SOVIET PHYSICS JETP VOLUME 19, NUMBER 2 AUGUST, 1964

QUANTUM PROCESSES IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE AND IN A CONSTANT FIELD. I

A. I. NIKISHOV and V. I. RITUS

Lebedev Physical Institute of the Academy of Sciences of the USSR

Submitted to JETP editor July 30, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 776-796

The effect of the field of a plane electromagnetic wave or of a constant electromagnetic field on various quantum phenomena is investigated. General formulas are derived for the probabilities for radiation of a photon by an electron, for pair production by photons and for single photon annihilation of an electron and positron in the field of a plane electromagnetic wave. For $ea/m \ll 1$ (a_μ is the potential amplitude) these formulas transform into the corresponding probabilities for processes in crossed field ($\mathbf{E}\cdot\mathbf{H} = 0$, $E^2 - H^2 = 0$) with a field strength $E \sin\psi$ averaged over the phase ψ. The exact formulas for the probabilities of processes in a crossed field depend on the invariant $\chi = e\sqrt{(F_{\mu\nu}p_\nu)^2}/m^3$ where p_ν is the particle momentum; they are applicable to an arbitrary constant field $F_{\mu\nu}$ provided $(F_{\mu\nu}p_\nu)^2/m^2 \gg F_{\mu\nu}^2, i\varepsilon_{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$, and the latter are much smaller than $(m^2/e)^2 = (1.3\times 10^{13}$ abs. Heaviside units$)^2$.

1. INTRODUCTION

THE availability of powerful light beams from lasers offers in principle new possibilities for investigation of various quantum process. The study of quantum effects in a very large electromagnetic wave field should make it possible to obtain new information on the nature of the interactions of the particles participating in these processes. The dependences of these processes on the field strength of the wave and on its frequency and polarization (i.e., on those parameters which are easily varied in an experiment) are such as to suggest that this method of investigation will be more detailed and sensitive than the investigation of quantum phenomenon in the coulomb field of the nucleus, etc.

This paper considers the effects both of the strong field of a plane electromagnetic wave and of a constant electromagnetic field on the behavior of various quantum phenomena. In this connection all quantum processes may be divided into two groups: 1) processes which are induced by the electromagnetic field and which do not occur without it; 2) processes which occur in the absence of the field and are altered by its presence.

The present paper treats several processes in the first group; viz. the emission of a photon by an electron, the formation of a pair by a photon, and single photon annihilation of an electron and a positron. We note that pair formation from a photon in a wave field was first studied by Reiss[1]. Quantum phenomenon in the second group, which includes in particular all decay processes, will be considered in a later paper.

The probabilities for the various processes are calculated by the general method of quantum transitions, which accurately takes account of the interactions of charged particles with an electromagnetic wave field. The remaining electromagnetic interactions between particles (including the weak interactions) are treated by perturbation theory.

We obtain general expressions for the probabilities of various processes in the field of a plane electromagnetic wave. These expressions are studied in two limiting cases which depend on the magnitude of the parameter ea/m, which plays a very important role in this theory. For $ea/m \ll 1$, the expressions for the probabilities transform into the corresponding probabilities of perturbation theory, in which the plane wave appears as a single photon. For $ea/m \gg 1$ the probabilities for the processes are essentially the probabilities for processes in a constant field in which the electric and magnetic fields are orthogonal and equal in magnitude; $\mathbf{E} \perp \mathbf{H}$, $E = H$. A detailed study is made of the probabilities for processes in this field. Since it is a relativistic and gauge invariant quantity, the total probability for a process depends only on the parameter $e^2(F_{\mu\nu}p_\nu)^2/m^6$.

The expressions obtained for probabilities may be generalized to the case of an arbitrary constant field. In the general case the probabilities depend on two more parameters, $e^2F_{\mu\nu}^2/m^4$ and $ie^2\varepsilon_{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}/m^4$. However since all fields known at present are considerably smaller than the critical field m^2/e, these additional parameters are very much smaller than unity. Similarly if the energy of the incident particles is sufficiently large ($p_0/m \gg 1$), the additional parameters are much smaller even than the parameter $e^2(F_{\mu\nu}p_\nu)^2/m^6$ and may be neglected. Thus the expressions obtained for the probabilities also describe phenomena in an arbitrary constant electromagnetic field if the incident particles have relativistic energies. In particular, if one takes the quantity $F_{\mu\nu}$ to be a magnetic field, one obtains expressions for the intensity of emission by an electron and for the probability of pair formation by a photon in a magnetic field which are the same as the results found by Klepikov [2] as well as the more detailed results obtained by Sokolov, Klepikov and Ternov [3] and by Schwinger [4].

2. SOLUTION OF THE DIRAC EQUATION FOR AN ELECTRON IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE[1)]

The field of a plane electromagnetic wave propagating along the direction $\mathbf{k}$ can be described by a four-potential $A_\mu = A_\mu(\varphi)$ which depends on the coordinates only via a single invariant variable $\varphi = (kx)$, where k_μ is a four-vector of zero length formed from $\mathbf{k}$: $k_\mu = (\mathbf{k}, i|\mathbf{k}|)$, $k^2 = 0$. It is assumed that the potential A_μ satisfies the Lorentz gauge condition, so that $(kA) = 0$. Moreover one can always require the potential A_μ to satisfy another condition; for example, one can require that its fourth component be zero in some coordinate system. It is convenient to write this additional condition in the form $(fA) = 0$, where f is one of the four-vectors which occur in the problem.

The exact solution of the Dirac equations for an electron in the field of the plane electromagnetic wave was found by Volkov [5] It may be written in the form

$$\psi_{pr}(x) = \left[1 + e\frac{\hat{k}\hat{A}}{2(kp)}\right]u(p, r) \times \exp\left[i\int_0^{(kx)}\left(\frac{e(pA)}{(kp)} - \frac{e^2A^2}{2(kp)}\right)d\varphi + i(px)\right], \tag{1}$$

where p_μ is a constant four-vector whose components are quantum numbers describing the motion of the particle in the field of the wave, $p^2 = -m^2$, and $u(p)$ is the usual bispinor satisfying the equation $(i\hat{p} + m)u(p) = 0$. The significance of the components of the vector p may be seen easily if one transforms to the coordinate system in which $A_0 = 0$, and in which $\mathbf{A}$ is directed along the 1-axis and $\mathbf{k}$ along the 3-axis. We will call this coordinate system the "special" coordinate system. In this special system the electric field of the wave is directed along the 1-axis, the magnetic field is along the 2-axis, and the wave is propagating along the 3-axis. It may be seen easily in this system that the operators $-i\partial/\partial x_1$, $-i\partial/\partial x_2$, and $i\partial/\partial t + i\partial/\partial x_3$ commute with the Hamiltonian of the Dirac equation and hence are conserved, and further that the solution (1) is an eigenfunction of these operators with eigenvalues p_1, p_2, $p_3 - p_0$. Thus in the system we are using, p_1 and p_2 are the components of the generalized momentum along the axes 1 and 2, and $p_0 - p_3$ is the difference between the total energy and the component of the generalized momentum along the 3-axis.

If we allow the operator of the four-kinetic momentum $\Pi_\mu = -i\partial/\partial x_\mu - eA_\mu$ to act on the state ψ_p, we obtain $\Pi_\mu\psi_p = \pi_\mu\psi_p$, where

$$\pi_\mu = p_\mu - eA_\mu + k_\mu\left(\frac{e(pA)}{(kp)} - \frac{e^2A^2}{2(kp)}\right). \tag{2}$$

It follows that the density of kinetic momentum in the state ψ_p equals the product of the particle number density $\psi_p^+\psi_p$ and the four vector π_μ: $\psi_p^+\Pi_\mu\psi_p = \pi_p\psi_p^+\psi_p$. Hence the quantity π_μ may be called the four-kinetic momentum of the particle at the point x (this is completely analogous to the classic situation, cf [6]). The particle current density in the field of the wave has the form

$$j_\mu = i\bar{\psi}_p\gamma_\mu\psi_p = c\pi_\mu, \qquad c = u^+u/p_0. \tag{3}$$

The constant c is a relativistic invariant and is determined by the normalization of spinor $u(p)$. Physically, it is the ratio of the number density of the particles to their kinetic energy $c = j_0/\pi_0$, or it may be considered to be the number density of the particles in the eigensystem divided by the mass of the particles m.

The functions ψ_p satisfy the following condition of orthogonality and normalization

$$\int\psi_{p'}^+\psi_p d^3x = (2\pi)^3 u^+u\delta(\mathbf{p}' - \mathbf{p}). \tag{4}$$

In what follows we will be dealing as a rule with a monochromatic plane wave, for which $A_\mu = a_\mu\cos\varphi$, $\varphi = (kx)$. In this case

1) We use the following notation: $p_\mu = (\mathbf{p}, ip_0)$, $(pq) = \mathbf{p}\cdot\mathbf{q} - p_0q_0$, $\hat{p} = (\gamma p) = \boldsymbol{\gamma}\cdot\mathbf{p} + i\gamma_4p_0$, where γ_μ are hermitian matrices. Finally, $h = c = 1$ and $e^2/4\pi = 1/137$.

$$\psi_p(x) = \left[1 + e\frac{\hat{k}\hat{a}\cos\varphi}{2(kp)}\right] u(p) \exp\left[i\frac{e(ap)}{(kp)}\sin\varphi - i\frac{e^2a^2}{8(kp)}\sin 2\varphi + i(qx)\right], \quad (5)$$

where

$$q_\mu = \bar{\pi}_\mu = p_\mu - \frac{e^2a^2}{4(kp)}k_\mu \quad (6)$$

is the averaged kinetic momentum or "quasi-momentum." We note that $q^2 = -m_*^2$, where $m_* = m\sqrt{1 + (ea/m)^2/2}$ is the "effective" mass of the particle. It is convenient to take the average number density of particles, excluding the incident particles, to be equal to unity (i.e. $\bar{j}_0 = 1$). Then the spinor $u(p)$ must be normalized by the relationship $u^+u = p_0/q_0$, i.e., $c = 1/q_0$, and the function ψ_p is normalized by the relation

$$\int \psi_{p'}^+ \psi_p d^3x = (2\pi)^3\,\delta(\mathbf{q}' - \mathbf{q}). \quad (4')$$

The average density of incident particles will be called n, so that we have $c = n/q_0$.

