cause a stationary mode to be established. In the present work we are considering a different problem, that of the stationary mode arising in a crystal when one wave of sufficiently large amplitude propagates. This problem has a direct bearing on the question not of generation, but of amplification of sound in piezoelectrics.

2. FUNDAMENTAL EQUATIONS OF THE PROBLEM

The system of equations of motion of a piezoelectric conducting medium has the following form (summation over repeated indices is implied)

$$\rho\ddot{u}_i = c_{iklm}\,\partial u_{lm}/\partial x_k + \beta_{l,ik}\,\partial E_l/\partial x_k + \eta_{iklm}\,\partial\dot{u}_{lm}/\partial x_k, \tag{2.1}$$

$$\varepsilon_{ik}\partial E_k/\partial x_i - 4\pi\beta_{i,kl}\,\partial u_{kl}/\partial x_i = 4\pi en, \tag{2.2}$$

$$e\partial n/\partial t + \operatorname{div}\mathbf{j} = 0. \tag{2.3}$$

Here u—vector of displacement of the continuous medium, c_{iklm}—tensor of the elastic moduli (at constant electric field **E**), η_{iklm}—viscosity-coefficient tensor, $\beta_{l,ik}$—piezoelectric coefficient tensor (symmetrical in the last two indices), ϵ_{ik}—dielectric constant (at a constant deformation tensor u_{ik}), n—excess electron density (compared with the stationary value n_0), e—electron charge, and **j**—current density. The continuity equation (2.3) holds if generation and recombination of current carriers is neglected. For simplicity we disregard the contribution made by heat conduction to the lattice absorption.

This system must be supplemented by a relation for the dependence of **j** on **E**, n, and $\partial n/\partial x_k$. We start from the simplest assumption[5)]:

$$j_i = \sigma_{ik}E_k - D_{ik}\partial n/\partial x_k. \tag{2.4}$$

The conductivity σ_{ik} is considered to be a linear function of the electron concentration

$$\sigma_{ik} = e(n_0 + n)\mu_{ik}, \tag{2.5}$$

[5)]When the main mechanism of electron energy relaxation is their scattering by acoustic phonons, the assumption that j is linearly dependent on **E** is not satisfactory, since when V is of the order of w, deviations from Ohm's law should already be noticeable[7]. However, if the energy relaxation is due to optical phonons, then the linearity assumption is justified. In Smith's experiments with CdS[6] a linear dependence of j on $\mathbf{E}_0$ was observed right up to the start of the acoustic instability.

where μ_{ik} is the electron mobility. Then the diffusion coefficient tensor D_{ik}, which is connected with μ_{ik} by the Einstein relation

$$D_{ik} = \mu_{ik}T/e \tag{2.6}$$

(where T—temperature in energy units), does not depend on n.

We consider further the simplest case of a longitudinal stationary wave propagating along some symmetry axis of the crystal, which we choose to be the x axis. The tensor indices of the tensor σ_{xx}, $\beta_{x,xx}$, etc., will be left out, and the x component will be represented in the form of a sum $E_x = E_0 - \partial\varphi/\partial x$, where E_0 is the dc component and $\partial\varphi/\partial x$ the ac component. The x derivatives with respect to the time are expressed in terms of the derivatives with respect to the coordinate x in accordance with the formula $\partial/\partial t = -w\partial/\partial x$ [6], where w is the wave phase velocity. Inserting in (2.3), following such a substitution, the expression (2.4) with account of (2.5) and (2.6), we reduce the system to the form

$$(\rho w^2 - c)\, u = -\beta\varphi - \eta w \partial u/\partial x, \tag{2.7}$$

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{4\pi\beta}{\varepsilon}\frac{\partial^2 u}{\partial x^2} - \frac{4\pi e}{\varepsilon}n, \tag{2.8}$$

$$\frac{\partial n}{\partial x} - \left[\varkappa^2\tau_M (V - w) - \frac{e}{T}\frac{\partial\varphi}{\partial x}\right] n + n_0\frac{e}{T}\frac{\partial\varphi}{\partial x} = \frac{e}{T}\left\langle n\frac{\partial\varphi}{\partial x}\right\rangle. \tag{2.9}$$

We assume here that $\langle n \rangle = \langle \varphi \rangle = 0$, where the symbol $\langle \ldots \rangle$ denotes averaging over the wavelength.

3. CASE OF NEGLIGIBLY SMALL VISCOSITY

When $\eta = 0$, eliminating u from (2.7) and (2.8), we get

$$a^{-2}\partial^2\psi/\partial x^2 = n/n_0, \tag{3.1}$$

where a is determined from the relation

$$\rho w^2 - c = 4\pi\beta^2 a^2/\varepsilon\,(a^2 + \varkappa^2), \tag{3.2}$$

and $\psi = e\varphi/T$.

