ON THE THEORY OF BREMSSTRAHLUNG AND PAIR PRODUCTION IN A MEDIUM

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Interactions between photons and a medium in connection with bremsstrahlung from extremely relativistic electrons are investigated. It is shown that for sufficiently high-energy electrons Compton scattering plays an important part in the soft portion of the spectrum. In this case the main contribution to the radiation comes from electrons of the medium which emit photons as a result of collisions with the fast particles.

1. INTRODUCTION

THE process of bremsstrahlung from high-energy electrons in a medium differs greatly from emission induced by isolated atoms. Landau and Pomeranchuk found [1] that multiple electron scattering in a medium reduces the probabilities of bremsstrahlung and pair production. For dense large-Z media the integral probability of these processes begins to be affected in this way at electron energies exceeding $10^{12}-10^{13}$ eV.

Ter-Mikaélyan showed [2] that the changed photon dispersion law resulting from polarization of the medium alters considerably the soft part of the bremsstrahlung spectrum. Migdal [3] has developed a quantum theory of bremsstrahlung and pair production in a medium at high energies, taking both of the aforementioned effects into account.

Polarization of the medium is not, however, the only factor that changes photon energies. The polarization effect considered in [2,3] results from photon interactions with electrons of the medium and is described by first-order terms in the coupling constant e^2 ($\hbar = c = 1$). Terms of the next order, which are proportional to e^4, add an imaginary quantity to the photon energy, as a result of Compton scattering by electrons of the medium.

It is shown in the present work that when Compton scattering is taken into account an additional term appears in the probability of bremsstrahlung. This term dominates all other terms at photon energies $k \gtrsim 10^7$ eV and electron energies $\mathscr{E}_0 \lesssim 10^5$ k. It is shown in Sec. 5 that this term describes photon emission by electrons of the medium. Thus the bremsstrahlung spectrum in the indicated region is given by Eqs. (49)—(51), which have been derived here for the first time, rather than by Ter-Mikaélyan's equation (42), which holds true when Compton scattering is neglected.

Interactions between photons and nuclei begin to play an appreciable part in the harder portion of the spectrum. Direct pair production by extremely relativistic electrons then accompanies the bremsstrahlung, and under certain conditions the two processes cannot be discriminated. Ternovskiĭ [4] has considered the way in which bremsstrahlung is affected by photon absorption resulting in pair production. In the present work the contribution of photon scattering on nuclei is also evaluated.

Multiple scattering becomes a prominent factor only when $k > 10^{-4} L_r^{-1}(\mathscr{E}_0/m)^2$, where L_r is the radiation length.

In the present work the problem of radiation processes induced by high-energy electrons in a medium is solved using the diagram technique of Konstantinov and Perel'. [5] The solution of such problems by means of the diagram technique possesses a number of advantages over Migdal's method, [3] which does not employ diagrams. The diagram method enables more rigorous averaging over atomic positions. Consequently, the general expression derived here for bremsstrahlung probability describes the transition region from multiple to single scattering more accurately then Migdal's expression. Furthermore, a relation is found between the probabilities of individual elementary processes and the kinetic equations describing the development of a shower in a medium; it thus becomes possible to derive these equations rigorously and consistently. Finally, the diagram technique facilitates the investigation of the role of photon energy renormalization in radiation processes. The foregoing shows that application of the diagram method to the calculation of the probabilities of radiation processes in a medium is of decided methodological interest and permits the construction of a more rigorous theory of these phenomena.

2. DIAGRAM TECHNIQUE FOR CALCULATING THE PROBABILITIES OF RADIATION PROCESSES IN A MEDIUM

An electron having at $t = 0$ the momentum p_0 and polarization σ_0 is present in a medium that consists of randomly located screened Coulomb centers. The density matrix of the system for $t > 0$ is given by

$$F(t) = e^{-iHt} a^+_{p_0\sigma_0} F_0 a_{p_0\sigma_0} e^{iHt}, \qquad (1)$$

where F_0 is the density matrix of the state in which electrons, positrons, and photons are absent, a, a^+ are the electron creation and annihilation operators, $H = H_0 + H_1$ is the complete Hamiltonian, H_0 is the Hamilton of free electron, positron, and photon fields,

