RELATIVISTIC EQUATION FOR THE S MATRIX IN THE p REPRESENTATION.
I. UNITARITY AND CAUSALITY CONDITIONS

V. G. KADYSHEVSKIĬ

Joint Institute for Nuclear Research

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A covariant formulation of the theory of the scattering matrix in the p representation is given. The theory contains the unitarity and causality conditions.

1. INTRODUCTION

THE aim of the present paper is a consistent covariant formulation of the theory of the S matrix in the p representation. Special attention will be paid to such properties of the S matrix as unitarity and causality.

It is well known that these properties of the S matrix are easily established in the x representation with the help of the Schrödinger equation or the representation of the scattering matrix in the form of a T exponential, which follows from the Schrödinger equation. However, if for example, we go over in the T exponential to the p representation and then try to prove its unitarity in the general form, we find that this is a very complicated task. The causality condition implies some definite analytic properties of the matrix elements in p space which are also in general extremely involved.

In considering generalizations of quantum field theory in p space[1,2] where the transition to a description in terms of x space quantities is impossible, there are no convenient criteria for unitarity and causality, which makes the construction of a new scheme very difficult and quite generally casts some doubt on the correctness of such a scheme. We therefore attempt below to solve the following problem: to reformulate the usual field theory in p space in such a way that the conditions of unitarity and causality of the S matrix have a compact form and are easy to demonstrate.

All further derivations will be carried out in the interaction representation. For simplicity we shall consider the self-interaction of a scalar field $\varphi(x)$ with mass m, where the interaction Lagrangian is chosen in the form[3]

$$\mathscr{L}(x) = g : \varphi^3(x) :. \tag{1.1}$$

The generalization to other interactions presents no difficulties.

2. EQUATIONS OF MOTION FOR THE S MATRIX IN p SPACE AND THE UNITARITY CONDITION

Let

$$S = T \exp\{i \int \mathscr{L}(x)\, dx\} \tag{2.1}$$

be the scattering matrix corresponding to the Lagrangian (1.1). Writing S in the form

$$S = 1 + i\mathscr{T}, \tag{2.2}$$

we have, according to (2.1),

$$\mathscr{T} = \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int T(\mathscr{L}(x_1)\ldots\mathscr{L}(x_n))\, dx_1 \ldots dx_n,$$

or

$$\mathscr{T} = \sum_{n=1}^{\infty} i^{n-1} \int \theta(x_1^0 - x_2^0)\ldots\theta(x_{n-1}^0 - x_n^0)\,\mathscr{L}(x_1) \ldots \mathscr{L}(x_n)\, dx_1 \ldots dx_n \equiv \sum_{n=1}^{\infty} \mathscr{T}_n. \tag{2.3}$$

We bring (2.3) into a completely fourdimensional form by replacing $\theta(x^0)$ by the invariant functions $\theta(\lambda x)$, where $\lambda x = \lambda_0 x^0 - \boldsymbol{\lambda}\mathbf{x}$, and

$$\lambda^2 = 1, \qquad \lambda_0 > 0. \tag{2.4}$$

As a result we obtain

$$\sum_{n=1}^{\infty} \mathscr{T}_n = \sum i^{n-1} \int \theta(\lambda(x_1 - x_2)) \ldots \theta(\lambda(x_{n-1} - x_n))\,\mathscr{L}(x_1)\ldots\mathscr{L}(x_n)\, dx_1 \ldots dx_n. \tag{2.5}$$

As is known, the dependence of the quantities $\mathscr{T}_n$ on λ in (2.5) is purely fictitious, since for $(x_i - x_{i+1})^2 > 0$ always $\theta(x_i^0 - x_{i+1}^0)$

$= \theta(\lambda(x_i - x_{i+1}))$, and for $(x_i - x_{i+1})^2 < 0$ the function $\theta(\lambda(x_i^0 - x_{i+1}^0))$ makes no contribution owing to the "locality" condition

$$[\mathscr{L}(x_i), \mathscr{L}(x_{i+1})]_- = 0, \qquad (2.6)$$

which holds in that region.