The equation for the electron density (2.9) is linear of first order. Its periodic solution has the form

$$n = n_0\left[Ae^{-\psi(x)}\int_0^{2\pi/q}\exp\{-\varkappa^2\tau_M (V - w)\,\xi + \psi(x + \xi)\}\,d\xi - 1\right]. \tag{3.3}$$

The constant A is determined from the condition $\langle n \rangle = 0$.

We now ascertain the limitation imposed on the quantity V − w by the requirement that there exist periodic solutions of the system (3.1) and (3.3). To this end we multiply (3.1) and (3.3) by $\partial\psi/\partial x$ and average over the period. We obtain

$$\left\langle\frac{\partial\psi}{\partial x}e^{-\psi(x)}\int_0^{2\pi/q}\exp\{-\varkappa^2\tau_M (V - w)\,\xi + \psi(x + \xi)\}\,d\xi\right\rangle = 0. \tag{3.4}$$

But

$$\left\langle\frac{\partial\psi}{\partial x}e^{-\psi(x)}\int_0^{2\pi/q}\exp\{-\varkappa^2\tau_M (V - w)\,\xi + \psi(x + \xi)\}\,d\xi\right\rangle$$
$$= \left\langle e^{-\psi(x)}\frac{\partial}{\partial x}\int_0^{2\pi/q}\exp\{-\varkappa^2\tau_M (V - w)\,\xi + \psi(x + \xi)\}\,d\xi\right\rangle = \varkappa^2\tau_M (V - w)$$
$$\times\left\langle\int_0^{2\pi/q}\exp\,[-\varkappa^2\tau_M (V - w)\,\xi]\,[e^{\psi(x+\xi)-\psi(x)} - 1]\,d\xi\right\rangle$$
$$= \varkappa^2\tau_M (V - w)\int_0^{2\pi/q}\exp\,[-\varkappa^2\tau_M (V - w)\,\xi]\,I(\psi)\,d\xi, \tag{3.5}$$

where

$$I(\psi) = \frac{q}{2\pi}\int_0^{2\pi/q}e^{\psi(x+\xi)-\psi(x)}\,dx - 1.$$

We shall show that the functional I, and together with it the integral in the last line of (3.5), is essentially positive. Then it follows from (3.4) that

$$V - w = 0. \tag{3.6}$$

Let us find the extremum of the functional I. To this end we equate to zero its first variation

$$\delta I = \langle e^{\psi(x+\xi)-\psi(x)}\,\delta\psi(x + \xi)\rangle - \langle e^{\psi(x+\xi)-\psi(x)}\,\delta\psi(x)\rangle$$
$$= \langle [e^{\psi(x)-\psi(x-\xi)} - e^{\psi(x+\xi)-\psi(x)}]\,\delta\psi(x)\rangle = 0.$$

For arbitrary $\delta\psi(x)$ the expression in the square brackets should vanish. Consequently for arbitrary ξ we have

$$\psi(x) = \tfrac{1}{2}\,[\psi(x + \xi) + \psi(x - \xi)].$$

This equation has a unique periodic solution

$$\psi(x) = \text{const}.$$

Investigating the second variation, we can readily show that when $\psi(x) = \text{const}$ the functional I reaches a minimum value. Since this value is equal to zero, our statement is proved. Combining (3.1) and (3.3) with condition (3.6) we obtain ultimately

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[6]An analogous procedure was used by A. Akhiezer and Lyubarskii[10] in an investigation of nonlinear plasma oscillations. We are grateful to L. P. Pitaevskii for pointing out this work.

$$a^{-2}\partial^2\psi/\partial x^2 = e^{-\psi}/\langle e^{-\psi}\rangle - 1. \qquad (3.7)$$

Our purpose is to find the dependence of the wave vector q on the frequency ω and the displacement amplitude U. But when $w = \omega/q$ relation (3.2) holds. It is therefore sufficient to determine a as a function of q and U.

The first integral of (3.7) is

$$\frac{1}{2a^2}\left(\frac{\partial\psi}{\partial x}\right)^2 = H - \psi - \frac{e^{-\psi}}{\langle e^{-\psi}\rangle}, \qquad (3.8)$$

where H is an integration constant, ψ varies within the limits ψ_1 to ψ_2, limits which are determined from the equations

$$H - \psi_i - e^{-\psi_i}/\langle e^{-\psi}\rangle = 0 \qquad (i = 1, 2). \qquad (3.9)$$

Introducing the amplitude of the dimensionless potential Ψ,

$$\psi_1 = \psi_0 - \Psi, \qquad \psi_2 = \psi_0 + \Psi, \qquad (3.10)$$

we obtain

$$H = \Psi \operatorname{cth} \Psi - \ln (\Psi\langle e^{-\psi}\rangle/\operatorname{sh} \Psi),$$
$$e^{-\psi_0}/\langle e^{-\psi}\rangle = \Psi/\operatorname{sh} \Psi. \qquad (3.11)^*$$