$$H_1 = \int U(\mathbf{x}) \psi^+(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} - \int \mathbf{j}(\mathbf{x}) \mathbf{A}(\mathbf{x}) d\mathbf{x} \qquad (2)$$

is the interaction operator, $\mathbf{j}(\mathbf{x}) = e\psi^+(\mathbf{x})\alpha\psi(\mathbf{x})$ is the Dirac current, $\mathbf{A}(\mathbf{x})$ is the quantized electromagnetic potential, ψ^+ and ψ are quantized operators of the electron-positron field, and

$$U(\mathbf{x}) = \sum_\nu V(\mathbf{x} - \mathbf{x}_\nu) - \left\langle \sum_\nu V(\mathbf{x} - \mathbf{x}_\nu) \right\rangle \qquad (3)$$

is the potential of the interaction between an electron and the medium. We sum over all atoms of the medium, and the carets denote averaging over their positions $(\mathbf{x}_\nu)$. The Hamiltonian H_1 includes for simplicity the electron interaction with only transverse photons. From (1) we determine the one-electron density matrix in the momentum representation:

$$F^{\sigma\sigma'}_{\mathbf{pp}'}(t) = \langle (0|\, a_{\mathbf{p}_0\sigma_0} S^+(t)\, (a^+_{\mathbf{p}\sigma} a_{\mathbf{p}'\sigma'})_t\, S(t)\, a^+_{\mathbf{p}_0\sigma_0}\,|0)\rangle,$$

$$S(t) = T \exp\left\{ -i \int_0^t e^{iH_0t'} H_1 e^{-iH_0t'} dt' \right\}, \qquad (4)$$

where $a^+_{p\sigma} a_{p'\sigma'_t})$ is the product of operators in the interaction representation, and $|0)$ is the vacuum state vector. One-positron and one-photon density matrices are obtained from (4) by replacing the electron operators in $(a^+_{p\sigma} a_{p'\sigma'_t})$ with positron and photon operators, respectively.

Performing a series expansion of S and S^+ with respect to the interaction as well as the Laplace integral transform

$$F^{\sigma\sigma'}_{\mathbf{pp}'}(s) = \int_0^\infty F^{\sigma\sigma'}_{\mathbf{pp}'}(t) e^{-St} dt, \qquad (5)$$

we represent the density matrix as an infinite set of modified diagrams.[5] Figure 1 shows certain irreducible diagrams. Wavy lines represent photons, dashed lines represent interactions with atoms, and solid lines represent electrons.

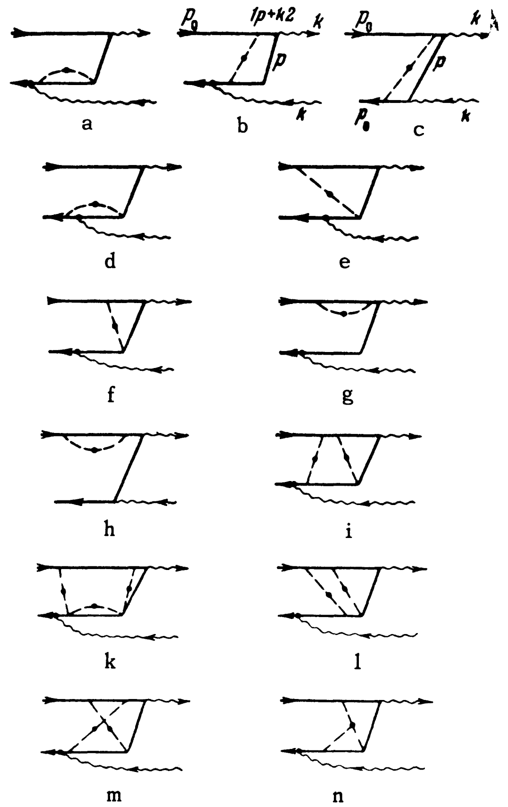

FIG. 1

The rules for associating analytic expressions by means of diagrams are easily established by considering the series expansion of $S(t)$ (see also [6]). For example, the vertex 1 in Fig. 1b corresponds to the factor $-iV_q(u_{\mathbf{p+k}\sigma}, u_{p_0\sigma_0})$; vertex 2 corresponds to $ie(2\pi/k)^{1/2} x(u_{p\sigma}, \alpha e_\nu u_{\mathbf{p+k}\sigma'})$, where u is a bispinor describing the electron state and $\mathbf{e}_\nu$ is the photon normalization unit vector. Vertical intersections correspond to factors of the form $(s + i\omega_{AB})^{-1}$, where $\omega_{AB} = \mathscr{E}_A - \mathscr{E}_B$ is the energy difference between left-to-right lines and right-to-left lines.