3. THE MATRIX ELEMENTS AND PROBABILITIES.

The matrix element for the transition of an electron from the state ψ_p to the state $\psi_{p'}$, with the emission of a photon having momentum k′ and polarization e′ is equal to

$$M = e\int (\bar{\psi}_{p'}\hat{e}'\psi_p)\frac{1}{\sqrt{2k_0'}}e^{-i(k'x)}d^4x. \quad (7)$$

We introduce the parameters

$$\alpha = e\left(\frac{(ap)}{(kp)} - \frac{(ap')}{(kp')}\right), \qquad \beta = \frac{e^2a^2}{8}\left(\frac{1}{(kp)} - \frac{1}{(kp')}\right) \quad (8)$$

and put

$$\cos^n\varphi\exp(i\alpha\sin\varphi - i\beta\sin 2\varphi) = \sum_{s=-\infty}^{\infty} A_n(s, \alpha, \beta)\,e^{is\varphi},$$

$$n = 0, 1, 2, \quad (9)$$

where, clearly,

$$A_n(s, \alpha, \beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^n\varphi\, e^{f(\varphi)}\,d\varphi,$$

$$f(\varphi) = i\alpha\sin\varphi - i\beta\sin 2\varphi - is\varphi. \quad (10)$$

Then the matrix element may be written in the form

$$M = \frac{e}{\sqrt{2k_0'}}\sum_s \bar{u}(p')\left\{\hat{e}'A_0 + e\left(\frac{\hat{a}\hat{k}\hat{e}'}{2(kp')} + \frac{\hat{e}'\hat{k}\hat{a}}{2(kp)}\right)A_1 - \frac{e^2a^2(ke')\hat{k}}{2(kp)(kp')}A_2\right\}u(p)\,(2\pi)^4\,\delta(sk + q - q' - k'). \quad (11)$$

Thus the matrix element is an infinite sum of terms. As can be seen from the arguments of the delta functions, each of these terms describes the absorption from the wave or the emission into the wave of a definite number s of photons with momentum k. This obvious interpretation will apply to all the expressions obtained. The properties of the functions $A_n(s, \alpha, \beta)$, in terms of which the matrix elements are expressed, are described in appendix A.

Since the structure of the matrix elements (11) is similar to that of the matrix elements for plane waves, many of the operations on the matrix elements, including summation over final states, and summation and averaging over polarizations, are carried out in the same way as in the usual theory of plane waves.

The square of the matrix element (11), summed over the polarizations of the electrons, is equal to $(e'' = e' - k(k'e')/(kk'))$

$$\sum_{r,r'}\frac{|M|^2}{VT} = \frac{e^2}{2q_0q_0'k_0'}\sum_s\Big\{[2(pe'')^2 - (pp') - m^2]A_0^2 + [\alpha(kk') - 4e(ae'')(pe'')]A_0A_1 + \left[\frac{e^2a^2(kk')^2}{2(kp)(kp')} + 2e^2(ae'')^2\right]A_1^2\Big\}(2\pi)^4\,\delta(sk + q - q' - k'). \quad (12)$$

Summing over the polarizations of the photon e′, and using the relation (A3) from appendix A, we find

$$\sum\frac{|M|^2}{VT} = \frac{e^2}{2q_0q_0'k_0'}\sum_s\left\{-2m^2A_0^2 + e^2a^2\left(2 + \frac{(kk')^2}{(kp)(kp')}\right)\times(A_1^2 - A_0A_2)\right\}(2\pi)^4\,\delta(sk + q - q' - k'). \quad (13)$$

Integrating this expression over final states and replacing the summation over the directions of the spin of initial electron by an average, we obtain the probability for photon emission by an electron, calculated per unit volume and per unit time:

$$W = \frac{e^2n}{16\pi^2q_0}\sum_{s=1}^{\infty}\int_0^{2\pi}d\varphi\int_0^{u_s}\frac{du}{(1+u)^2}\times\left\{-2m^2A_0^2 + e^2a^2\left(2 + \frac{(kk')^2}{(kp)(kp')}\right)(A_1^2 - A_0A_2)\right\}; \quad (14)$$

Here $u = (kk')/(kq')$, $u_s = -2s(kq)/m_*^2$ and φ is the c.m.s. angle between the planes $(\mathbf{k}, \mathbf{q}')$ and $(\mathbf{k}, \mathbf{a})$. Because of the conservation laws, the summation is over values of $s \geq 1$. In analogy with (13) one obtains the probability for pair formation by a photon with momentum l.

$$W = \frac{e^2n}{32\pi^2l_0}\sum_{s>s_0}^{\infty}\int_0^{2\pi}d\varphi\int_1^{s/s_0}\frac{du}{u\sqrt{u(u-1)}}\times\left\{2m^2A_0^2 + e^2a^2\left(\frac{(kl)^2}{(kp)(kp')} - 2\right)(A_1^2 - A_0A_2)\right\}, \quad (15)$$

where $u = (kl)^2/4(kq)(kq')$, $s_0 = -2m_*^2/(kl)$, and φ is the same as in (14).

Because of the complexity of the functions A_0^2 and $A_1^2 - A_0A_2$, we can not in general carry out the integrations in (14) and (15). Hence we will consider two special cases determined by the magnitude of the invariant parameter $x = ea/m$ (we note that $x = Bm/B_0\omega$, where B is the magnitude of the field strength, ω is the frequency, and $B_0 = m^2/e$ is the critical field). It may be shown that for $x \ll 1$, for processes which do not occur in the absence of the field,

$$\alpha \sim xs, \qquad \beta \sim x\alpha. \tag{16}$$

In this case it is clear from the expression (A1) of appendix A that $A_0(0, \alpha, \beta) = 1$, $A_0(1, \alpha, \beta) = \alpha/2$, accurate to terms of second order. Then for $s = 1$ we have $A_0^2 = \alpha^2/4$, $A_1^2 - A_0A_2 = 1/4$, and the terms with $s > 1$ may be neglected. If in addition we put $a_\mu = e_\mu\sqrt{2n_\gamma/k_0}$ where n_γ is the photon number density in the wave and e_μ is the unit polarization vector, we obtain from (14) and (15) the probabilities for the Compton effect and for pair formation by two photons calculated by perturbation theory.

We obtain the other limiting case with $ea/m \gg 1$. The parameter $x \equiv ea/m$ may be large, for example, owing to a decrease of the frequency at fixed field strength. It is clear therefore that the case $x \gg 1$ leads essentially to consideration of processes in a constant field whose electric and magnetic field strengths are orthogonal and equal ($\mathbf{E} \perp \mathbf{H}$, $E = H$; expressed in the language of invariant fields: $F_{\mu\nu}^2 = 0$, $\varepsilon_{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} = 0$). We call such a field a constant crossed field.

We now show the connection between the probability $F(B)$ for an arbitrary process in a constant crossed field of magnitude B and the probability $W(B)$ for the same process in the alternating field of a plane wave with strength $B \sin\varphi$ for $x \gg 1$. It is clear that in a slowly varying field the probability of a process, occurring at the instant when the phase of the wave is equal to ψ, is the same as the probability for the process in the constant crossed field of strength $B \sin\psi$, that is, it is equal to $f(B \sin\varphi)$; hence the average probability in the alternating field, $W(B)$, is the average of $F(B \sin\psi)$ over the phases ψ, i.e.,

$$W(B) = \frac{2}{\pi}\int_0^{\pi/2} F(B\sin\psi)\, d\psi. \tag{17}$$

If the function $W(B)$ is known, then the above equation is an integral equation of the Schlömilch type (see [7]) for the function $F(B)$ which has the single solution

$$F(B) = W(0) + B\int_0^{\pi/2} W'(B\sin\psi)\, d\psi. \tag{18}$$

Thus if the probability of the process is known for an alternating field in the case $x \gg 1$, the probability of the same process in a constant crossed field is also known.

We now consider the transition to a slowly varying field ($x \to \infty$) for a more general process in which a charged particle with quasi-momentum q (and mass m), interacting with the field, is transformed into a different charged particle with quasi-momentum q' (and mass m') and several neutral particles with total momentum l. The total probability for such a process can be put in the form $W = (n/q_0)\sum_s \int wd^3q'/q'q_0'$, where w is a scalar function of the four-vectors a, k, q, q' and the invariant variable s. Because of the relativistic and gauge invariance, and because of the conservation law $sk + q = q' + l$, the function w depends on six independent invariants, which may be chosen as follows

$$s,\ \alpha,\ \beta,\ (kq),\ (kq'),\ l^2. \tag{19}$$

We are interested in the probability W for $x \to \infty$ and for fixed amplitude of the field $B = \omega a$. In this case the integrand w depends on x primarily through $a \sim x$, and $k \sim x^{-1}$. Moreover, w may depend implicitly on x through the variables of integration q' and summation s, since the effective values of these variables may vary with changing x.