In order to express H and $\langle \exp(-\psi)\rangle$ in terms of Ψ, it is necessary to have in addition to (3.1) another relation between these quantities; this is obtained from the condition $\langle\psi\rangle = 0$. To this end we multiply the identity

$$2d\,[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2} = -\frac{d\psi}{[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2}} + \frac{e^{-\psi}/\langle e^{-\psi}\rangle}{[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2}}\,d\psi \qquad (3.12)$$

by ψ and integrate with respect to ψ from ψ_1 to ψ_2. Taking (3.8) into account, as well as the relation

$$dx = \pm\, 2^{-1/2}\,[H - \psi - e^{-\psi}/\langle e^{-\psi}\rangle]^{-1/2}\, d\psi, \qquad (3.13)$$

we get

$$2\int_{\psi_1}^{\psi_2} \psi d\left[H - \psi - \frac{e^{-\psi}}{\langle e^{-\psi}\rangle}\right]^{1/2} = -\frac{1}{a^2}\int_{\psi_1}^{\psi_2}\left(\frac{\partial\psi}{\partial x}\right)^2 \frac{d\psi}{[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2}}$$
$$= -\frac{\pi\sqrt{2}}{a^2}\frac{a}{q}\left\langle\left(\frac{\partial\psi}{\partial x}\right)^2\right\rangle = -2\pi\sqrt{2}\,\frac{a}{q}(H-1).$$

We took into consideration here the fact that when integrating over one half cycle (that is, from ψ_1 to ψ_2), the root must be taken with a positive sign, and when integrating over the second half cycle (from ψ_2 to ψ_1) it should be taken with a negative sign

$$\int_{\psi_1}^{\psi_2} \frac{\psi\, d\psi}{[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2}} = \pi\sqrt{2}\,\frac{a}{q}\langle\psi\rangle = 0,$$

from which follows the required relation

$$2\sqrt{2}\pi\frac{a}{q}(H-1) = -\int_{\psi_1}^{\psi_2}\frac{\psi e^{-\psi}/\langle e^{-\psi}\rangle}{[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2}}\,d\psi. \qquad (3.14)$$

Integrating (3.13) over the period ψ, we get

$$2\pi\frac{a}{q} = \sqrt{2}\int_{\psi_1}^{\psi_2}[H - \psi - e^{-\psi}/\langle e^{-\psi}\rangle]^{-1/2}d\psi. \qquad (3.15)$$

This relation together with (3.11) and (3.14) solves our problem. The answer can be represented in a more compact form. To this end we transform the integral in the right half of (3.15)*

$$2\pi\frac{a}{q} = \sqrt{2}\int_{\psi_0-\Psi}^{\psi_0+\Psi}\left[H - \psi - \frac{e^{-\psi}}{\langle e^{-\psi}\rangle}\right]^{-1/2} d\psi$$
$$= \int_{-\Psi}^{\Psi}\left[\psi_0 + \frac{e^{-\psi_0}}{\langle e^{-\psi}\rangle}\operatorname{ch}\Psi - \psi_0 - y - \frac{e^{-\psi_0}}{\langle e^{-\psi}\rangle}e^{-y}\right]^{-1/2}dy$$

and then, taking (3.11) into account

$$a = \frac{q}{\pi\sqrt{2}}\int_{-\Psi}^{\Psi}\left[\Psi\operatorname{cth}\Psi - y - \frac{\Psi}{\operatorname{sh}\Psi}e^{-y}\right]^{-1/2}dy. \qquad (3.16)$$

Let us consider two limiting cases.

1. $\Psi \ll 1$.

We expand the integral in the right half of (3.16) in powers of Ψ:

$$a = \frac{q}{\pi\sqrt{2}}\int_{-\Psi}^{\Psi}\Big[1 + \frac{\Psi^2}{3} - \frac{\Psi^4}{45} - y - \left(1 - y + \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^4}{24}\right)$$
$$\times\left(1 - \frac{\Psi^2}{6} + \frac{7\Psi^4}{360}\right)\Big]^{-1/2}dy = \frac{q}{\pi}\int_{-1}^{1}\Big\{(1-z^2)$$
$$\times\left[1 - \frac{\Psi z}{3} - \frac{\Psi^2}{12}(1-z^2)\right]\Big\}^{-1/2}dz = q\left(1 + \frac{\Psi^2}{12}\right).$$

We have neglected here terms of order Ψ^4. We substitute the last result in (3.2):

$$\rho w^2 - c = \frac{4\pi\beta^2}{\varepsilon}\frac{q^2}{q^2+\varkappa^2}\left(1 + \frac{1}{12}\frac{\varkappa^2}{q^2+\varkappa^2}\Psi^2\right). \qquad (3.17)$$