Separating the irreducible diagrams and summing formally the reducible diagrams, we obtain a system of equations describing the development of a shower in the medium. Definite sets of irreducible diagrams will denote the probabilities of elementary processes: bremsstrahlung, pair production, and elastic scattering.

3. PROBABILITY OF BREMSSTRAHLUNG IN A MEDIUM

The diagrams in Fig. 1, a—h, correspond in the Born approximation to the differential probability of bremsstrahlung summed over final electron states. To these we must add the complex conjugate diagrams obtained by reflecting the diagrams of Fig. 1 in the horizontal axis. The sum of all these diagrams together with the correct expression for the probability of bremsstrahlung contains superfluous terms not describing radiation. These terms are proportional to δ functions of the form $\delta(\mathscr{E}_{\mathbf{p}} - \mathscr{E}_{\mathbf{p+q}})$ having arguments that do not include photon energies. An example is provided by the quantity

$$-2\pi n \sum_{\mathbf{p}} \frac{|H'_{\mathbf{p},\,\mathbf{p+k}}|^2 \, |V_{\mathbf{q}}|^2}{(\mathscr{E}_{\mathbf{p+k}} - \mathscr{E}_{\mathbf{p}} - k)^2} \, \delta(\mathscr{E}_{\mathbf{p}_0} - \mathscr{E}_{\mathbf{p}_0-\mathbf{q}}), \tag{6}$$

taken from diagrams of the type of Fig. 1c.

The appearance of superfluous terms such as (6) is not associated with the specific form of perturbation theory employed in the present work, but can also occur in other cases. For example, similar terms appear when the probability of bremsstrahlung is calculated in the Born approximation using the formal scattering theory of Gell-Mann and Goldberger[7] or conventional perturbation theory. In order to obtain from the diagrams of Fig. 1 a correct expression for the radiation probability it is necessary to distinguish and omit all those pole terms where the position of the pole is independent of the photon energy k. For example, in the case of Fig. 1c the energy denominators can be transformed for $s \to 0$ as follows:

$$\begin{aligned}&[s + i(\mathscr{E}_{\mathbf{p+k}} - \mathscr{E}_{\mathbf{p}} - k)]^{-1} [s + i(\mathscr{E}_{\mathbf{p}_0} - \mathscr{E}_{\mathbf{p}} - k)]^{-1} \\ &\times [s + i(\mathscr{E}_{\mathbf{p}_0} - \mathscr{E}_{\mathbf{p}_0-\mathbf{q}})]^{-1} = (\mathscr{E}_{\mathbf{p+k}} - \mathscr{E}_{\mathbf{p}} - k)^{-2} \\ &\times \{[s + i(\mathscr{E}_{\mathbf{p}_0} - \mathscr{E}_{\mathbf{p}} - k)]^{-1} - [s + i(\mathscr{E}_{\mathbf{p}_0} - \mathscr{E}_{\mathbf{p}_0-\mathbf{q}})]^{-1}\},\end{aligned} \tag{7}$$

as a result of which the pole factors on the right-hand side of (7) were separated, since $\mathscr{E}_{\mathbf{p+k}} - \mathscr{E}_{\mathbf{p}} - k \neq 0$. The contribution of the second term in the curly brackets of (7) may be dropped. A similar procedure should be followed for all other diagrams containing intersections not crossing photon lines.