We now make the important assumption that as $x \to \infty$ in the "special" system of coordinates the values of the variables will be effectively $q_1' \sim x$, $q_2' \sim$ const and $q_0' - q_3' \sim$ const. Moreover we assume that the values of the invariant variables are essentially $s \sim x^3$ and $l^2 \sim$ const. These assumptions will turn out later to have been justified. Using these assumptions we express the six variables (19) in terms of x and five new invariant variables which are independent of x when for $x \to \infty$. It is convenient to choose these new variables, except l^2, as follows

$$\chi = -\frac{(kq)}{m^2}x, \quad \chi' = -\frac{(kq')}{m^2}x, \quad \varkappa = -\frac{(kl)}{m^2}x,$$
$$\sigma = x^2\left[\frac{s}{4\beta} - \frac{1}{2} - \left(\frac{\alpha}{8\beta}\right)^2\right], \quad \cos\psi = \frac{\alpha}{8\beta}. \tag{20}$$

Only four of these five variables are independent, since $\kappa = \chi - \chi'$. We now express these variables in the "special" coordinate system and also introduce their invariant expressions in terms of the field amplitude $F_{\mu\nu}$:

$$\chi = \frac{eB\gamma}{m^3} = \frac{e\sqrt{(F_{\mu\nu}q_\nu)^2}}{m^3}, \qquad \chi' = \frac{eB\gamma'}{m^3} = \frac{e\sqrt{(F_{\mu\nu}q'_\nu)^2}}{m^3},$$

$$\varkappa = \frac{eB\lambda}{m^3} = \frac{e\sqrt{(F_{\mu\nu}l_\nu)^2}}{m^3},$$

$$\cos\psi = \frac{\gamma q'_1 - \gamma' q_1}{xm\lambda} = \frac{eF_{\mu\nu}q'_\mu q_\nu}{xm^4\varkappa},$$

$$\tau = \frac{\gamma q'_2 - \gamma' q_2}{m\lambda} = -\frac{eF^*_{\mu\nu}q'_\mu q_\nu}{m^4\varkappa},$$

$$\sigma = \frac{\gamma m'^2 - \gamma' m^2}{m^2\lambda} - \frac{l^2\gamma\gamma'}{m^2\lambda^2} + \frac{(\gamma q'_2 - \gamma' q_2)^2}{m^2\lambda^2} = \frac{\chi m'^2 - \chi' m^2}{m^2\varkappa} - \frac{l^2\chi\chi'}{m^2\varkappa^2} + \frac{e^2(F^*_{\mu\nu}q'_\mu q_\nu)^2}{m^8\varkappa^2}. \tag{21}$$

Here

$$\gamma = q_0 - q_3, \quad \gamma' = q'_0 - q'_3, \quad \lambda = l_0 - l_3,$$

$$F^*_{\mu\nu} = {}^1/_2\, i\, \varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$

($F^*_{\mu\nu}$ is obtained from $F_{\mu\nu}$ by replacing **H** by **E** and **E** by **−H**). Only four of these variables are independent, for example χ, χ', $\cos\psi$, and τ; the remaining variables κ and σ are functions of the others. The first equation of (21) show that the new variables actually are independent of x because of the assumptions made previously; the second equations will be very important in what follows when we turn to a description of processes occurring in an arbitrary constant field $F_{\mu\nu}$.

4. THE EMISSION OF A PHOTON BY AN ELECTRON

The emission of a photon by an electron is a special case of a process having the conservation law $sk + q = q' + l$. The probabilities for processes with this type of kinematics have the form

$$W = \frac{n}{(2\pi)^2 q_0}\sum_s \int \frac{d^3q'}{q'_0 l_0}\,\delta(sk_0 + q_0 - q'_0 - l_0)w, \tag{22}$$

where $l_0 = \sqrt{-l^2 + (\mathbf{sk} + \mathbf{q} - \mathbf{q}')^2}$, and where w is an invariant function. Keeping in mind the limiting case $x \to \infty$, we transform to new variables of integration in (22). Since for $x \to \infty$ we have essentially $s \sim x^3$ in the sum over s, we may replace the sum over s by an integral. Integrating over s and then transforming from the variable q'_3 to the variable $\gamma' \equiv q'_0 - q'_3$, we obtain

$$\sum_s \int \frac{d^3q'}{q'_0 l_0}\,\delta(sk_0 + q_0 - q'_0 - l_0)\ldots = -\int \frac{d^3q'}{q'_0(kl)}\ldots$$

$$= -\int_{-\infty}^{\infty} dq'_1 \int_{-\infty}^{\infty} dq'_2 \int_0^{\gamma} \frac{d\gamma'}{\gamma'(kl)}\ldots \tag{23}$$

The limits of integration in the last integral are chosen to correspond with the conservation laws $q_1 = q'_1 + l_1$, $q_2 = q'_2 + l_2$, $\gamma = \gamma' + \lambda$.

Using (21), we now transform from the variables q'_1, q'_2, γ' to the variables $\cos\psi$, τ, χ' respectively. Then if we take account of the fact that the function in the integrand is even in ψ and τ we obtain

$$W = \frac{n}{(2\pi)^2 q_0}\int_0^{\pi/2} d\psi \int_0^{\chi} d\chi' \int_0^{\infty} d\tau\, \frac{4x^2 \sin\psi(\chi - \chi')}{\chi^2\chi'}w. \tag{24}$$

Finally putting $\tau = \sinh u$, $\chi/\chi' = \cosh^2 v$, we obtain instead of (22)

$$W = \frac{n}{(2\pi)^2 q_0}\int_0^{\pi/2} d\psi \int_0^{\infty} dv \int_0^{\infty} du\, \frac{8x^2 \sin\psi\, \mathrm{th}^3 v\, \mathrm{ch}\, u}{\chi}w. \tag{25}^*$$

For the emission of a photon by an electron, the function w is determined by (13). We put this function in (25) using the asymptotic expressions for the functions A_0^2 and $A_1^2 - A_0A_2$, given by (B13) and (B21) in Appendix B. Since we have in the present case $\kappa/\chi' = \sinh^2 v$, $\sigma = 1 + \tau^2 = \cosh^2 u$, the probability per unit volume and per unit time for the emission of a photon by an electron will then be

$$W(\chi) = \frac{2e^2m^2n}{\pi^3 q_0}\int_0^{\pi/2} d\psi \int_0^{\infty} dv \int_0^{\infty} du\, \frac{\mathrm{sh}\, v}{\mathrm{ch}^3 v}\sqrt{y}\,\Big\{-2\Phi^2(y)$$

$$+ \mathrm{ch}^2 u\,(\mathrm{ch}^2 v + \mathrm{ch}^{-2} v)\Big[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\Big]\Big\}, \tag{26}$$

where $y = (\sinh^2 v/2\chi\sin\psi)^{2/3}\cosh^2 u$. Thus the probability for emission depends only on the parameter χ and is of the type (17). Since (17) has a simple solution, the probability for the emission of a photon by an electron in a crossed field is equal to

$$F(\chi) = \frac{e^2m^2c}{\pi^2}\int_0^{\infty} dv \int_0^{\infty} du\, \frac{\mathrm{sh}\, v}{\mathrm{ch}^3 v}\sqrt{y}\,\Big\{-2\Phi^2(y)$$

$$+ \mathrm{ch}^2 u\,(\mathrm{ch}^2 v + \mathrm{ch}^{-2} v)\Big[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\Big]\Big\}, \tag{27}$$

where $y = (\sinh^2 v/2\chi)^{2/3}\cosh^2 u$.

We now investigate the function $F(\chi)$ for both small and large χ. To do this it is convenient to change from the variable v to the variable $t = (2/3)y^{3/2}$, replacing the Airy function and its derivative by the Bessel functions $K_{1/3}(t)$, and $K_{2/3}(t)$. The function $F(\chi)$ then becomes a sum of three integrals, whose integrands depend on χ only through the factor $(1 + 3\chi t/\cosh^3 u)^{-n}$,

*sh = sinh, ch = cosh, th = tanh.

where n = 1, 2, 3. For $\chi \ll 1$ these factors may be developed in a series

$$(1+at)^{-n} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+k)}{\Gamma(n)\Gamma(k+1)} (at)^k$$

in powers of χ. Then for $F(\chi)$ we also obtain a series in powers of χ, whose coefficients are integrals which have been tabulated [8]. Hence for $\chi \ll 1$ we obtain the asymptotic series

$$F(\chi) = \frac{e^2 m^2 c\chi}{32\sqrt{3}\pi^2} \sum_{k=0}^{\infty} (-1)^k (3k^2+3k+10) \times \Gamma\left(\frac{k}{2}+\frac{1}{6}\right)\Gamma\left(\frac{k}{2}+\frac{5}{6}\right)(3\chi)^k = \frac{5e^2m^2c\chi}{8\sqrt{3}\pi}\left\{1-\frac{8}{15}(3\chi)+\frac{7}{18}(3\chi)^2-\ldots\right\}. \qquad (27')$$

For $\chi \gg 1$, in place of the series expansion of the quantity $(1+at)^{-n}$, we use the representation [8]

$$(1+at)^{-n} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk \, \frac{\Gamma(-k)\Gamma(n+k)}{\Gamma(n)} (at)^k.$$

Then clearly in place of the sum (27′) we obtain the integral

$$F(\chi) = \frac{e^2m^2c\chi}{32\sqrt{3}\pi^2}\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk\,(3k^2+3k+10)\,\Gamma(-k) \times \Gamma(k+1)\,\Gamma\left(\frac{k}{2}+\frac{1}{6}\right)\Gamma\left(\frac{k}{2}+\frac{5}{6}\right)(3\chi)^k.$$

