ON THE POSSIBILITY OF APPEARANCE OF LARGE SCALE INHOMOGENEITIES IN THE EXPANDING UNIVERSE

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Gravitational instability of an expanding uniform isotropic universe may lead to the formation of cosmic bodies of various sizes, up to galaxy clusters with dimensions of the order of several hundred megaparsec.

It is shown in the present paper that arbitrarily small density fluctuations, accompanied by small fluctuations of the metric, can grow in a homogeneous isotropic expanding universe and reach at the present time values on the order of unity, provided they arise at a sufficiently early stage of the expansion of the universe.

The usual refutation of the possibility of such a gravitational instability was based on the following deductions from a paper by Lifshitz [1].

The fluctuations produced at an early stage of the expansion of the universe, when the equation of state $P = \rho/3$ was applicable (we shall call this period the first stage of expansion) [1)] remain small up to maximum values of the time, limiting the period of applicability of the indicated equation of state. Large-scale statistical fluctuations produced at a later stage of expansion, when the equation of state $P = 0$ is valid (second stage of expansion) have not had the time to grow to considerable magnitude at present. Zel'dovich [3] has shown that the reason for the large amplitude of initial perturbations can be a phase transition in expanding cold hydrogen. However, as shown by a more detailed calculation of Zel'dovich, this phenomenon likewise can not lead to the formation of large-scale fluctuations of the density which would be sufficient for the formation of stars (let alone objects of larger size).

However, if we recognize that a small fluctuation produced sufficiently early will grow by the end of the first stage and will appreciably exceed the statistical density fluctuations which can arise during that time (although it still remains absolutely small), continuing its growth during the second stage, then we arrive at the statement formulated at the beginning of this article. The maximum spatial dimensions of the fluctuation which has had a chance to reach a magnitude on the order of unity at the present time is ~ 300 Megaparsec.

Let us prove these statements. In a homogeneous isotropic expanding universe the square of the interval can be written in the form

$$ds^2 = a^2(\eta)\,[d\eta^2 - d\chi^2 - S^2(\chi)\,(d\theta^2 + \sin^2\theta\, d\varphi^2)],$$

where $S(\chi) = \sin\chi$ for a space of constant positive curvature and $S(\chi) = \sinh\chi$ for a space of constant negative curvature. The fundamental velocity is assumed equal to unity. During the first stage of expansion $P = \rho/3$. A solution of Einstein's equations gives for this stage

$$a = b_0\eta; \quad \eta \ll 1,$$

where b_0 is a constant. In this case the dependence of the time t on η is determined by the relation $t = b_0\eta^2/2$.

We denote by δg_{ik} the deviation from a homogeneous and isotropic metric[2)]. It is always possible to choose a reference frame such that $\delta g_{00} = 0$ and $\delta g_{\alpha 0} = 0$. The perturbation of the velocity of matter will be denoted by δv^α, and the density perturbation by $\delta\rho$. In regions of space that are small compared with the radius of curvature a, the perturbations can be expanded in plane waves. As shown by Lifshitz [4], small perturbations of the metric, connected with perturbations of the density and velocity, can be written in the form

$$\delta g_\alpha^\beta = [\mu(\eta)\tfrac{1}{3}\delta_\alpha^\beta + \lambda(\eta)(\tfrac{1}{3}\delta_\alpha^\beta - n_\alpha n^\beta/n^2)]\,e^{i\mathbf{nr}},$$

where $\mathbf{r}$ and $\mathbf{n}$ are respectively the radius vector and the wave vector, measured in units of a, while λ and μ depend in the specially chosen co-

[1)]At densities $\rho \gg 10^{14}$ g/cm^3, the equation of state has possibly the form $P = \rho$ (see [2]). Account of this fact does not change the result.

[2)]The Latin indices run through values 0, 1, 2, 3 while the Greek ones run through 1, 2, 3.

ordinate frame on η in the following manner:

$$\lambda = 3C_1/\eta + C_2(1 + n^2\eta^2/9),$$

$$\mu = -\tfrac{2}{3} n^2 C_1 \eta + C_2(1 - n^2\eta^2/6).$$

Here C_1 and C_2 are constants, whose values are examined by the initial perturbations. It is assumed that $1/\eta \gg n \gg 1$. For the density and velocity fluctuations we obtain the following expressions:

$$\delta\rho/\rho = \tfrac{1}{9} n^2 (C_1\eta + C_2\eta^2) e^{inr},$$

$$\delta v^\alpha = -\frac{in}{12}\left(3C_1 + C_2\frac{n^2}{9}\eta^3\right)\frac{n^\alpha}{n} e^{inr}.$$

The smallness of the initial perturbations of the metric (λ and μ), of the density ($\delta\rho/\rho$), and of the velocity (δv^α) necessitate $C_1 \ll \eta_0$ and $C_2 \ll 1$, where η_0 corresponds to the instant of occurrence of the perturbation[3]. Taking these relations into account, we find at the end of the first stage[4], corresponding to the value $\eta = \eta_1$ ($\eta_1 \gg \eta_0$), the perturbations will have the following values:

$$\lambda + \mu = 2C_2, \qquad \lambda - \mu = \tfrac{5}{18} n^2 C_2 \eta_1^2,$$

$$\frac{\delta\rho}{\rho} = \frac{n^2}{9} C_2 \eta_1^2 e^{inr},$$

$$\delta v^\alpha = -\frac{in}{12}\left(3C_1 + C_2\frac{n^2}{9}\eta_1^3\right)\frac{n^\alpha}{n} e^{inr}. \qquad (1)$$