With increasing electron energy a growing contribution comes from the diagrams of Fig. 1, i and k, containing intersections crossing only two electron lines and a photon line. Taking the atomic potential to be

$$V_{\mathbf{q}} = 4\pi Z e^2/(q^2 + \varkappa^2), \qquad \varkappa = 1/a_0, \tag{8}$$

we obtain for the ratio l of contributions from the diagrams of Fig. 1, i and b, the following estimate in the extreme relativistic case:

$$l \approx (Z/137)^2 n a_0^2 \Lambda p_0 (p_0 - k)/mk, \tag{9}$$

where $a_0 = 137/mZ^{1/3}$ is the atomic radius and Λ is the Compton wavelength.

For $l \lesssim 1$ corresponding to quite large values of p_0 or small k, it is necessary to sum an infinite set of diagrams of the type of Fig. 1, i and k, that is, to take into account the multiple scattering of an electron in connection with the emission of a photon. Diagrams l and m with superposed or intersecting interaction lines contain, compared with the main diagrams, the small factor $(Z/137)^2 n a_0^3 \lesssim 10^{-4}$, and can be neglected. The addition of each new interaction line in a bundle (Fig. 1n) gives an additional Born factor z/137.

The diagrams taking into account the effect of multiple scattering on bremsstrahlung are shown in Fig. 2. The shaded blocks represent the electron density matrix $f_{\mathbf{k}}(\mathbf{p}_0; \mathbf{p}, \omega)$ (where $\omega = k + is$) averaged over a large number of collisions (Fig. 3). The corresponding equation follows directly from Fig. 3, and was first derived by Migdal[8] (see also [6]).

Among the diagrams c—h of Fig. 2 we must distinguish the terms not describing bremsstrahlung, as was indicated above. The probability of photon emission then becomes

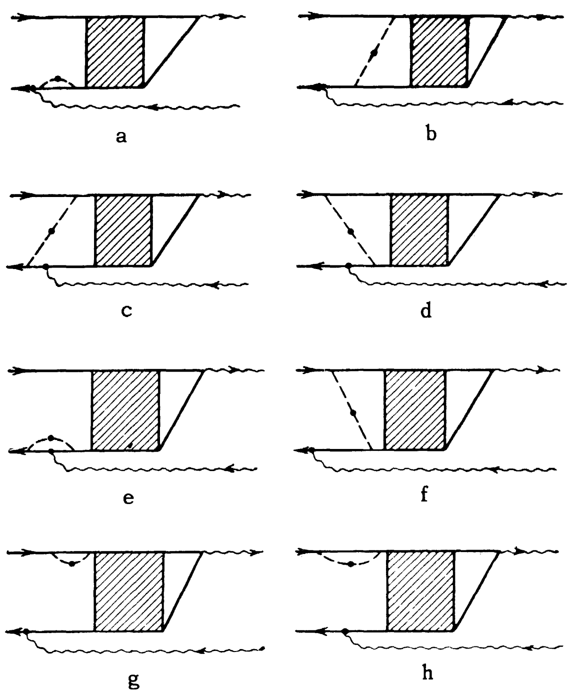

FIG. 2

$$W(p_0, k)\,dk$$

$$= \frac{ne^2\,dk}{(2\pi)^2 k} \operatorname{Re} \int_0^\infty d\tau e^{ik\tau} \int d\Omega_{\mathbf{k}} \sum_{\mathbf{p}_0', \mathbf{p}} \frac{|V_{\mathbf{p}_0-\mathbf{p}_0'}|^2}{s + i(\mathscr{E}_{\mathbf{p}_0} - \mathscr{E}_{\mathbf{p}_0'-\mathbf{k}} - k)}$$

$$\times [f_{\mathbf{k}}(\mathbf{p}_0'; \mathbf{p}, \tau) - f_{\mathbf{k}}(\mathbf{p}_0; \mathbf{p}, \tau)]$$

$$\times \left\{ \frac{\mathscr{L}(\mathbf{p}_0, \mathbf{p})}{i(\mathscr{E}_{\mathbf{p}_0} - \mathscr{E}_{\mathbf{p}_0-\mathbf{k}} - k)} - \frac{\mathscr{L}(p_0', p)}{i(\mathscr{E}_{\mathbf{p}_0'} - \mathscr{E}_{\mathbf{p}_0'-\mathbf{k}} - k)} \right\}, \quad (10)$$

$$\mathscr{L}(\mathbf{p}_0, \mathbf{p}) = \sum_{\mu\nu\sigma_0} (u_{\mathbf{p}_0\sigma_0}, \boldsymbol{\alpha}\,\mathbf{e}_\nu u_{\mathbf{p}_0-\mathbf{k}\mu})(u_{\mathbf{p}-\mathbf{k}/2\mu}, \boldsymbol{\alpha}\,\mathbf{e}_\nu u_{\mathbf{p}+\mathbf{k}/2\sigma_0}). \quad (11)$$