Using the relation between the amplitudes

$$\Psi = \frac{\rho w^2 - c}{\beta}\frac{e}{T}U, \qquad (3.18)$$

which follows from (2.7), we get

$$\rho w^2 - c = \frac{4\pi\beta^2}{\varepsilon}\frac{q^2}{q^2+\varkappa^2}\left[1 + \frac{1}{12}\frac{q^4\varkappa^2}{(q^2+\varkappa^2)^3}\left(\frac{4\pi\beta}{\varepsilon}\frac{e}{T}U\right)^2\right]. \qquad (3.19)$$

This result was obtained in [5] by iteration with respect to the amplitude.

2. $\Psi \gg 1$.

We shall seek asymptotic formulas for the case $H \gg 1$. We shall see that $H \sim \Psi$ (see (3.23) below), i.e., we obtain precisely the result that we need.

*sh = sinh; cth = coth.

*ch = cosh.

Let us determine the asymptotic behavior of the integrals in formulas (3.14) and (3.15). We make use of the identity (3.12):

$$\int_{\psi_1}^{\psi_2}\frac{d\psi}{[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2}}=\frac{1}{\langle e^{-\psi}\rangle}\int_{\psi_1}^{\psi_2}\frac{e^{-\psi}d\psi}{[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2}}$$

$$=\int_{y_1}^{y_2}\frac{e^{-y}dy}{[H+\ln\langle e^{-\psi}\rangle-y-e^{-y}]^{1/2}}.$$

The integration is carried out between the roots of the integrand. In the last integral the essential region of integration is near the lower limit. In this region the radicand is approximately equal to $H + \ln\langle\exp(-\psi)\rangle - e^{-y}$. Consequently, the integral itself is asymptotically equal to

$$\int_{-\ln(H+\ln\langle e^{-\psi}\rangle)}^{\infty}\frac{e^{-y}dy}{[H+\ln\langle e^{-\psi}\rangle-e^{-y}]^{1/2}}=2\sqrt{H+\ln\langle e^{-\psi}\rangle}.$$

We obtain analogously

$$\frac{1}{\langle e^{-\psi}\rangle}\int_{\psi_1}^{\psi_2}\frac{\psi e^{-\psi}d\psi}{[H-\psi-e^{-\psi}/\langle e^{-\psi}\rangle]^{1/2}}$$

$$\approx-2\sqrt{H+\ln\langle e^{-\psi}\rangle}\ln\langle e^{-\psi}\rangle.$$

It now follows from (3.14) and (3.15) that:

$$\sqrt{2}\pi\,(a/q)\,H=\sqrt{H+\ln\langle e^{-\psi}\rangle}\ln\langle e^{-\psi}\rangle, \qquad (3.20)$$

$$\sqrt{2}\pi\,(a/q)=2\sqrt{H+\ln\langle e^{-\psi}\rangle}. \qquad (3.21)$$

Consequently

$$\ln\langle e^{-\psi}\rangle=2H. \qquad (3.22)$$

Substituting (3.22) in (3.11), we get

$$H=\tfrac{2}{3}\Psi. \qquad (3.23)$$

Further, on the basis of (3.21) and (3.23),

$$a=2q\sqrt{\Psi}/\pi. \qquad (3.24)$$

Finally,

$$\rho w^2-c=\frac{4\pi\beta^2}{\varepsilon}\left[1-\frac{\pi^2}{4}\frac{\varkappa^2}{q^2}\frac{\varepsilon T}{4\pi\beta e U}\right]. \qquad (3.25)$$

Thus, the nonlinear effects of electronic origin become strong when $\Psi \gg 1$ or $U \gg (\epsilon T/4\pi\beta e)\times(1+\pi^2\kappa^2/4q^2)$. The nonlinear effects of elasticity theory can be neglected when $qU \ll 1$. The region of values of the amplitudes corresponding to both inequalities exists if

$$\varkappa^2/q_c\ll q\ll q_c,\qquad q_c=(4\pi\beta/\varepsilon)\,(e/T). \qquad (3.26)$$

When $\beta\sim10^5$, $\epsilon\sim10$, $T\sim10^{-14}$ we get $q_c\sim10^9$ cm^{-1}. Thus, q is practically unbounded from above, and the limitation from below is also not very significant for reasonable values of κ.

From (3.25) we see that the phase velocity cannot exceed w_1. Thus, if $E_0 > w_1/\mu$, the existence of stationary waves at vanishingly low viscosity is impossible. In this case the system (2.7)—(2.9) has only growing solutions.