We now make a Fourier transfromation of (2.5), setting

$$\theta(\lambda x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\tau(\lambda x)}}{\tau - i\varepsilon} d\tau, \qquad (2.7)$$

$$\tilde{\mathscr{L}}(p) = \int e^{-ipx} \mathscr{L}(x)\, dx = \frac{g}{\sqrt{2\pi}} \int \delta(p - k_1 - k_2 - k_3) : \varphi \times (k_1)\, \varphi(k_2)\, \varphi(k_3) : dk_1 dk_2 dk_3, \qquad (2.8)$$

$$\varphi(k) = \frac{1}{(2\pi)^{5/2}} \int e^{-ikx} \varphi(x)\, dx. \qquad (2.9)$$

As a result we have

$$\mathscr{T}_1 = \tilde{\mathscr{L}}(0),$$

$$\mathscr{T}_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathscr{L}}(-\lambda\tau) \frac{d\tau}{\tau - i\varepsilon} \tilde{\mathscr{L}}(\lambda\tau),$$

. . . .

$$\mathscr{T}_n = \frac{1}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \tilde{\mathscr{L}}(-\lambda\tau_1) \frac{d\tau_1}{\tau_1 - i\varepsilon} \tilde{\mathscr{L}}(\lambda\tau_1 - \lambda\tau_2) \frac{d\tau_2}{\tau_2 - i\varepsilon} \cdots \frac{d\tau_{n-1}}{\tau_{n-1} - i\varepsilon} \tilde{\mathscr{L}}(\lambda\tau_{n-1}). \qquad (2.10)$$

The expressions (2.10) can be regarded as successive iterations of the linear integral equation

$$R(\lambda\tau) = \tilde{\mathscr{L}}(\lambda\tau) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathscr{L}}(\lambda\tau - \lambda\tau') \frac{d\tau'}{\tau' - i\varepsilon} R(\lambda\tau'). \qquad (2.11)$$

with the condition[1)]

$$\mathscr{T} = \sum_{n=1}^{\infty} \mathscr{T}_n = R(0). \qquad (2.12)$$

Equation (2.11) is the four-dimensional equation of motion for the scattering matrix in p space. It is completely analogous to the corresponding equation for the S matrix in the E representation (see, e.g., [4]). It is also clear that (2.11) contains the same information on the scattering matrix as (2.1).

For example, let us show the unitarity of the S matrix with the help of (2.11).[2)] For this purpose we rewrite (2.11) in the form

$$R(\lambda\tau) = \tilde{\mathscr{L}}(\lambda\tau) + \int \tilde{\mathscr{L}}(\lambda\tau') F(\tau - \tau')\, d\tau', \qquad (2.13)$$

$$F(\tau - \tau') = \frac{1}{2\pi} \frac{R(\lambda\tau - \lambda\tau')}{\tau - \tau' - i\varepsilon}. \qquad (2.14)$$

Taking it into account that under hermitian conjugation

$$\tilde{\mathscr{L}}^+(\lambda\tau) = \tilde{\mathscr{L}}(-\lambda\tau), \qquad (2.15)$$

the equation conjugate to (2.13) can be written as

$$R^+(-\lambda\tau) = \tilde{\mathscr{L}}(\lambda\tau) + \int F^+(\tau' - \tau) \tilde{\mathscr{L}}(\lambda\tau')\, d\tau', \qquad (2.16)$$

where

$$F^+(\tau' - \tau) = \frac{1}{2\pi} \frac{R^+(\tau' - \tau)}{\tau' - \tau + i\varepsilon}. \qquad (2.17)$$

From (2.13) and (2.16) we obtain

$$R(\lambda\tau) - R^+(-\lambda\tau) = \int \tilde{\mathscr{L}}(\lambda\tau') F(\tau - \tau')\, d\tau' - \int F^+(\tau' - \tau) \tilde{\mathscr{L}}(\lambda\tau')\, d\tau'. \qquad (2.18)$$

On the other hand, one can show with the help of the same equations after some simple transformations that

$$\int R^+(-\lambda\tau') F(\tau - \tau')\, d\tau' - \int F^+(\tau' - \tau) R(\lambda\tau')\, d\tau' = \int \tilde{\mathscr{L}}(\lambda\tau') F(\tau - \tau')\, d\tau' - \int F^+(\tau' - \tau) \tilde{\mathscr{L}}(\lambda\tau')\, d\tau' + \Phi_1(\tau) - \Phi_2(\tau), \qquad (2.19)$$

where

$$\Phi_1(\tau) = \int F^+(\tau' - \tau'') \tilde{\mathscr{L}}(\lambda\tau') F(\tau - \tau'')\, d\tau' d\tau'',$$

$$\Phi_2(\tau) = \int F^+(\tau'' - \tau) \tilde{\mathscr{L}}(\lambda\tau') F(\tau'' - \tau')\, d\tau' d\tau''. \qquad (2.20)$$