The path of the integration can be closed on the left, thereby transforming the integral into a sum of the residues at the poles of the integrand which lie in the left-hand plane. We then obtain

$$F(\chi) = \frac{e^2m^2c\chi}{8\sqrt{3}} \sum_{n=0}^{\infty} \left\{ \frac{4(18n^2-3n+14)}{9n!\,\Gamma(n+1/3)}(3\chi)^{-2n-1/3} + \frac{4(18n^2+21n+20)}{9n!\,\Gamma(n+5/3)}(3\chi)^{-2n-5/3} - \frac{(3n^2+3n+10)(3\chi)^{-n-1}}{(2+(-1)^n)\,\Gamma(n/2+2/3)\,\Gamma(n/2+4/3)} \right\}$$
$$= \frac{7\Gamma(2/3)\,e^2m^2c}{54\pi}(3\chi)^{2/3}\left\{1-\frac{45}{28\Gamma(2/3)}(3\chi)^{-2/3} + \frac{15\Gamma(1/3)}{7\Gamma(2/3)}(3\chi)^{-4/3} - \frac{216\sqrt{3}}{35\Gamma(2/3)}(3\chi)^{-5/3}+\ldots\right\}. \qquad (27'')$$

We now calculate the emission intensity from an electron in the field of a wave and in a crossed field. To this end we consider the four-momentum of the radiation $I_\mu = \sum_s \int k'_\mu \, dW_s$, emitted per unit volume and per unit time. Since I_μ depends only on p, k, and a, and since it must be gauge-invariant, it has the form

$$I_\mu = p_\mu A + k_\mu B + \left(a_\mu - k_\mu \frac{(ap)}{(kp)}\right) C,$$

where the invariant functions A, B, and C are easy to find by multiplying the right and left hand sides of this relation by p, k, and a. We then obtain

$$I_\mu = \sum_s \int \left\{ p_\mu - e\left(a_\mu - k_\mu \frac{(ap)}{(kp)}\right)\frac{\alpha}{8\beta} - k_\mu \frac{e^2a^2}{(kp)}\left(\frac{s}{8\beta}-\frac{1}{4}-\frac{1}{x^2}\right)\right\} \frac{(kk')}{(kp)} dW_s. \qquad (28)$$

For $x \to \infty$ we have $\alpha/8\beta \to \cos\psi$, $s/4\beta - 1/2 \to \cos^2\psi$, since the expression in curly brackets has the value $-k_\mu e^2a^2\cos^2\psi/2(kp)$, we find for the emission intensity I_0 per unit volume the following expression:

$$I_0(\chi) = \frac{2}{\pi}\int_0^{\pi/2} n(\psi)\, I(\chi\sin\psi)\, d\psi, \qquad (29)$$

where $n(\psi) = n\cdot 2\cos^2\psi$ is the particle number density when the phase of the wave is ψ, and $I(\chi)$ is the emission intensity due to a single particle in the crossed field:

$$I(\chi) = \frac{e^2m^2}{\pi^2}\int_0^\infty dv \int_0^\infty du \, \frac{\mathrm{sh}^3 v}{\mathrm{ch}^5 v}\sqrt{y} \times \left\{-2\Phi^2 + \mathrm{ch}^2 u\,(\mathrm{ch}^2 v + \mathrm{ch}^{-2} v)\left(\Phi^2+\frac{1}{y}\Phi'^2\right)\right\},$$
$$y = \left(\frac{\mathrm{sh}^2 v}{2\chi}\right)^{2/3}\mathrm{ch}^2 u. \qquad (30)$$

$I(\chi)$ is an invariant quantity and differs from the probability $F(\chi)$ by an additional factor $c^{-1}\tanh^2 v$ under the integral sign. The behavior of $I(\chi)$ for both small and large χ may be found in the same way as was done for $F(\chi)$. Hence for $\chi \ll 1$

$$I(\chi) = \frac{3e^2m^2\chi^2}{32\sqrt{3}\pi^2}\sum_{k=0}^{\infty}(-1)^k(k+1)(k^2+2k+8) \times \Gamma\left(\frac{k}{2}+\frac{2}{3}\right)\Gamma\left(\frac{k}{2}+\frac{4}{3}\right)(3\chi)^k = \frac{e^2m^2\chi^2}{6\pi}\left\{1-\frac{55\sqrt{3}}{48}(3\chi)+\frac{16}{3}(3\chi)^2-\ldots\right\}. \qquad (30')$$

The first term of this series does not contain Planck's constant and hence is the classical emission intensity for an electron in a crossed field[2)] (cf. Sec. 73 in [6]). For $\chi \gg 1$

[2)]The classical formula $e^2m^2\chi^2/6\pi$ is valid for the emission intensity from an electron in an arbitrary constant field, if $F_{\mu\nu}$ is taken in the invariant $\chi = e\sqrt{(F_{\mu\nu}p_\nu)^2}/m^3$ to be the field strength and p_ν is the electron momentum[6]. We show in Sec. 7 that our quantum formulae, including the formula for $I(\chi)$, are also valid for an arbitrary constant field under certain restrictions.

$$I(\chi)=\frac{3e^2m^2\chi^2}{8\sqrt{3}}\sum_{n=0}^{\infty}\left\{\frac{8(6n+1)(9n^2+3n+16)}{81n!\,\Gamma(n+1/3)}(3\chi)^{-2n-4/3}\right.$$
$$+\frac{8(6n+5)(9n^2+15n+22)}{81n!\,\Gamma(n+5/3)}(3\chi)^{-2n-8/3}$$
$$\left.-\frac{n(n^2+7)}{(2-(-1)^n)\Gamma(n/2+1/6)\Gamma(n/2+5/6)}(3\chi)^{-n-1}\right\}$$
$$=\frac{8\Gamma(2/3)e^2m^2}{243\pi}(3\chi)^{2/3}\left\{1-\frac{81}{16\Gamma(2/3)}(3\chi)^{-2/3}\right.$$
$$\left.+\frac{165\Gamma(1/3)}{16\Gamma(2/3)}(3\chi)^{-4/3}-\frac{2673\sqrt{3}}{80\Gamma(2/3)}(3\chi)^{-8/3}+\ldots\right\}. \qquad (30'')$$

We note that for $-s(kq)/m_*^2 \ll 1$ we obtain from the quantum mechanical formula (28) the classical expression for the intensity of the emission of an electron in the field of a plane polarized wave which, in the coordinate system in which the electron is on the average at rest ($\mathbf{q}=0$, $q_0=m_* = m\sqrt{1+x^2/2}$), has the form

$$I^{\mathrm{cl}}=\frac{e^2\omega^2}{8\pi^2(1+x^2/2)}\sum_{s=1}^{\infty}s^2\int\{-A_0^2(s,\alpha,\beta)$$
$$+x^2[A_1^2(s,\alpha,\beta)-A_0(s,\alpha,\beta)A_2(s,\alpha,\beta)]\}\,d\Omega, \qquad (28')$$

where

$$\alpha=-s\frac{x}{\sqrt{1+x^2/2}}\sin\theta\cos\varphi,\quad \beta=s\frac{x^2}{8(1+x^2/2)}(1-\cos\theta),$$

and where θ is the angle between $\mathbf{k}'$ and $\mathbf{k}$ and φ the angle between the plane $(\mathbf{k},\mathbf{k}')$ and $\mathbf{a}$. The intensity of the first two harmonics ($s=1, 2$) for $x \ll 1$ was discussed in [12], but the results do not agree with ours.[3)]

5. PAIR PRODUCTION BY A PHOTON

Pair production by a photon is a process with a conservation law $sk + l = q + q'$. The probability for processes of this type has the form

$$W=\frac{n}{(2\pi)^2 l_0}\sum_s\int\frac{d^3q'}{q_0'q_0}\delta(sk_0+l_0-q_0-q_0')w, \qquad (31)$$

where $q_0=\sqrt{-q^2+(sk+l-q')^2}$, and where w is an invariant function. Changing to new variables of integration as in Sec. 4, we obtain

$$\sum_s\int\frac{d^3q'}{q_0'q_0}\delta(sk_0+l_0-q_0-q_0')\ldots=-\int\frac{d^3q'}{q_0'(kq)}\ldots \qquad (32)$$
$$=-\int_{-\infty}^{\infty}dq_1'\int_{-\infty}^{\infty}dq_2'\int_0^{\lambda}\frac{d\gamma'}{\gamma'(kq)}\ldots$$

The limits of integration in the last integral are chosen to correspond with the conservation laws $l_1 = q_1 + q_1'$, $l_2 = q_2 + q_2'$, $\lambda = \gamma + \gamma'$. We now change from the variables q_1', q_2', γ' to the variables $\cos\psi$, τ, χ' by using Eq. (21), in which, to satisfy the conservation law, we must make the changes $l \to -l$, $q \to -q$ and hence $\gamma \to -\gamma$, $\chi \to -\chi$, $\kappa \to -\kappa$. We then obtain

$$W=\frac{n}{(2\pi)^2 l_0}\int_0^{\pi/2}d\psi\int_0^{\varkappa}d\chi'\int_0^{\infty}d\tau\,\frac{4x^2\sin\psi}{\chi\chi'}w. \qquad (33)$$

In the important particular case where w is symmetric in χ and χ', the integral over χ' between 0 and κ is equal to twice the integral over χ' from 0 to $\kappa/2$. In this case we put $\chi' = (1+\tanh v)/2$, and also $\tau = \sinh u$, and we obtain

$$W=\frac{n}{(2\pi)^2 l_0}\int_0^{\pi/2}d\psi\int_0^{\infty}dv\int_0^{\infty}du\,\frac{16x^2\sin\psi\,\mathrm{ch}\,u}{\varkappa}w. \qquad (34)$$

For the production of a pair from a photon with momentum l and polarization e', the function w is determined by (12) if we make in this formula, in accordance with a well known rule [9], the substitutions $k' \to -l$ and $q \to -q$, and change the sign. We consider two cases; first, the case in which the polarization of the incident photon is parallel to the polarization of the wave of the "special" system and, second, the case in which the polarization of the incident photon is perpendicular to the polarization of the wave in the "special" system. In the first case we take $e_{\|}' = e$, and in the second case we take $e'_{\perp\mu} = -i\varepsilon_{\mu\nu\lambda\sigma}e_\nu k_\lambda l_\sigma/(kl)$, where e is the unit polarization vector of the wave. We put $e_{\|}'$ and $e_\perp'$ in (16), using the fact that $(e_{\|}'k) = (e_{\|}'l) = (e_\perp'k) = (e_\perp'l) = 0$, and also using expression (A3) from appendix A. We then obtain the two invariant functions $w_{\|}$ and $w_\perp$ which describe the production of a pair by photons polarized respectively parallel and perpendicular to the polarization of the wave:

$$w_{\|}=e^2m^2\left\{\sigma A_0^2+x^2\left(\frac{\varkappa^2}{4\chi\chi'}-1\right)(A_1^2-A_0A_2)\right\},$$
$$w_\perp=e^2m^2\left\{(1-\sigma)A_0^2+x^2\frac{\varkappa^2}{4\chi\chi'}(A_1^2-A_0A_2)\right\}. \qquad (35)$$

We have used here the variables (20). It should be noted that the functions $w_{\|}$ and $w_\perp$ contain only two combinations of the functions A_n, namely A_0^2 and $A_1^2 - A_0A_2$, rather than three combinations as was the case for the initial function w for arbitrary polarization e'.