4. CASE OF LOW VISCOSITY

An exact solution of the system (2.7)—(2.9) with $\eta \neq 0$ cannot be obtained. In this section we investigate the role of viscosity in those cases when it distorts weakly the character of motion when $\eta = 0$. The values of $\varphi(x)$ and w acquire in this case small increments, proportional to η and η^2 respectively, which are of no interest to us. The energy losses to viscosity should be compensated in a stationary wave by the influx of energy from the external electric field which acts on the electron system. In the preceding section we have seen that the condition for the existence of stationary waves in the case $\eta = 0$ has the form $V - w = 0$. As will be seen below, this is the condition for the absence of an influx of energy due to the external field. In the presence of viscous losses the difference $V - w$ should obviously be different from zero, and in the present section we determined it from the law of energy conservation.

The relation expressing this law can be readily obtained from Eqs. (2.7) and (2.8), and is of the form

$$e\,\langle n\,\partial\varphi/\partial x\rangle=\eta w\,\langle(\partial^2u/\partial x^2)^2\rangle. \qquad (4.1)$$

We expand expression (3.3) for n in powers of $\kappa^2\tau_M(V-w)/q$, accurate to terms of first order:

$$n=n_0\left(\frac{e^{-\psi}}{\langle e^{-\psi}\rangle}-1\right)-n_0\frac{\varkappa^2\tau_M(V-w)}{\langle e^{-\psi}\rangle\langle e^{\psi}\rangle}e^{-\psi(x)}\frac{q}{2\pi}\left\{\int_0^{2\pi/q}e^{\psi(x+\xi)}\xi\,d\xi\right.$$

$$\left.-\left\langle e^{-\psi(x)}\int_0^{2\pi/q}e^{\psi(x+\xi)}\xi d\xi\right\rangle\Big/\langle e^{-\psi}\rangle\right\}. \qquad (4.2)$$

Using a derivation similar to that of (3.5), we get

$$e\left\langle n\frac{\partial\varphi}{\partial x}\right\rangle=-n_0T\frac{\varkappa^2\tau_M(V-w)}{\langle e^{\psi}\rangle\langle e^{-\psi}\rangle}\frac{q}{2\pi}\left\langle\frac{\partial\psi}{\partial x}e^{-\psi(x)}\int_0^{2\pi/q}e^{\psi(x+\xi)}\xi d\xi\right\rangle$$

$$=n_0T\varkappa^2\tau_M(V-w)\left(1-\frac{1}{\langle e^{\psi}\rangle\langle e^{-\psi}\rangle}\right). \qquad (4.3)$$

The right-hand side of (4.1) is of the form

$$\eta w\left\langle\left(\frac{\partial^2u}{\partial x^2}\right)^2\right\rangle=\eta w\left(\frac{\beta}{\rho w^2-c}\frac{T}{e}\right)^2\left\langle\left(\frac{\partial^2\psi}{\partial x^2}\right)^2\right\rangle. \qquad (4.4)$$

In accordance with the foregoing, the values of $\psi(x)$ and $\rho w^2 - c$, which should be substituted in these formulas, is given by the theory with $\eta = 0$.

Let us consider two limiting cases.

1. $\Psi \ll 1$.

Iterations of Eq. (3.7), carried out by the method described in [9], yield, accurate to fourth order terms [7],

$$\psi = \Psi \cos q(x - wt) - \left(\tfrac{1}{12}\Psi^2 - \tfrac{5}{1728}\Psi^4\right)\cos 2q(x - wt) + \tfrac{1}{96}\Psi^3 \cos 3q(x - wt) - \tfrac{13}{8640}\Psi^4 \cos^4 q(x - wt). \quad (4.5)$$

Hence

$$1 - 1/\langle e^{\psi}\rangle\langle e^{-\psi}\rangle = \tfrac{1}{2}\Psi^2 - \tfrac{11}{72}\Psi^4,$$

$$\langle(\partial^2\psi/\partial x^2)^2\rangle = q^4\left(\tfrac{1}{2}\Psi^2 + \tfrac{1}{18}\Psi^4\right).$$

Substituting (4.3) and (4.4) in (4.1), we obtain with the aid of these expressions, and also of (3.18) and (3.25)

$$\tau_M(Vq - \omega) = \frac{\eta\omega}{4\pi\beta^2/\varepsilon}\left[\left(1 + \frac{q^2}{\varkappa^2}\right)^2 + \frac{1}{12}\frac{q^2}{\varkappa^2}\frac{q^2}{q^2 + \varkappa^2}\left(3 + 5\frac{q^2}{\varkappa^2}\right)\left(\frac{4\pi\beta}{\varepsilon}\frac{e}{T}U\right)^2\right]. \quad (4.6)$$

The conditions for the applicability of this relation include the smallness of the right half compared with $1 + q^2/\kappa^2$ and smallness of the second term compared with the first. Apart from this, if $q/\kappa \lesssim 1$ and $4\pi\beta^2/\epsilon c \ll 1$, then q in the right half can be replaced by $q_0 = \omega/w_0$.