Let us now make the following consecutive changes of integration variables in (2.20):

$$\tau' - \tau'' \to \xi, \quad \tau - \tau'' \to \eta, \quad (\text{in } \Phi_1(\tau)), \qquad (2.21)$$
$$\tau'' - \tau' \to \eta, \quad \tau'' - \tau \to \xi \quad (\text{in } \Phi_2(\tau)).$$

As a result we find

$$\Phi_1(\tau) = \Phi_2(\tau) = \int F^+(\xi) \tilde{\mathscr{L}}(\lambda(\tau + \xi - \eta)) F(\eta)\, d\xi d\eta. \qquad (2.22)$$

It now follows from (2.18) and (2.19) with account of (2.22) that

$$R(\lambda\tau) - R^+(-\lambda\tau) = \int [R^+(-\lambda\tau') F(\tau - \tau') - F^+(\tau' - \tau) R(\lambda\tau')]\, d\tau'$$

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1)We note that, instead of $R(\lambda\tau)$, it would be more correct to write $R(\lambda\tau, \lambda)$. However, since the operator R does not depend on λ for $\tau = 0$, we shall not write the second argument for simplicity.

2)The derivations below are essentially the same as those of [4].

or

$$R(\lambda\tau) - R^{+}(-\lambda\tau) = \frac{1}{2\pi}\int \frac{d\tau'}{\tau' - i\varepsilon}\,[R^{+}(-\lambda\tau')\,R(\lambda\tau - \lambda\tau') + R^{+}(\lambda\tau' - \lambda\tau)\,R(\lambda\tau')]. \quad (2.23)$$

Setting $\tau = 0$ in (2.23) and using

$$\frac{1}{\tau - i\varepsilon} = \mathrm{P}\,\frac{1}{\tau} + i\pi\delta(\tau), \quad (2.24)$$

we obtain

$$R(0) - R^{+}(0) = iR^{+}(0)\,R(0), \quad (2.25)$$

i.e., the unitarity condition for the matrix $S = 1 + iR(0)$. Hence (2.23) is the unitarity condition for the S matrix for $\tau \neq 0$.

The unitarity of the scattering matrix can also be derived from (2.11) in a different fashion. For this purpose, let us employ (2.24) to write (2.11) in the form

$$R(\lambda\tau) = \tilde{\mathscr{L}}(\lambda\tau)\left[1 + \frac{i}{2}R(0)\right] + \frac{1}{2\pi}\mathrm{P}\int \tilde{\mathscr{L}}(\lambda\tau - \lambda\tau')\frac{d\tau'}{\tau'}R(\lambda\tau'), \quad (2.26)$$

and multiply both sides on the right by the operator $[1 + (i/2)R(0)]^{-1}$. We then find

$$R(\lambda\tau) - \frac{i}{2}K(\lambda\tau)\,R(0) = K(\lambda\tau), \quad (2.27)$$

where the operator $K(\lambda\tau)$ is defined by the equation

$$K(\lambda\tau) = \tilde{\mathscr{L}}(\lambda\tau) + \frac{1}{2\pi}\mathrm{P}\int \tilde{\mathscr{L}}(\lambda\tau - \lambda\tau')\frac{d\tau'}{\tau'}K(\lambda\tau'), \quad (2.28)$$

or by the expansion

$$K(\lambda\tau) = \tilde{\mathscr{L}}(\lambda\tau) + \sum_{n=1}^{\infty}\frac{1}{(2\pi)^n}\mathrm{P}\int \tilde{\mathscr{L}}(\lambda\tau - \lambda\tau_1)\frac{d\tau_1}{\tau_1}\cdots\frac{d\tau_n}{\tau_n}\mathscr{L}(\lambda\tau_n). \quad (2.29)$$

Since $\tilde{\mathscr{L}}(0)^{+} = \mathscr{L}(0)$ from (2.15), it follows from (2.29) that

$$K^{+}(0) = K(0). \quad (2.30)$$

Setting $\tau = 0$ in (2.27) and recalling the definition $S = 1 + iR(0)$, we obtain

$$S = \left(1 + \frac{i}{2}K(0)\right)\Big/\left(1 - \frac{i}{2}K(0)\right), \quad (2.31)$$

from where we find $SS^{+} = 1$ on account of (2.30).