Starting with the instant η_1 we can put $P = 0$ and the dependence of a on t is determined (for the open model) by the relation*

$$a = a_0(\operatorname{ch}\tilde{\eta} - 1), \qquad t = a_0(\operatorname{sh}\tilde{\eta} - \tilde{\eta}),$$

where $\tilde{\eta}$ is a new parameter, differing from η by an additive constant. The condition for the continuity of the quantity a and its first derivative on going from the first stage to the second yields

$$\tilde{\eta} = \eta + \eta_1, \qquad a_0 = b_0/2\eta_1.$$

The development of the perturbations during the second stage is determined by the formulas (see [4])†

$$\lambda + \mu = -\tilde{C}_1(\varphi - 1) - A\Psi,$$

$$\lambda - \mu = \frac{2n^2}{3}\tilde{C}_1\varphi - 2n^2\tilde{C}_2\left(\operatorname{cth}\frac{\tilde{\eta}}{2} - \frac{1}{3}\operatorname{cth}^3\frac{\tilde{\eta}}{2}\right) + A\Psi + \frac{4n^2}{3}A\operatorname{cth}\frac{\tilde{\eta}}{2} + B,$$

$$\frac{\delta\rho}{\rho} = \left[\frac{n^2}{6}(\tilde{C}_1\varphi + \tilde{C}_2\Psi) + \frac{A}{2}\Psi\right] e^{inr},$$

$$\delta v^\alpha = \frac{in}{6}(A - \tilde{C}_2)\frac{1}{\operatorname{sh}^2(\tilde{\eta}/2)}\frac{n^\alpha}{n} e^{inr}, \qquad (2)$$

where

$$\varphi(\tilde{\eta}) = \frac{3}{\operatorname{sh}^2(\tilde{\eta}/2)}\left(1 - \frac{\tilde{\eta}}{2}\operatorname{cth}\frac{\tilde{\eta}}{2}\right) + 1, \quad \Psi(\tilde{\eta}) = \frac{\operatorname{ch}(\tilde{\eta}/2)}{\operatorname{sh}^3(\tilde{\eta}/2)}.$$

The constants $\tilde{C}_1$, $\tilde{C}_2$, A, and B should be determined from the condition of continuity of the values of λ, μ, $\delta\rho/\rho$, and δv^α on going from the first stage of expansion of the universe to second. We shall be interested only in the value of the coefficient $\tilde{C}_1$, since this is precisely the coefficient determining the growth of the density fluctuations in time. Equating for $\eta = \eta_1$ the corresponding expressions for the first stage (1) and the second stage (2), and solving the resultant relations for $\tilde{C}_1$, $\tilde{C}_2$, A and B, we find that $\tilde{C}_1 \approx 2\tilde{C}_2$.

Thus, the density fluctuation which is produced at the beginning of the first stage of expansion and continues its growth during the second stage will be a quantity of the following order:

$$\delta\rho/\rho \approx \tfrac{1}{3} n^2 C_2 \varphi(\tilde{\eta}).$$

Assuming the current values $h = 0.25 \times 10^{-17}$ $\sec^{-1}$ for the Hubble constant and $\rho = 10^{-30}$ g/cm^3, we obtain $\tilde{\eta} \approx 6$ and $\varphi(6) \approx 1$[5]. Consequently, the fluctuation of the density could have grown by now to a value

$$\delta\rho/\rho \approx n^2 C_2/3. \qquad (3)$$

The initial perturbation of the density can be arbitrarily small. It must only occur sufficiently early (for $t \to 0$ and $C_1 = 0$, $\delta\rho/\rho$ tends to zero like t, δv^α like $t^{3/2}$, while λ and μ remain equal to C_2).

Assuming for C_2 a value 0.001, we find from (3) that the minimum value for n for which $\delta\rho/\rho$ can grow to unity at the present is of order of 55. accordingly, the largest fluctuation scale is

$$L = 2\pi a/n = \frac{2\pi\, 10^{28}\,\text{cm}}{55} \approx 10^{27}\,\text{cm} \approx 300\ \text{Mps}.$$

The quantity a is obtained for the values of h and ρ indicated above. The estimate of L depends on

[3]We note that small perturbations of density and velocity, which cover sufficiently large regions of space, cause metric perturbations which are not small.

[4]In the case of small entropy, the end of the first stage corresponds to the instance when nuclear density is attained.

*sh = sinh, ch = cosh.

†cth = coth.

[5]We note that $\varphi(\tilde{\eta})$ reaches a value of 0.5 already when t amounts to a tenth of the present-day value.

the assumed value of C_2 as $\sqrt{C_2}$. It is interesting to note that according to Ambartsumyan [5] the largest known galactic clusters measure ~ 20 megaparsec, and beginning with a scale ~ 100 megaparsec, the distribution of matter can be regarded as homogeneous. This result follows from the papers of Zwicky [6].

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