2. $\Psi \gg 1$.

Using the procedures of the preceding section, we can readily obtain the following relations:

$$\langle e^{\psi}\rangle = \tfrac{1}{2}\sqrt{\pi/2\Psi}\,e^{\Psi},$$

$$\left\langle\left(\frac{\partial^2\psi}{\partial x^2}\right)^2\right\rangle = a^4\left(\frac{\langle e^{-2\psi}\rangle}{\langle e^{-\psi}\rangle^2} - 1\right) = \frac{64}{3\pi^4}q^4\Psi^3.$$

Hence

$$\tau_M(Vq - \omega) = \frac{64}{3\pi^4}\frac{\eta\omega}{4\pi\beta^2/\varepsilon}\frac{q^2}{\varkappa^2}\left(\frac{4\pi\beta}{\varepsilon}\frac{e}{T}U - \frac{\pi^2}{4}\frac{\varkappa^2}{q^2}\right)^3. \quad (4.7)$$

This relation is valid if

$$\frac{\eta\omega}{4\pi\beta^2/\varepsilon}\left(\frac{4\pi\beta}{\varepsilon}\frac{e}{T}U\right)^3 \ll 1, \qquad \frac{4\pi\beta}{\varepsilon}\frac{e}{T}U - \frac{\pi^2}{4}\frac{\varkappa^2}{q^2} \gg 1.$$

Thus, the difference V − w is proportional to the square of the amplitude for small amplitudes and to the cube for large amplitudes.

5. CASE OF ARBITRARY VISCOSITY. METHOD OF ITERATIONS

In the linear approximation the displacement vector can be represented in the form

$$u = Ue^{i(qx - \omega t)},$$

where ω is real and q is complex. If the damping is small (that is, |Im q| ≪ |Re q|), then the absorption coefficient is

$$\Gamma = 2\,\mathrm{Im}\,q = \frac{4\pi\beta^2}{\varepsilon c}q\left[\delta - \frac{\tau_M(Vq - \omega)}{\tau_M^2(Vq - \omega)^2 + (1 + q^2/\varkappa^2)^2}\right], \quad (5.1)$$

where

$$\delta = \varepsilon\eta\omega/4\pi\beta^2.$$

The condition for the existence of the stationary mode is of the form Γ = 0. If $\delta < \delta_0 = \kappa^2/2(q^2 + \kappa^2)$, then this condition determines two values of V, at which a stationary wave with infinitesimally small amplitude exists:

$$V_{1,2} = w + [1 \mp \sqrt{1 - 4(1 + q^2/\varkappa^2)^2\delta^2}]/2\tau_M q\delta. \quad (5.2)$$

When $V < V_1$ or $V > V_2$ we will have Γ > 0. If $V_1 < V < V_2$, then Γ < 0 and the wave with the given frequency will become amplified. If $\delta \to \delta_0$, then the interval (V_1, V_2) becomes narrower. When $\delta = \delta_0$ there exists a unique value $V_0 = w + 1/2\tau_M q\delta_0$, at which there is a stationary wave of infinitesimally small amplitude. If $\delta > \delta_0$, the viscosity turns out to be so large that the waves of small amplitude should attenuate.

The solution of the equations (2.7)–(2.9) of the stationary mode with $\eta \neq 0$ can be investigated in two limiting cases. First, in the case of low viscosity, when $\delta \ll \delta_0$, it becomes possible to consider low-amplitude waves for values of V that are close to V_1 and V_2. Second, in the case of high viscosity, we get $\Delta \ll 1$ when $\delta = \delta_0(1 - \Delta)$. In this case it becomes possible to determine the maximum amplitude of the stationary wave. In addition, we shall assume in this section that

$$q \lesssim \varkappa, \qquad 4\pi\beta^2/\varepsilon c \ll 1.$$

Let us proceed to iterations of the system (2.7)–(2.9). This method is perfectly analogous to the one developed in [9], so that we can confine ourselves only to a brief description of the main stages of the computation.

In the first approximation we put

$$u_{\mathrm{I}} = U\cos q(x - wt). \quad (5.3)$$

Then we get from (2.7) and (2.9)

$$\varphi_{\mathrm{I}} = -\frac{4\pi\beta}{\varepsilon}U[\alpha\cos q(x - wt) - \delta\sin q(x - wt)], \quad (5.4)$$

$$n_{\mathrm{I}} = n_0\frac{4\pi\beta}{\varepsilon}\frac{e}{T}U\left[\frac{\alpha + \nu\delta}{\nu^2 + 1}\cos q(x - wt) + \frac{\alpha\nu - \delta}{\nu^2 + 1}\sin q(x - wt)\right], \quad (5.5)$$