Evidently $K(0)$ is the known reactance matrix of Wigner, and (2.27) is the analog of the Heitler equation.[5,6] If we set $\tau = 0$ and $\lambda = 0$ in (2.29) and go over to x space, we arrive at an expression for the K matrix which has been given, e.g., in [6]:

$$K = \sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^{n-1}\int \mathscr{L}(x_1)\,\varepsilon(x_1^0 - x_2^0)\,\mathscr{L}(x_2)\ldots\varepsilon(x_{n-1}^0 - x_n^0)\,\mathscr{L}(x_n)\,dx_1\ldots dx_n,$$

where

$$\varepsilon(x^0) = \mathrm{sign}\,(x^0).$$

The expansion (2.12) can be summed up formally by introducing matrix notation in (2.10):

$$\tilde{\mathscr{L}}(\lambda\tau - \lambda\tau') = \langle\tau|\,\hat{\mathscr{L}}\,|\tau'\rangle,$$

$$R(\lambda\tau; \lambda\tau') = \langle\tau|\,\hat{R}\,|\tau'\rangle,$$

$$R(\lambda\tau) \equiv R(\lambda\tau; 0) = \langle\tau|\,\hat{R}\,|0\rangle,$$

$$2\pi\tau\delta(\tau - \tau') = \langle\tau|\,\hat{T}\,|\tau'\rangle,$$

$$\frac{1}{2\pi}\frac{1}{\tau - i\varepsilon}\delta(\tau - \tau') = \left\langle\tau\left|\frac{1}{\hat{T} - i\varepsilon}\right|\tau'\right\rangle, \quad (2.32)$$

where $\hat{\mathscr{L}}$, $\hat{R}$, and $\hat{T}$ are operators in the "state space" $|\tau\rangle$. Using the formula[7]

$$\frac{1}{A - B} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \ldots \quad (2.33)$$

which is valid for arbitrary operators A and B, we obtain from (2.10) and (2.12)

$$\mathscr{I} = \Sigma\mathscr{T}_n = \langle 0|\,\hat{R}\,|0\rangle = \left\langle 0\left|\,\hat{\mathscr{L}} + \hat{\mathscr{L}}\frac{1}{\hat{T} - \hat{\mathscr{L}} - i\varepsilon}\hat{\mathscr{L}}\,\right|0\right\rangle. \quad (2.34)$$

It is easy to see that $\hat{R}$ satisfies the equation

$$\hat{R} = \hat{L} + \hat{L}\frac{1}{\hat{T} - i\varepsilon}\hat{R}, \quad (2.35)$$

which is the matrix form of the equation for $R(\lambda\tau_1; \lambda\tau_2)$:

$$R(\lambda\tau_1; \lambda\tau_2) = \mathscr{L}(\lambda\tau_1 - \lambda\tau_2) + \frac{1}{2\pi}\int \mathscr{L}(\lambda\tau_1 - \lambda\tau)\frac{d\tau}{\tau - i\varepsilon}R(\lambda\tau; \lambda\tau_2). \quad (2.36)$$

Evidently, (2.36) goes over into (2.11) for $\tau_2 = 0$.

It is not difficult to see that the unitarity condition for the operator $R(\lambda\tau_1; \lambda\tau_2)$ is of the form

$$R(\lambda\tau_1; \lambda\tau_2) - R^{+}(-\lambda\tau_1; -\lambda\tau_2) = \frac{1}{i(2\pi)^2}\int\frac{d\tau'd\tau''}{(\tau' - i\varepsilon)(\tau'' - i\varepsilon)}$$
$$\times[R(\lambda\tau'; \lambda\tau'')\,R^{+}(\lambda\tau' - \lambda\tau_1; \lambda\tau'' - \lambda\tau_2) + R(\lambda\tau'; \lambda\tau_2 - \lambda\tau'')\,R^{+}(\lambda\tau' - \lambda\tau_1; -\lambda\tau'')$$
$$+ R(\lambda\tau_1 - \lambda\tau'; \lambda\tau'')\,R^{+}(-\lambda\tau'; \lambda\tau'' - \lambda\tau_2) + R(\lambda\tau_1 - \lambda\tau'; \lambda\tau_2 - \lambda\tau'')\,R^{+}(-\lambda\tau', -\lambda\tau'')]. \quad (2.37)$$