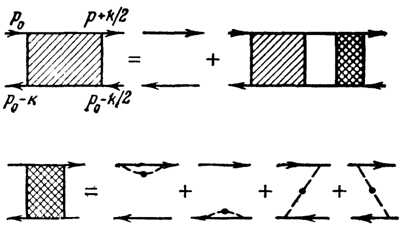

FIG. 3

Here $f_{\mathbf{k}}(p_0; p, \tau)$ is the density matrix in the time representation; we sum over final electron states and photon polarizations, average over initial electron polarizations, and integrate over photon emission angles.

Equation (10) differs in general from Migdal's initial expression, [3] which, after separation of the factor describing multiple electron scattering before photon emission, becomes

$$W(p_0, k)\,dk = \frac{e^2 k\,dk}{(2\pi)^2} \operatorname{Re} \int_0^\infty d\tau e^{ik\tau} \int d\Omega_{\mathbf{k}} \sum_{\mathbf{p}} f_{\mathbf{k}}(\mathbf{p}_0; \mathbf{p}, \tau)\, \mathscr{L}(\mathbf{p}_0, \mathbf{p}). \quad (12)$$

The difference becomes manifest in the case of single scattering. In this limit (12) corresponds to the consideration of diagrams a, b, f, and g of Fig. 1 and their complex conjugates, the sum of which does not yield the correct expression for radiation probability when k is comparable in magnitude to p_0.

In the region of considerable multiple scattering Eqs. (10) and (12) coincide. To show this, we assume in Migdal's approximation

$$[s + i(\mathscr{E}_{\mathbf{p}_0} - \mathscr{E}_{\mathbf{p}_0'-\mathbf{k}} - \mathbf{k})]^{-1} \approx \pi\delta(p_0 - p_0') \quad (13)$$

and take the angular part of $f_{\mathbf{k}}$:

$$f_{\mathbf{k}}(\mathbf{p}_0; \mathbf{p}, \tau)\, dp/(2\pi)^3 = \delta(p - p_0 + k/2)\, v(\boldsymbol{\eta}_0; \boldsymbol{\eta}, \tau)\, dp\, d\boldsymbol{\eta} \quad (14)$$

where the notation of [3] is used. The angular func-function $v(\eta_0; \eta, \tau)$ satisfies

$$\frac{dv}{d\tau} + i\left(a - \frac{1}{2} b\eta^2\right) v = \frac{n(p_0 - k/2)^2}{(2\pi)^2} \int |V(\boldsymbol{\eta}' - \boldsymbol{\eta})|^2 [v(\boldsymbol{\eta}_0; \boldsymbol{\eta}', \tau)$$

$$- v(\boldsymbol{\eta}_0; \boldsymbol{\eta}, \tau)]\, d\boldsymbol{\eta}', \quad (15)$$

where

$$a = k[1 - m^2/2p_0(p_0 - k)], \qquad b = k(p_0 - k/2)^2/p_0(p_0 - k).$$

From (10) and (13)–(15) we obtain the bremsstrahlung spectrum

$$W(p_0, k)$$

$$= \frac{e^2 k(p_0 - k/2)^2}{(2\pi)^2 p_0^2} \operatorname{Re} \int_0^\infty d\tau e^{ik\tau} \int \mathscr{L}(\boldsymbol{\eta}_0, \boldsymbol{\eta})\, v(\boldsymbol{\eta}_0; \boldsymbol{\eta}, \tau)\, d\boldsymbol{\eta}\, d\boldsymbol{\eta}_0. \quad (16)$$

The same result follows from (12).