[7] No account is taken here of the difference between the amplitude of the dimensionless potential, defined in accordance with (3.10), and the amplitude of its first harmonic, inasmuch as this would lead to inclusion of higher-order terms in the final result.

where

$$\alpha = (\rho w^2 - c)\, \varepsilon/4\pi\beta^2, \qquad \nu = (\varkappa^2/q^2)\, \tau_M (Vq - \omega).$$

Substituting (5.3)—(5.5) in (2.8) we obtain the following relations:

$$\tau_M^2 (Vq - \omega)^2 \delta - \tau_M (Vq - \omega) + (1 + q^2/\varkappa^2)^2 \delta = 0, \quad (5.6)$$

$$\rho w^2 - c = \frac{4\pi\beta^2}{\varepsilon} \frac{q^2}{(q^2 + \varkappa^2)} [1 + \delta \tau_M (Vq - \omega)]. \quad (5.7)$$

The first of these relations is the condition $\Gamma = 1$ which was mentioned earlier.

In order to find the second approximation, let us calculate the nonlinear terms in Eq. (2.9) with the aid of (5.4) and (5.5), and solve the resultant system of linear inhomogeneous equations. The solution of this system will contain second harmonics, that is, terms of the form $A \cos 2q(x - wt)$ and $B \sin 2q(x - wt)$. Setting up in analogous fashion the third-approximation equations, we note that the inhomogeneous terms contain first and third harmonics.

We seek essentially a solution in the form of Fourier series, in which the Fourier coefficients, in view of the use of the iteration method, are obtained in the form of expansions in powers of the amplitude u of the first harmonic. The presence of the first harmonics in the inhomogeneous terms of the third-approximation equations demonstrates that it is necessary to introduce corrections in the coefficients of the first harmonics φ and n, and thus also in relations (5.6) and (5.7), since these have been obtained through the use of these coefficients. These corrections lead to corrections for the quantities V and w. Leaving out the unwieldy manipulations, we present the final results which can be represented in lucid form only for the aforementioned limiting cases.

a) $\delta \ll \delta_0$. First mode (V close to V_1). For the quantity $\tau_M(Vq - \omega)$ we obtain, as expected, the expression (4.6). The dispersion relation neglecting terms of order δ^2 is of the form (3.19).

Second mode (V close to V_2).

$$\tau_M (Vq - \omega) = \frac{1}{\delta}\left[1 - \left(1 + \frac{q_0^2}{\varkappa^2}\right)^2 \delta^2\right] + \frac{5}{12} \frac{q_0^4}{\varkappa^4} \delta \left(\frac{4\pi\beta}{\varepsilon}\right)^2 \left(\frac{eU}{T}\right)^2, \quad (5.8)$$

$$\rho w^2 - c = \frac{4\pi\beta^2}{\varepsilon} [1 - (1 + q_0^2/\varkappa^2)\, \delta^2] - \frac{12 q_0^4}{\varkappa^4}\left(4 + 11 \frac{q_0^2}{\varkappa^2}\right) \delta^4 \left(\frac{4\pi\beta}{\varepsilon}\right)^2 \left(\frac{eU}{T}\right)^2. \quad (5.9)$$

Let us consider the question of the stability of both modes relative to small changes in amplitude. Γ is a function of V and U^2. The stability condition takes the form

$$\partial\Gamma/\partial U^2 > 0. \quad (5.10)$$

For V close to V_1 or V_2 and for a small amplitude we have

$$\Gamma = (\partial\Gamma/\partial V)(V - V_i) + (\partial\Gamma/\partial U^2)\, U^2 \qquad (i = 1, 2). \quad (5.11)$$

The derivatives are taken at the point $V = V_1$, $U = 0$.

From (5.1) we have

$$\frac{\partial\Gamma}{\partial V} = \frac{4\pi\beta^2}{\varepsilon c} q_0^2 \tau_M \frac{\tau_M^2 (Vq - \omega) - (1 + q_0^2/\varkappa^2)^2}{[\tau_M^2 (Vq - \omega)^2 + (1 + q_0^2/\varkappa^2)^2]^2}. \quad (5.12)$$

Equating (5.11) to zero, we should obtain the connection between V and U^2 for the stationary mode. Inasmuch, on the other hand, as this connection is determined by relations (4.6) and (5.8), we obtain by the same token $\partial\Gamma/\partial U^2$. The first mode is stable:

$$\frac{\partial\Gamma}{\partial U^2} = \frac{4\pi\beta^2}{\varepsilon c}\left(\frac{4\pi\beta}{\varepsilon}\frac{e}{T}\right)^2 \frac{q_0}{12} \frac{q_0^4 \varkappa^2}{(q_0^2 + \varkappa^2)^3}\left[\left(3 + 5\frac{q_0^2}{\varkappa^2}\right)\delta + \frac{4\pi\beta^2}{\varepsilon c}\frac{\omega\tau_M}{2}\frac{q_0^2 \varkappa^4}{(q_0^2 + \varkappa^2)^3}\right]. \quad (5.13)$$