Introducing the operators $\hat{K}$ and $\mathrm{P}\hat{T}^{-1}$ with the condition

$$K(\lambda\tau) = \langle\tau|\hat{K}|0\rangle,$$

$$\frac{1}{2\pi}\mathrm{P}\frac{1}{\tau}\delta(\tau-\tau') = \frac{1}{2\pi}\left\langle\tau\left|\mathrm{P}\frac{1}{\hat{T}}\right|\tau'\right\rangle, \quad (2.38)$$

we find in complete analogy to the foregoing

$$\hat{K} = \hat{L} + \hat{L}\left(\mathrm{P}\frac{1}{\hat{T}}\right)\hat{K}, \quad (2.39)$$

$$K(\lambda\tau) = \left\langle\tau\left|\hat{L} + \hat{L}\left(\mathrm{P}\frac{1}{\hat{T}-\hat{L}}\right)\hat{L}\right|0\right\rangle. \quad (2.40)$$

Using (2.40) it is easy to show the hermiticity of the matrix K(0) without recourse to perturbation theory.

3. TRANSITION TO THE σ REPRESENTATION

Let us make a Fourier transformation of (2.11) with respect to the variable τ, setting

$$L(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathscr{L}}(\lambda\tau)e^{i\tau\sigma}d\tau, \quad (3.1)$$

$$\mathscr{T}(\sigma; -\infty) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\tau}{\tau - i\varepsilon}R(\lambda\tau)e^{i\tau\sigma}. \quad (3.2)$$

As a result we have

$$d\mathscr{T}(\sigma; -\infty)/d\sigma = L(\sigma) + iL(\sigma)\mathscr{T}(\sigma; -\infty),$$

or

$$dS(\sigma; -\infty)/d\sigma = iL(\sigma)S(\sigma; -\infty), \quad (3.3)$$

where, by definition,

$$S(\sigma; -\infty) = 1 + i\mathscr{T}(\sigma; -\infty). \quad (3.4)$$

It follows from (2.8) and (3.1) that $\mathscr{L}(x)$ and $L(\sigma)$ are related to one another through the so-called Radon transformation:[8]

$$L(\sigma) = \int\delta(\sigma - \lambda x)\mathscr{L}(x)dx. \quad (3.5)$$

It is seen from this that $S(\sigma; -\infty)$ is the scattering matrix defined on the space-like plane $\lambda x = \sigma$, which in the special case $\lambda = 0$ goes over into the plane x_0 = const.

It can be shown with the help of (3.2) and (3.4) that (2.23) is equivalent to the unitarity condition for $S(\sigma; -\infty)$:

$$S(\sigma; -\infty)S^{+}(\sigma; -\infty) = 1. \quad (3.6)$$

It is clear that the description of the S matrix in terms of σ is a simple consequence of the replacement of the function $\theta(x^0)$ by $\theta(\lambda x)$. However, we go into this point in particular detail, because the generalization of this formalism, which will be our concern in what follows, is connected with the invariant parameter σ in an essential way.

We note that the quantity σ is equal to the distance from the origin to the plane $\lambda x = \sigma$ owing to the condition $\lambda^2 = 1$. Using the known formula

$$\frac{1}{2\pi i}\frac{e^{i\tau\sigma}}{\tau - i\varepsilon} \to \begin{cases}\delta(\tau) & \sigma\to+\infty\\ 0 & \sigma\to-\infty\end{cases}, \quad (3.7)$$

we can show from (3.2) that

$$\mathscr{T}(\infty; -\infty) = R(0) = \mathscr{T}, \qquad \mathscr{T}(-\infty; -\infty) = 0. \quad (3.8)$$

Let us now consider the motion of some system of particles which for $\sigma = -\infty$ has the total four-momentum P_n. We choose a vector λ_n such that $\lambda_n \parallel P_n$. Since $\lambda^2 = 1$, this vector will have the physical meaning of the four-velocity of the system, and hence the parameter σ will be its proper time, as always $x^0 = \sigma$ for $\boldsymbol{\lambda} = \mathbf{P} = 0$. Equation (3.3) is then appropriately called the proper time Schrödinger equation.