We now take the limit as $x \to \infty$ and put the functions $w_{\|}$ and $w_\perp$ in (34), using for A_0^2 and $A_1^2 - A_0A_2$ their asymptotic expressions (B13) and (B21) (cf. appendix B), and putting [as in (34)] $\sigma = 1 + \tau^2 = \cosh^2 u$, $\chi' = \kappa(1 + \tanh v)/2$. Then we have

[3)]The corrected results recently published by Vachaspati (Phys. Rev. 130, 2598, 1963) agree with ours.

$$W_{\parallel}(\varkappa) = \frac{4e^2m^2n}{\pi^3 l_0}\int_0^{\pi/2} d\psi \int_0^\infty dv \int_0^\infty du \frac{\operatorname{ch}^2 u}{\operatorname{ch}^2 v}\sqrt{y}$$
$$\times\left\{\Phi^2(y) + \operatorname{sh}^2 v\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\right\},$$

$$W_{\perp}(\varkappa) = \frac{4e^2m^2n}{\pi^3 l_0}\int_0^{\pi/2} d\psi \int_0^\infty dv \int_0^\infty du \frac{\operatorname{ch}^2 u}{\operatorname{ch}^2 v}\sqrt{y}$$
$$\times\left\{-\operatorname{th}^2 u\Phi^2(y) + \operatorname{ch}^2 v\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\right\}, \quad (36)$$

where $y = (2\cosh^2 v/\kappa \sin\psi)^{2/3}\cosh^2 u$. For small κ in the integrals (36) an effective role is played by u, v, and $\pi/2 - \psi \ll 1$. Expanding the corresponding functions in series and using the asymptotic representation of the Airy functions for large values of the argument, we obtain:

$$W_{\parallel} = \frac{3e^2m^2n}{32 l_0}\left(\frac{\varkappa}{2\pi}\right)^{1/2} e^{-8/3\varkappa}, \quad W_{\perp} = 2W_{\parallel}, \quad \varkappa \ll 1. \quad (36')$$

For large values of κ we essentially have $u \gg 1$ in the integrals (36). Using this fact, we obtain

$$W_{\parallel} = \frac{27\Gamma^7(2/3)\, e^2m^2n}{56\pi^5 l_0}\left(\frac{3\varkappa}{2}\right)^{2/3}, \quad W_{\perp} = \frac{3}{2}W_{\parallel}, \quad \varkappa \gg 1. \quad (36'')$$

The probabilities $W_{\parallel}$ and $W_{\perp}$ for small κ were obtained recently by Reiss [1].

The expressions (36) for the probabilities $W_{\parallel}$ and $W_{\perp}$ have the form (17), and hence the probabilities for pair production by a photon in a crossed field are

$$F_{\parallel}(\varkappa) = \frac{2e^2m^2n}{\pi^2 l_0}\int_0^\infty dv \int_0^\infty du \frac{\operatorname{ch}^2 u}{\operatorname{ch}^2 v}\sqrt{y}\left\{\Phi^2(y)\right.$$
$$\left. + \operatorname{sh}^2 v\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\right\},$$

$$F_{\perp}(\varkappa) = \frac{2e^2m^2n}{\pi^2 l_0}\int_0^\infty dv \int_0^\infty du \frac{\operatorname{ch}^2 u}{\operatorname{ch}^2 v}\sqrt{y}$$
$$\times\left\{-\operatorname{th}^2 u\,\Phi^2(y) + \operatorname{ch}^2 v\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\right\}, \quad (37)$$

where $y = (2\cosh^2 v/\kappa)^{2/3}\cosh^2 u$. The limiting values of $F_{\parallel}$ and $F_{\perp}$ for small and for large κ are

$$F_{\parallel} = \sqrt{\frac{3}{2}}\frac{e^2m^2n}{32\pi l_0}\varkappa e^{-8/3\varkappa}, \quad F_{\perp} = 2F_{\parallel}, \quad \varkappa \ll 1. \quad (37')$$

$$F_{\parallel} = \frac{3\Gamma^4(2/3)\, e^2m^2n}{28\pi^3 l_0}(3\varkappa)^{2/3}, \quad F_{\perp} = \frac{3}{2}F_{\parallel}, \quad \varkappa \gg 1. \quad (37'')$$

The probability of pair production by a photon in a crossed field was obtained for $\kappa \ll 1$ by Toll and Wheeler (cf. [1]). In conclusion we note that the probability of pair production by an unpolarized photon is equal to $F = (F_{\parallel} + F_{\perp})/2$, and the probability of pair formation by a photon with arbitrary polarization e' is equal to $F(e') = F_{\perp} + e^2(F_{\mu\nu} l_\mu e'_\nu)^2(F_{\parallel} - F_{\perp})/m^6\kappa^2$.

6. SINGLE PHOTON ANNIHILATION OF AN ELECTRON AND A POSITRON

Single photon annihilation of a pair is a process with a conservation law $q + q' = sk + l$. Probabilities for such processes have the form

$$W = 2\pi\frac{nn'}{q_0 q_0'}\sum_s \frac{1}{l_0}\delta(q_0 + q_0' - sk_0 - l_0)w, \quad (38)$$

where $l_0 = \sqrt{-l^2 + (q + q' - sk)^2}$ and w is an invariant function. For $x \to \infty$ the sum over s may be replaced by an integral. We then obtain

$$W = -2\pi\frac{nn'}{q_0 q_0'}\frac{w}{(kl)}. \quad (39)$$

For single photon annihilation of a pair, the function w is equal to the function w for the production of a pair by a photon but multiplied by one fourth, owing to averaging rather than a summing over the polarizations of the electron and positron. If one is not interested in the polarization of the emitted photon, one must take $w = (w_{\parallel} + w_{\perp})/4$ where $w_{\parallel}$ and $w_{\perp}$ are given by (35). Putting this function in (39) and using the asymptotic expressions (B13) and (B21) from appendix B for A_0^2 and $A_1^2 - A_1A_2$, we obtain

$$W = \frac{e^2nn'}{x q_0 q_0'}\frac{\sigma}{\varkappa y}\left\{\Phi^2(y) + \left(\frac{\varkappa^2}{2\chi\chi'} - 1\right)\sigma\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\right\}. \quad (40)$$

Here, as follows from (21),

$$\sigma = 1 + e^2(F^*_{\mu\nu}q_\mu q'_\nu)^2/m^8\varkappa^2, \qquad y = (\varkappa/2\chi\chi')^{2/3}\sigma,$$

where $\cos\psi \to 0$ and $\sin\psi \to 1$ as $x \to \infty$, for in contrast to the previous cases the quantities q_1, q_2, γ, and q_1', q_2', γ' are now fixed for both charged particles.

Expression (40) shows that the annihilation probability per unit volume per unit time goes to zero as $x \to \infty$. Physically this is connected with the fact that the interaction between the electron and the positron depends critically on the phase of the field, and for $x \to \infty$ vanishes if $\psi \neq \pi/2$. However, if one introduces the annihilation probability per unit volume over the half-period $T_{1/2}$ of the field vibration, the probability will be finite for $x \to \infty$ and equal to

$$WT_{1/2} = \frac{\pi e m n n'}{B q_0 q_0'}\frac{\sigma}{\varkappa y}\left\{\Phi^2(y) + \sigma\frac{\chi^2 + \chi'^2}{2\chi\chi'}\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\right\}. \quad (41)$$

The reason why it is natural to consider the annihilation probability over a half-period of the vibration of the wave rather than a full period is that classically the total trajectory of the particle in the crossed field is equivalent to half of a figure-eight trajectory of the particle in an oscillating field as $\omega \rightarrow 0$. Then expression (41) represents the probability for annihilation in a crossed field of 1 cm^3 over all time. In other words, in a constant crossed field the annihilation probability depends on the time, and the integral over the time is given by (41). Since the time dependence enters in the theory only in the combination $x_3 - t$, the integration over t is equivalent to an integration over x_3. Hence expression (41) (multiplied by the velocity of light) is also the probability for annihilation per second in an infinite cylindrical volume whose axis lies along the vector $\mathbf{E} \times \mathbf{H}$ and whose base has area 1 cm^2.