The second mode is unstable:

$$\frac{\partial\Gamma}{\partial U^2} = -\frac{4\pi\beta^2}{\varepsilon c}\left(\frac{4\pi\beta e}{\varepsilon T}\right)^2 \frac{5 q_0}{12} \frac{q_0^4}{\varkappa^4} \delta^3. \quad (5.14)$$

b) $\delta = \delta_0(1 - \Delta)$, $\Delta \ll 1$. With the aid of calculations similar to those for a), we obtain

$$\tau_M (Vq - \omega) = \left(1 + \frac{q_0^2}{\varkappa^2}\right)(1 \mp \sqrt{2\Delta}) \mp \frac{1}{\sqrt{2\Delta}} f(q_0^2/\varkappa^2) \left(\frac{4\pi\beta}{\varepsilon}\right)^2 \left(\frac{eU}{T}\right)^2, \quad (5.15)$$

$$\rho w^2 - c = \frac{4\pi\beta^2}{\varepsilon}\left[1 - \frac{1}{2}\frac{\varkappa^2}{q_0^2 + \varkappa^2}(1 \pm \sqrt{2\Delta}) \pm \frac{1}{\sqrt{2\Delta}}\frac{1}{2}\left(\frac{\varkappa^2}{q_0^2 + \varkappa^2}\right)^2 f\left(\frac{q_0^2}{\varkappa^2}\right)\left(\frac{4\pi\beta}{\varepsilon}\frac{e}{T}U\right)^2\right], \quad (5.16)$$

where the upper sign pertains to the first mode and the lower to the second, and where

$$f(x) = \frac{1}{24}\frac{x^2}{(1 + x)^2}\frac{24x^3 + 38x^2 + 23x + 4}{8x^2 + 4x + 1}.$$

The expansion analogous to (5.11) can be written in this case in the form

$$\Gamma(V, \delta, U^2) = \frac{(V - V_0)^2}{2}\frac{\partial^2\Gamma}{\partial V^2} - \frac{\partial\Gamma}{\partial\delta}\delta_0\Delta + \frac{\partial\Gamma}{\partial U^2}U^2. \quad (5.17)$$

The derivatives are taken at the point $\delta = \delta_0$, $V = V_0$, $U = 0$. From (5.1) we have

$$\frac{1}{2}\frac{\partial^2\Gamma}{\partial V^2} = \frac{4\pi\beta^2}{\varepsilon c}\tau_M^2 q_0^3 \frac{1}{4}\frac{\varkappa^6}{(q_0^2 + \varkappa^2)^3}, \qquad \frac{\partial\Gamma}{\partial\delta} = \frac{4\pi\beta^2}{\varepsilon c} q_0.$$

Equating (5.17) to zero, we get

$$V = V_0 \pm \left[\frac{(\partial\Gamma/\partial\delta)\,\delta_0\Delta - (\partial\Gamma/\partial U^2)\,U^2}{\frac{1}{2}(\partial^2\Gamma/\partial V^2)}\right]^{1/2}. \qquad (5.18)$$

Let us expand (5.18) in powers of U^2 near $V = V_1$:

$$V = V_1 + \frac{1}{2}\left[\frac{1}{2}\frac{\partial^2\Gamma}{\partial V^2}\frac{\partial\Gamma}{\partial\delta}\delta_0\Delta\right]^{-1/2}\left(\frac{\partial\Gamma}{\partial U^2}\right)U^2. \qquad (5.19)$$

Equating (5.19) and (5.15), we get

$$\frac{\partial\Gamma}{\partial U^2} = \frac{4\pi\beta^2}{\varepsilon c}\frac{q_0}{2}\frac{\varkappa^4}{(q_0^2+\varkappa^2)^2} f\left(\frac{q_0^2}{\varkappa^2}\right)\left(\frac{4\pi\beta}{\varepsilon}\frac{e}{T}\right)^2. \qquad (5.20)$$

Now on the basis of (5.18) we can determine the maximum amplitude of the stationary wave at a given value of the viscosity

$$U_{max}^2 = \frac{(1+q_0^2/\varkappa^2)\,\Delta}{(4\pi\beta e/\varepsilon\Gamma)^2 f(q_0^2/\varkappa^2)}. \qquad (5.21)$$

We call attention to the fact that in the case of high viscosity both modes are stable. Therefore there should exist a value of the viscosity $\eta_c = \delta_c(4\pi\beta^2/\epsilon)$ such that the second mode is unstable when $\eta = \eta_c$. The value of η_c is determined by the condition $\partial\Gamma/\partial U^2 = 0$. The explicit form of this equation is quite complicated, and we do not write it out.

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