In concluding this section, we quote, without derivation, a formula which connects the operator $R(\lambda\tau_1; \lambda\tau_2)$ introduced above with the operator $\mathscr{T}(\sigma_1; \sigma_2)$ defined on the two finite planes $\lambda x = \sigma_1$ and $\lambda x = \sigma_2$:

$$\mathscr{T}(\sigma_1; \sigma_2) = \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}\frac{d\tau_1 d\tau_2}{(\tau_1 - i\varepsilon)(\tau_2 - i\varepsilon)}e^{i\tau_1\sigma_1 - i\tau_2\sigma_2}R(\lambda\tau_1; \lambda\tau_2). \quad (3.9)$$

4. CAUSALITY CONDITION

In this section it will be our aim to obtain a relation for the operator $R(\lambda\tau)$ which would be equivalent to the relativistic causality condition.

The operator $R(\lambda\tau)$ itself will in this section be denoted by $R^{(-)}(\lambda\tau)$, in view of the fact that the imaginary part of the denominator in (2.11) is negative.

We also introduce the operator $R^{(+)}(\lambda\tau)$ by the defining equation

$$R^{(+)}(\lambda\tau) = \tilde{\mathscr{L}}(\lambda\tau) + \frac{1}{2\pi}\int\tilde{\mathscr{L}}(\lambda\tau - \lambda\tau')\frac{d\tau'}{\tau' + i\varepsilon}R^{(+)}(\lambda\tau'). \quad (4.1)$$

It is easy to show with the help of (2.36) and (4.1) that

$$[R^{(+)}(\lambda\tau)]^{+} = R(0; \lambda\tau), \quad (4.2)$$

so that

$$[R^{(+)}(0)]^{+} = R(0; 0) \equiv R(0). \quad (4.3)$$

In the following we need an equation for the operator $(R^{(+)}(-\lambda\tau))^{+}$, which is easily seen to have the form [3]

$$[R^{(+)}(-\lambda\tau)]^{+} = \widetilde{\mathscr{L}}(\lambda\tau)$$
$$-\frac{1}{2\pi}\int [R^{(+)}(-\lambda\tau')]^{+}\frac{d\tau'}{\tau'+i\varepsilon}\widetilde{\mathscr{L}}(\lambda\tau-\lambda\tau'). \qquad (4.4)$$

The development to follow is wholly reminiscent of the derivation of the unitarity condition given in Sec. 2. We first write (2.11) and (4.4) in the form

$$R^{(-)}(\lambda\tau) = \widetilde{\mathscr{L}}(\lambda\tau) + \frac{1}{2\pi}\int \widetilde{\mathscr{L}}(\lambda\tau')F_1(\tau-\tau')\,d\tau', \qquad (4.5)$$
$$(R^{(+)}(-\lambda\tau))^{+} = \widetilde{\mathscr{L}}(\lambda\tau) + \frac{1}{2\pi}\int F_2(\tau'-\tau)\widetilde{\mathscr{L}}(\lambda\tau')\,d\tau', \qquad (4.6)$$

where

$$F_1(\tau-\tau') = \frac{1}{2\pi}\frac{R^{(-)}(\tau-\tau')}{\tau-\tau'-i\varepsilon},$$
$$F_2(\tau'-\tau) = \frac{1}{2\pi}\frac{[R^{(+)}(\tau'-\tau)]^{+}}{\tau'-\tau-i\varepsilon}. \qquad (4.7)$$

From (4.5) and (4.6) we find

$$R^{(-)}(\lambda\tau) - [R^{(+)}(-\lambda\tau)]^{+}$$
$$= \int \widetilde{\mathscr{L}}(\lambda\tau')F_1(\tau-\tau')\,d\tau' - \int F_2(\tau'-\tau)\widetilde{\mathscr{L}}(\lambda\tau')\,d\tau'. \qquad (4.8)$$

On the other hand,

$$\int \widetilde{\mathscr{L}}(\lambda\tau')F_1(\tau-\tau')\,d\tau' - \int F_2(\tau'-\tau)\widetilde{\mathscr{L}}(\lambda\tau')\,d\tau'$$
$$= \int [R^{(+)}(-\lambda\tau')]^{+}F_1(\tau-\tau')\,d\tau'$$
$$-\int F_2(\tau'-\tau)R^{(-)}(\lambda\tau')\,d\tau', \qquad (4.9)$$

since

$$\int F_2(\tau''-\tau)\widetilde{\mathscr{L}}(\lambda\tau')F_1(\tau''-\tau')\,d\tau'd\tau''$$
$$= \int F_2(\tau'-\tau'')\widetilde{\mathscr{L}}(\lambda\tau')F_1(\tau-\tau'')\,d\tau'd\tau''$$
$$= \int F_2(\xi)\widetilde{\mathscr{L}}(\lambda(\tau+\xi-\eta))F_1(\eta)\,d\xi d\eta.$$