7. PROCESSES OCCURRING IN A CONSTANT ELECTROMAGNETIC FIELD

In the previous sections we obtained results for the probabilities of various processes in the field of the plane electromagnetic wave, and it was shown that for $x \ll 1$ these probabilities go over into the corresponding formulas of perturbation theory, and that for $x \gg 1$ they go into the probabilities for a crossed field of strength $B \sin \psi$, averaged over the phase ψ. We then obtained exact formulas for the probabilities in a crossed field, written in relativistic and gauge invariant form in terms of parameters containing the electromagnetic field tensor $F_{\mu\nu}$ and the vectors p_μ, p'_μ, characterizing the state of the particle in this field,

The question arises whether one may use the formulas obtained for probabilities in a crossed field to describe processes in an arbitrary constant electromagnetic field, using in the above formulas for the quantity $F_{\mu\nu}$ an arbitrary constant field rather than the crossed field, and using instead of the vectors p_μ and p'_μ, the quantum numbers describing the states of the particle in the given field.

In this connection we note that the state of the particle in an arbitrary constant field is determined by the same number of quantum numbers as the state of the free particle, and that these quantum numbers may always be chosen so that when the field is turned off they transform into the quantum numbers of the free particle, i.e., into the four-momentum of the free particle p_μ ($p^2 = -m^2$) and the projection r of its spin. Thus the state of the particle in a constant field can be described by a constant four-vector p_μ and a discrete number r.

We now consider for simplicity these processes in a constant field which are caused by a single particle. The total probability for such a process (averaged over r) may depend only on the field $F_{\mu\nu}$ and on the vector p_μ. If one is dealing with a probability calculated per unit volume and per unit time, such a probability must be an invariant quantity and hence can depend only on the invariants $(F_{\mu\nu}p_\nu)^2$, $F^2_{\mu\nu}$, $F^*_{\mu\nu}F_{\mu\nu}$. Moreover since in a constant field the electric charge e can occur in the probability only in combination with the field $F_{\mu\nu}$, we find that

$$W = W(\chi, f, g), \tag{42}$$

where $\chi^2 = e^2(F_{\mu\nu}p_\nu)^2/m^6$, $f = e^2F^2_{\mu\nu}/m^4$, $g = e^2 F^*_{\mu\nu}F_{\mu\nu}/m^4$ are dimensionless parameters. The function $W(\chi, f, g)$ is unknown. However we have found the exact function W for the crossed field, where $f = g = 0$, i.e., the function $W(\chi, 0, 0)$. Therefore if

$$f, g \ll 1, \qquad \chi^2 \gg f, g, \tag{43}$$

then the function $W(\chi, 0, 0)$ obtained for the process in the crossed field will be a good approximation to the function $W(\chi, f, g)$. The condition $f, g \ll 1$ reflects the smallness of the constant field $F_{\mu\nu}$ in comparison to the critical field $m^2/e \sim 10^{13}$ absolute Heaviside units. Since the presently available fields are several orders of magnitude smaller than the critical field, the condition $f, g \ll 1$ can always be assumed to hold. The condition $\chi^2 \gg f, g$ is satisfied for particles with relativistic energies, i.e., for $p_0 \gg m$.

The invariant χ may be considered to be the eigenvalue of a certain operator which is conserved. Actually, we consider the operator $(F_{\mu\nu}\Pi_\nu)^2$, where $\Pi_\nu = -i\partial/\partial x_\nu - eA_\nu$ is the kinetic momentum operator. In a crossed field the operator $(F_{\mu\nu}\Pi_\nu)^2$ is conserved and its eigenvalues are equal to $(F_{\mu\nu}p_\nu)^2$. In an arbitrary constant field the operator $(F_{\mu\nu}\Pi_\nu)^2$ is conserved for scalar particles and is not conserved for spinor particles. In this latter case however its commutator with the operator $\hat{\Pi}$ is proportional to f, g and if these latter quantities are small, the operator $(F_{\mu\nu}\Pi_\nu)^2$ may be regarded as conserved even for spinor particles. To the same degree of accuracy, the eigenvalues of the operator $(F_{\mu\nu}\Pi_\nu)^2$ are equal to $(F_{\mu\nu}p_\nu)^2$. Similarly the components of the vector p_ν, contained in the invariant χ, may be considered to be the corresponding components of the kinetic momentum. Since $(F_{\mu\nu}p_\nu)^2 = (\mathbf{p} \times \mathbf{H} + p_0\mathbf{E})^2 - (\mathbf{pE})^2$, we have in a crossed field $(F_{\mu\nu}p_\nu)^2 = B^2(p_0 - p_3)^2$, and $p_0 - p_3$ is equal to the difference between the kinetic energy of the particle and the component of its kinetic momentum along the axis $\mathbf{E} \times \mathbf{H}$, while in a magnetic field we have $(F_{\mu\nu}p_\nu)^2 = H^2 p_\perp^2$, and $p_\perp$ is equal to the component of the kinetic momentum perpendicular to the magnetic field, etc.

Thus the probability found in Secs. 4 and 5 for the emission of a photon by an electron and the probability for pair production by a photon exactly describe these same processes in the crossed field and approximately describe these processes in an arbitrary constant field; the degree of approximation is given by the conditions (43). In particular if one takes $F_{\mu\nu}$ to be a constant magnetic field, we obtain both the probability for the corresponding processes in a magnetic field and the applicable invariant conditions (43). The emission intensity from an electron and the probability for pair production by an unpolarized photon in a magnetic field found in this way agree with the formulas derived by Klepikov [2].

So far we have been dealing with processes involving a single particle. If the process involves two particles, as for example in the case of the annihilation of an electron-positron pair, the total probability of such a process will depend on the total number of invariants. In this case the probability of a process calculated per unit volume and per unit time may be zero, since the interaction depends in a sensitive manner on their relative motion. However, in this case there may be physical meaning to a probability of a non-invariant sort, for example the probability per unit volume, etc. (cf. Sec. 6). These probabilities may be put in the form of a product of an invariant function and a certain non-invariant multiplying factor. The invariant function must be the same for all constant fields, and when conditions of the type (43) are satisfied, this function may be taken to be the one found for the crossed field. However, the non-invariant multiplying factor may change in going from one field to another. Therefore, for processes involving two particles the connection between the probability for the crossed field and the probability for an arbitrary field is more complicated and will not be discussed here

In conclusion we thank I. E. Tamm, V. L. Ginzburg, and D. A. Kirzhnits for fruitful discussions of this work and for valuable comments.

APPENDIX A

Certain Properties of the Function $A_n(s, \alpha, \beta)$

It follows from (10) that the functions A_n are real. Using the relationship $e^{i\alpha \sin\varphi} = \sum_{l=-\infty}^{\infty} J_l(\alpha) e^{il\varphi}$ and formula (9) it is not difficult to show that

$$A_0(s, \alpha, \beta) = \sum_{l=-\infty}^{\infty} J_{s+2l}(\alpha) J_l(\beta). \tag{A1}$$

It follows from the definition (10) that the functions A_n are connected with the function A_0 by the relations

$$A_1(s, \alpha, \beta) = \tfrac{1}{2}[A_0(s-1, \alpha, \beta) + A_0(s+1, \alpha, \beta)],$$
$$A_2(s, \alpha, \beta) = \tfrac{1}{4}[A_0(s-2, \alpha, \beta) + 2A_0(s, \alpha, \beta) + A_0(s+2, \alpha, \beta)]. \tag{A2}$$

Since the integrand in (10) is periodic with period 2π, we have $\int_{-\pi}^{\pi} \exp f(\varphi)\, df(\varphi) = 0$, from which follows the important relation

$$(s - 2\beta) A_0 - \alpha A_1 + 4\beta A_2 = 0. \tag{A3}$$

From (A1) and (A2) it follows that

$$A_n(s, -\alpha, \beta) = (-1)^{s+n} A_n(s, \alpha, \beta). \tag{A4}$$

Therefore the combinations A_0^2 and $A_1^2 - A_0A_2$ are even functions of α. Putting $\varphi = 0$ in (9) we obtain

$$\sum_{s=-\infty}^{\infty} A_0(s, \alpha, \beta) = 1. \tag{A5}$$

Multiplying (9) by its complex conjugate and integrating with the weighting factor $e^{ik\varphi}$ over φ between 0 and 2π we obtain

$$\sum_{s=-\infty}^{\infty} A_0(s, \alpha, \beta) A_0(s+k, \alpha, \beta) = \delta_{k0}. \tag{A6}$$

APPENDIX B

Asymptotic Expressions for the Functions A_0, A_0^2 and $A_1^2 - A_0A_2$ for $x \to \infty$

We now find an asymptotic expression for the function

$$A_0(s, \alpha, \beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{f(\varphi)} d\varphi = \operatorname{Re}\frac{1}{\pi}\int_0^{\pi} e^{f(\varphi)}\, d\varphi,$$
$$f(\varphi) = -i\alpha\sin\varphi + i\beta\sin 2\varphi + is\varphi \qquad \text{(B1)}$$

As $x \to \infty$, using the method of steepest descent. To do this we transform the integral (B1) into an integral over a contour C which goes through the saddle point $\varphi = \varphi_0$, where $f'(\varphi_0) = 0$, and at the ends of which $\operatorname{Re} f(\varphi) = -\infty$. We expand $f(\varphi)$ in a series around the saddle point:

$$f(\varphi) = f(\varphi_0) + \frac{1}{2!}f''(\varphi_0)(\varphi - \varphi_0)^2 + \frac{1}{3!}f'''(\varphi_0)(\varphi-\varphi_0)^3 + \ldots \qquad \text{(B2)}$$