Hence we have from (4.7), (4.8), and (4.9)

$$R^{(-)}(\lambda\tau) - (R^{(+)}(-\lambda\tau))^{+} =$$
$$= \frac{1}{2\pi}\int \frac{d\tau'}{\tau'-i\varepsilon}[(R^{(+)}(\lambda\tau'-\lambda\tau))^{+}R^{(-)}(\lambda\tau')$$
$$-(R^{(+)}(\lambda\tau'))^{+}R^{(-)}(\lambda\tau'+\lambda\tau)]. \qquad (4.10)$$

The relation (4.10), considered for any λ satisfying (2.4), will be called the causality condition. This name will become understandable when we go over to the σ representation in (4.10). Using the definitions (3.2), (3.9) and the relation (4.2), we have instead of (4.10)

$$S(\infty;\sigma)\,S(\sigma;-\infty) = S(\infty;-\infty), \qquad (4.11)$$

i.e., the so-called group property of the scattering matrix. Of fundamental importance is here the circumstance that the plane $\lambda x = \sigma$ on which the operators $S(\infty;\sigma)$ and $S(\sigma;-\infty)$ are defined is, in view of the arbitrariness in the choice of λ, an arbitrary space-like plane through the point x. From this follows the relativistic invariance of the condition (4.11) and hence of (4.10).

The possibility of choosing freely the vector λ in (4.11) guarantees the correct S matrix description not only of events which are causally connected but also of events in mutually space-like regions of four-space. The first assertion is obvious and we shall therefore discuss only the second.

Let $g_1(x)$ and $g_2(x)$ be regions in which "the interaction is switched on"[9] such that all points g_1 are space-like relative to the points g_2:

$$g_1(x) \sim g_2(x). \qquad (4.12)$$

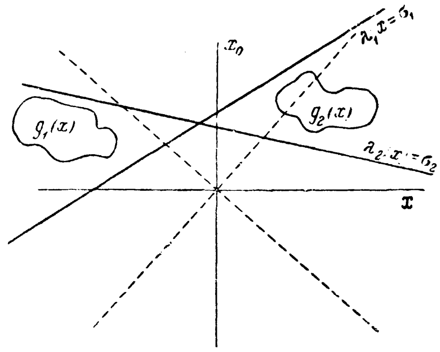
Owing to (4.12) one can always find two vectors λ_1 and λ_2 such that the regions g_1 and g_2 lie on different sides of each of the planes $\lambda_1 x = \sigma_1$ and $\lambda_2 x = \sigma_2$ [cf. the figure]. This leads to the obvious equations

$$S(g_1) = S(\infty;\sigma_1), \qquad S(g_2) = S(\sigma_1;-\infty). \quad (4.13)$$
$$S(g_1) = S(\sigma_2;-\infty), \qquad S(g_2) = S(\infty;\sigma_2). \quad (4.14)$$

Using (4.13) and (4.14), we finally find from (4.11)

$$S(g_1+g_2) = S(g_1)\,S(g_2) = S(g_2)\,S(g_1), \quad (4.15)$$

if $g_1 \sim g_2$.



Thus the relation (4.10) is indeed equivalent to the relativistic causality condition. We note that (4.10) reduces to the identity 0 = 0 on the mass shell $\tau = 0$.[4] This is in accordance with the known circumstance that the formulation of the

[3]Substituting (4.2) in (4.4), we can obtain an equation for $R(0;\lambda\tau)$:

$$R(0;\lambda\tau) = \widetilde{\mathscr{L}}(-\lambda\tau) + \frac{1}{2\pi}\int R(0;\lambda\tau')\frac{d\tau'}{\tau'-i\varepsilon}\widetilde{\mathscr{L}}(\lambda\tau'-\lambda\tau).$$

[4]To show this, use must be made of Eq. (4.3).

causality principle in field theory requires going into the nonphysical region.

In a subsequent paper we shall consider the diagram technique for solving Eq. (2.11) by perturbation theory.

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