We will show below that when $x \to \infty$ we have $\operatorname{Re} f(\varphi_0) \to \text{const}$ and $f''(\varphi_0) \to \infty$ like x^2, whereas $f'''(\varphi_0)$ and all higher derivatives at the point φ_0 approach infinity like x^3. It is clear from this that the effective region of integration over $\varphi - \varphi_0$ is $\sim x^{-1}$ and that the second and third derivatives make comparable contributions to the integral; at the same time it is clear that the contribution of the remaining derivatives approaches 0 as $x \to \infty$. Hence we need retain only the first three terms in (B2) and can replace the contour of integration C by a line L, at both ends of which $\operatorname{Re} f'''(\varphi_0)(\varphi - \varphi_0)^3 = -\infty$. This means that the ends of the contour L go to infinity in those sectors of the complex plane of the variable φ in which $\operatorname{Re} f'''(\varphi_0)(\varphi - \varphi_0)^3 < 0$ (they are hatched in Fig. 1). Then, designating by f, f″, and f‴ the values of the function $f(\varphi)$ and its derivatives at the saddle point $\varphi = \varphi_0$, and using the notation

$$f_0(\varphi) = f + \frac{1}{2}f''(\varphi-\varphi_0)^2 + \frac{1}{6}f'''(\varphi-\varphi_0)^3,$$

we obtain

$$\int_C e^{f(\varphi)} d\varphi \approx \int_L e^{f_0(\varphi)} d\varphi$$
$$= \exp\{f + f''^3/3f'''^2\}\int_L d\varphi \exp\{f'''u^3/6 - f''^2u/2f'''\}, \qquad \text{(B3)}$$
$$u = \varphi - \varphi_0 + \frac{f''}{f'''}.$$

The contour L may be deformed into the straight line L_1 which is parallel to the boundary of the sectors and which goes through the point $u = 0$ (cf. Fig. 1). We put $u = re^{i\eta}$ and $f''' = |f'''|e^{i\theta}$. Then if the contour L goes from sector to sector in the positive direction (i.e., counterclockwise as in Fig. 1) and then along the straight line L_1) we have $3\eta + \theta = \pi/2$, $f'''u^3 = i|f'''|r^3$. If however the contour L goes from sector to sector in the negative direction, then along the line L_1 we have $3\eta + \theta = 3\pi/2$, $f'''u^3 = -i|f'''|r^3$. Hence

$$\int_{L_1} d\varphi e^{f'''u^3/6 - f''^2/2f'''} = e^{i\eta}\int_{-\infty}^{\infty} dr \exp\left(\pm i\frac{|f'''|}{6}r^3 - \frac{f''^2}{2f'''}e^{i\eta}r\right)$$
$$= 2\left(\frac{2}{|f'''|}\right)^{1/3} e^{i\eta}\int_0^{\infty} dt\cos\left(\frac{t^3}{3} + yt\right), \qquad \text{(B4)}$$

where $y = \pm i f''^2/2f'''(2/|f'''|)^{1/3}e^{i\eta}$ and the $\pm$ signs correspond to the positive or negative direction of the contour L.

Using the Airy function $\Phi(y)$ [10,11], which is finite for real values of its argument,

$$\Phi(y) = \frac{1}{\sqrt{\pi}}\int_0^{\infty}\cos\left(\frac{t^3}{3} + yt\right)dt, \qquad \text{(B5)}$$

we obtain

$$\int_C e^{f(\varphi)}\, d\varphi \approx \int_L e^{f_0(\varphi)}\, d\varphi = 2\sqrt{\pi}\sqrt[3]{\frac{2}{|f'''|}}\, e^{f + \frac{f''^3}{3f'''^2} + i\eta}\,\Phi(y),$$
$$y = \pm i\frac{f''^2}{2f'''}\sqrt[3]{\frac{2}{|f'''|}}e^{i\eta}. \qquad \text{(B6)}$$

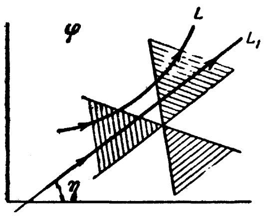

FIG. 1

Here η is the angle of the straight line L_1 obtained by straightening the contour L, and the $\pm$ signs correspond to the positive or negative directions of the contour L.

We now turn to the actual calculation of the function A_0. We use the second representation (B1). As in Appendix A we take $\alpha > 0$. Moreover, we note that $\beta > 0$ always, since it can be put in the form $\beta = -e^2a^2(kf)/8(kq)(kq')$ where f is the total number of neutral particles, and $(kf) < 0$ for time-like vectors f. From the condition $f'(\varphi) = 0$ at the saddle points we obtain

$$\cos\varphi_{1,2} = \frac{\alpha}{8\beta} \pm\left[\left(\frac{\alpha}{8\beta}\right)^2 + \frac{1}{2} - \frac{s}{4\beta}\right]^{1/2} = \frac{\alpha}{8\beta} \pm i\frac{\sqrt{\sigma}}{x}. \qquad \text{(B7)}$$

As was shown in the text, as $x \to \infty$ the quantities $\alpha/8\beta$ and σ approach constant values $(\alpha/8\beta)_0$ and σ_0.

We assume that $0 \le (\alpha/8\beta)_0 \le 1$, $\sigma_0 > 0$. Then for the saddle points we can write

$$\varphi_{1,2} = \psi \mp i\varepsilon, \qquad \cos\varphi_{1,2} = \cos\psi \operatorname{ch}\varepsilon \pm i\sin\psi\operatorname{sh}\varepsilon,$$
$$\sin\varphi_{1,2} = \sin\psi\operatorname{ch}\varepsilon \mp i\cos\psi\operatorname{sh}\varepsilon, \qquad \text{(B8)}$$

where the new variables ψ and ε are connected with α, β, and s by the relations

$$\frac{\alpha}{8\beta}=\cos\psi\,\mathrm{ch}\,\varepsilon,$$

$$\left[\frac{s}{4\beta}-\frac{1}{2}-\left(\frac{\alpha}{8\beta}\right)^2\right]^{1/2}=\sin\psi\,\mathrm{sh}\,\varepsilon,$$

$$\frac{s}{2\beta}=1+2\cos^2\psi+2\,\mathrm{sh}^2\,\varepsilon. \tag{B9}$$

Thus in this case we have two saddle points $\varphi_{1,2}$, lying respectively above and below the half-plane; moreover, as $x\to\infty$ we have $0\le \mathrm{Re}\,\varphi_{1,2}\le\pi/2$, and $\mathrm{Im}\,\varphi_{1,2}\equiv\mp\varepsilon\to 0$ (cf. Fig. 2).

Let the integration in (B1) be transformed to the contour $C'+C+C''$ which goes through the saddle point φ_2 as shown in Fig. 2. Since the vertical portions C' and C'' give pure imaginary contributions, only the integral along the contour C, at whose ends $\mathrm{Re}\,f(\varphi)=-\infty$, remains.

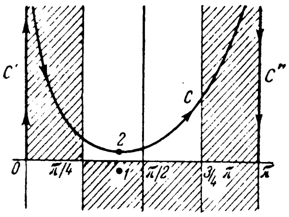

FIG. 2

Formula (B6) can be applied to this integral. With the help of (B8) and (B9) we find the values of f, f'', and f''' at the saddle point φ_2 in terms of ψ, ε, and s and their limiting values for $x\to\infty$ ($\varepsilon\to 0$):

$$\mathrm{Re}\,f=s\left[\frac{\mathrm{sh}\,\varepsilon\,\mathrm{ch}\,\varepsilon\,(1+2\cos^2\psi)}{1+2\cos^2\psi+2\,\mathrm{sh}^2\,\varepsilon}-\varepsilon\right]_{\varepsilon\to 0}\to-\frac{4s\varepsilon^3\sin^2\psi}{3(1+2\cos^2\psi)},$$

$$\mathrm{Im}\,f=-s\left[\frac{\sin\psi\cos\psi\,(1+2\,\mathrm{ch}^2\,\varepsilon)}{1+2\,\mathrm{ch}^2\varepsilon-2\sin^2\psi}-\psi\right]_{\varepsilon\to 0}$$

$$\to -s\left[\frac{3\sin\psi\cos\psi}{3-2\sin^2\psi}-\psi\right],$$

$$f''=-\frac{4s\sin\psi\,\mathrm{sh}\,\varepsilon}{1+2\cos^2\psi+2\,\mathrm{sh}^2\varepsilon}\,[\sin\psi\,\mathrm{ch}\,\varepsilon+i\cos\psi\,\mathrm{sh}\,\varepsilon]_{\varepsilon\to 0}$$

$$\to-\frac{4s\varepsilon\sin^2\psi}{1+2\cos^2\psi},$$

$$f'''=\left[\frac{4s\,[-\sin\psi\cos\psi\,\mathrm{sh}\,\varepsilon\,\mathrm{ch}\,\varepsilon+i\,(\sin^2\psi-\cos^2\psi\,\mathrm{sh}^2\varepsilon)]}{1+2\cos^2\psi+2\,\mathrm{sh}^2\,\varepsilon}\right]$$

$$\times\underset{\varepsilon\to 0}{\to} i\,\frac{4s\sin^2\psi}{1+2\cos^2\psi}. \tag{B10}$$

Since $s\sim x^3$ and $\varepsilon\sim x^{-1}$ as $x\to\infty$, we have $\mathrm{Re}\,f\to$ const, $\mathrm{Im}\,f\sim x^3$, $f''\sim x^2$, and $f'''\sim x^3$ as $x\to\infty$. The remaining derivatives at the saddle point will behave like x^3 as $x\to\infty$.

Using (B6) to write the integral over the contour C, for which the angle $\eta=0$ and for which the direction of traversal is positive, we obtain

$$\int_C e^{f(\varphi)}\,d\varphi=2\sqrt{\pi}\,\sqrt[3]{\frac{1+2\cos^2\psi}{2s\sin^2\psi}}\,e^{i\,\mathrm{Im}\,f}\,\Phi(y),$$

$$y=\varepsilon^2\left(\frac{2s\sin^2\psi}{1+2\cos^2\psi}\right)^{2/3}. \tag{B11}$$

We point out that $\mathrm{Re}\,f+f''^3/3f'''^2=0$ when $x\to\infty$.

Using (B11) and the variables σ and ψ we write

$$y=\left(\frac{\varkappa}{2\chi\chi'\sin\psi}\right)^{2/3}\sigma,\qquad\left(\frac{1+2\cos^2\psi}{2s\sin^2\psi}\right)^{1/3}=\frac{1}{x\sin\psi}\sqrt{\frac{\sigma}{y}},$$

$$\mathrm{Im}\,f=\xi=s\left(\psi-\frac{3\sin\psi\cos\psi}{3-2\sin^2\psi}\right). \tag{B12}$$

where

$$A_0=\frac{2}{\sqrt{\pi}\,x\sin\psi}\sqrt{\frac{\sigma}{y}}\,\Phi(y)\cos\xi,$$

$$A_0^2=\frac{2\sigma}{\pi x^2\,y\sin^2\psi}\,\Phi^2(y)(1+\cos 2\xi). \tag{B13}$$

We now calculate the combinations $A_1^2-A_0A_2$ for $x\to\infty$. Since to a first approximation in x^{-1} all of the A_n are equal, we can not use the asymptotic expressions already obtained, but must begin instead from the exact expression.

$$\pi^2\,(A_1^2-A_0A_2)=\left(\mathrm{Re}\int_C\cos\varphi\,e^{f(\varphi)}d\varphi\right)^2 \tag{B14}$$

$$-\,\mathrm{Re}\int_C e^{f(\varphi)}\,d\varphi\cdot\mathrm{Re}\int_C\cos^2\varphi\,e^{f(\varphi)}d\varphi.$$

Inserting in this expression the expansion of $\cos\varphi$ about the saddle point φ_2: $\cos\varphi=\cos\varphi_2-\sin\varphi_2\cdot(\varphi-\varphi_2)-(1/2)\cos\varphi_2\cdot(\varphi-\varphi_2)^2+\ldots$, and taking account of the fact that $\mathrm{Re}\cos\varphi_2$, $\mathrm{Re}\sin\varphi_2\sim 1$, and $\mathrm{Im}\cos\varphi_2$, $\mathrm{Im}\sin\varphi_2\sim x^{-1}$, $(\varphi-\varphi_2)_{\mathrm{eff.}}\sim x^{-1}$ as $x\to\infty$, we obtain, accurate to terms of order $x^{-3}J_0^2$,

$$\pi^2\,(A_1^2-A_0A_2)$$

$$=(\mathrm{Im}\cos\varphi_2)^2\,|J_0|^2+(\mathrm{Re}\sin\varphi_2)^2\,[(\mathrm{Re}\,J_1)^2$$

$$-\,\mathrm{Re}\,J_0\cdot\mathrm{Re}\,J_2]+2\,\mathrm{Re}\sin\varphi_2\cdot\mathrm{Im}\cos\varphi_2\,[\mathrm{Im}\,J_0\cdot\mathrm{Re}\,J_1$$

$$-\,\mathrm{Re}\,J_0\cdot\mathrm{Im}\,J_1], \tag{B15}$$

where

$$J_0=\int_C e^{f(\varphi)}\,d\varphi,\qquad J_1=\int_C(\varphi-\varphi_2)\,e^{f(\varphi)}\,d\varphi,$$

$$J_2=\int_C(\varphi-\varphi_2)^2\,e^{f(\varphi)}\,d\varphi. \tag{B16}$$

In place of the integrals J_0, J_1 and J_2 in (B15) we can use their asymptotic expressions. The asymptotic expression for J_0 is given by formula (B6). The asymptotic expression for J_2 can be obtained by differentiating the right and left sides of (B6) with respect to f''. We then obtain

$$J_2 \approx \int_L (\varphi - \varphi_2)^2 e^{f_0(\varphi)}\, d\varphi$$
$$= \frac{8\sqrt{\pi}}{f''}\left(\frac{2}{|f'''|}\right)^{1/3} \exp\left\{f + \frac{f''^3}{3f'''^2} + i\eta\right\}$$
$$\times [y^{3/2}\Phi(y) + y\Phi'(y)]. \qquad \text{(B17)}$$

We note further that since $\operatorname{Re} f_0(\varphi) = -\infty$ at the ends of the contour L, we get

$$\int_L e^{f_0(\varphi)}\, df_0(\varphi) = 0, \qquad \text{(B18)}$$

from which follows a relation between the asymptotic values of the integrals J_1 and J_2:

$$2f'' \int_L (\varphi - \varphi_2)\, e^{f_0(\varphi)}\, d\varphi + f''' \int_L (\varphi - \varphi_2)^2 e^{f_0(\varphi)}\, d\varphi = 0. \qquad \text{(B19)}$$

With the help of (B17) and (B19) we obtain

$$J_1 \approx \int_L (\varphi - \varphi_2)\, e^{f_0(\varphi)}\, d\varphi$$
$$= -\frac{4\sqrt{\pi} f'''}{f''^2}\left(\frac{2}{|f'''|}\right)^{1/3} \exp\left\{f + \frac{f''^3}{3f'''^2} + i\eta\right\}$$
$$\times [y^{1/2}\Phi(y) + y\Phi'(y)]. \qquad \text{(B20)}$$

Putting these asymptotic expressions for J_0, J_1, and J_2 in (B15) and using (B8), (B10) and (B12) we obtain

$$A_1^2 - A_0A_2 = \frac{2\sigma^2}{\pi x^4 y \sin^2\psi}\left\{\Phi^2(y) + \frac{1}{y}\Phi'^2(y) + \left[\Phi^2(y) - \frac{1}{y}\Phi'^2(y)\right]\cos 2\xi\right\}. \qquad \text{(B21)}$$

We now assume that $0 \le (\alpha/8\beta)_0 \le 1$, $\sigma_0 < 0$. In this case, as follows from (B7), we may write for the saddle points

$$\varphi_{1,2} = \psi \mp \varepsilon, \quad \cos\varphi_{1,2} = \cos\psi\cos\varepsilon \pm \sin\psi\sin\varepsilon,$$
$$\sin\varphi_{1,2} = \sin\psi\cos\varepsilon \mp \cos\psi\sin\varepsilon, \qquad \text{(B8')}$$

where the variables ψ and ε are related to α, β, and s by

$$\frac{\alpha}{8\beta} = \cos\psi\cos\varepsilon, \quad \left[\left(\frac{\alpha}{8\beta}\right)^2 + \frac{1}{2} - \frac{s}{4\beta}\right]^{1/2} = \sin\psi\sin\varepsilon,$$
$$\frac{s}{2\beta} = 1 + 2\cos^2\psi - 2\sin^2\varepsilon. \qquad \text{(B9')}$$

The saddle points $\varphi_{1,2}$ are located on the real axis, and for $x \to \infty$ we have $0 \le \psi \le \pi/2$ and $\varepsilon \to 0$ (cf. Fig. 3). The function A_0 is the real part of the sum of two integrals along the contours C_1 and C_2 going through the saddle points $\varphi_{1,2}$ as shown in Fig. 3. Using (B6) to calculate the asymptotic form of these integrals, using the angle $\eta_{1,2} = \pm\pi/3$ and the negative direction of traversal of the contours, and using the relation

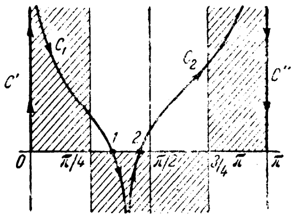

FIG. 3

$$e^{-i\pi/3}\Phi(-ye^{-i\pi/3}) + e^{i\pi/3}\Phi(-ye^{i\pi/3}) = \Phi(y)$$

we obtain for A_0 and for A_0^2 the same expressions (B13) as in the case $\sigma > 0$, if by y and ξ we mean the functions given by (B12). One may show analogously that the asymptotic expression for $A_1^2 - A_0A_2$ in the case under consideration is given by the same formula (B21) as in the case $\sigma > 0$.

Thus in the case $0 \le (\alpha/8\beta)_0 \le 1$, the functions A_0^2 and $A_1^2 - A_0A_2$ are given asymptotically by (B13) and (B21), regardless of the sign of σ. It is not difficult to see that in the case $(\alpha/8\beta)_0 > 1$ the functions A_0^2 and $A_1^2 - A_0A_2$ decrease, like as $\exp(-Cx^3)$ as $x \to \infty$, and that we can therefore limit ourselves to the region $0 \le (\alpha/8\beta)_0 \le 1$. In conclusion we note that in integrating expressions involving A_0^2 and $A_1^2 - A_0A_2$, the oscillating terms may be omitted since they give a vanishing contribution as $x \to \infty$.

[1] H. R. Reiss, J. Math. Phys. **3**, 59 (1962).
[2] N. P. Klepikov, JETP **26**, 19 (1954).
[3] Sokolov, Klepikov, and Ternov, JETP **24**, 249 (1953).
[4] J. Schwinger, Proc. Nat. Acad. Sci. **40**, 132 (1954).
[5] D. M. Volkov, Z. Physik **94**, 250 (1935); JETP **7**, 1286 (1937).
[6] L. D. Landau and E. M. Lifshitz, Teoriya Polya (Field Theory), Fizmatgiz, 1962.
[7] E. T. Whittaker and G. N. Watson, A Course in Modern Analysis, ch. 1 Cambridge, 1940.
[8] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov, i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1962.
[9] J. M. Rauch and F. Rohrlich, The Theory of Photons and Electrons, Addison-Wesley, 1955.
[10] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Gostekhizdat, 1948.
[11] V. A. Fock, Tablitsy funktsiĭ Eiri (Tables of Airy Functions), Inform. Otdel NII-108, 1946.
[12] Vachaspati, Phys. Rev. **128**, 664 (1962).

Translated by J. A. Armstrong
108