4. RENORMALIZATION OF PHOTON ENERGY

Equation (16) is inexact because the interaction between photons and the medium is neglected. In order to take this interaction into account the photon lines must include different polarization parts, i.e., the thin wavy lines must be replaced with thick lines. This corresponds to replacing ik with $i\omega - \gamma/2$ in the exponential included within the integrand of (16); here

$$\mathscr{E}_\gamma(k) = \omega - i\gamma/2 \quad (17)$$

denotes the energy of a photon in the medium.

We shall now calculate the added photon energy $\Delta\mathscr{E}_\gamma^{(1)}$ due to the interaction with nuclei. The zeroth-order retarded Green's function of a photon,

$$d^0_{\mu\nu}(\mathbf{k}, t - t') \quad (18)$$

$$= \frac{2\pi}{k} \theta(t - t') \sum_{\lambda\lambda'} e_\mu^{(\lambda)} e_\nu^{(\lambda')} (0|(c_{\mathbf{k}\lambda})_t (c^+_{\mathbf{k}\lambda'})_{t'}|0),$$

corresponding to the wavy lines in the diagrams, is

$$d^0_{\mu\nu}(\mathbf{k}, t) = d^0(k, t)\,\delta_{\mu\nu}, \qquad d^0(k, t) = 2\pi k^{-1}\theta(t)\, e^{ikt},$$

$$\mu, \nu = 1, 2. \quad (19)$$

The Green's function $d_{\mu\nu}(\mathbf{k}, t) = d(k, t)\delta_{\mu\nu}$ for photon-nucleus interactions satisfies the equation

$$d_{\mu\nu}(\mathbf{k}, t) = d^0_{\mu\nu}(\mathbf{k}, t)$$

$$+ \int_0^t d\tau' \int_0^{\tau'} d\tau\, d^0_{\mu\mu'}(\mathbf{k}, \tau)\, \Pi_{\mu'\nu'}(\mathbf{k}, \tau' - \tau)\, d_{\nu'\nu}(\mathbf{k}, t - \tau'), \quad (20)$$

where $\Pi_{\mu\nu}(\mathbf{k}, \tau) = \Pi(k, \tau)\delta_{\mu\nu}$ is the polarization operator which in the first nonvanishing approximation corresponds to the four loops (without external lines) in Fig. 4.

Using the integral transform (5) with $s = -i\omega + \nu$, $\nu \to +0$ along with (19), we obtain from (20) the result

$$d(k, s) = 2\pi/[k(s + ik) - 2\pi\Pi(k, s)]. \quad (21)$$

The photon energy $\mathscr{E}_\gamma$ is is_γ, where s_γ is the

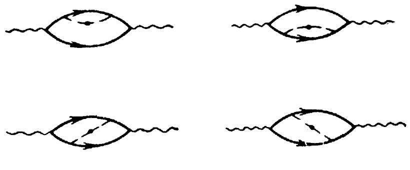

FIG. 4

value of s for which the Green's function (21) has a pole. Assuming $s = -ik + \nu$ in the correction term, we have

$$\mathscr{E}_\gamma = k + \Delta\mathscr{E}_\gamma^{(1)} = k + 2\pi i k^{-1}\,\Pi\,(k, -ik+\nu). \quad (22)$$

It is easily shown that $\Pi(k, -ik+\nu)$ is proportional to the amplitude $f(k, 0) = a_1(k, 0) + ia_2(k, 0)$ of photon scattering at zero angle in the atomic field:

$$\Pi\,(k, -ik+\nu) = in\,f\,(k, 0). \quad (23)$$

The photon scattering amplitude $f(k, 0)$ has been calculated several times (in [9], for example). For $k \gg 2m$ (but with $k \ll 137\,Z^{-1/3}m$, so that there is no screening effect) we have

$$a_1\,(k, 0) = \frac{7}{18}\bar{\varphi}k, \qquad a_2\,(k, 0) = \frac{7}{9\pi}\bar{\varphi}k\ln\frac{2k}{m}, \quad (24)$$

where $\bar{\varphi} = Z^2r_0^2/137$ and r_0 is the classical electron radius.

For $k \gg 137\,Z^{-1/3}m$ (complete screening) we have

$$a_1\,(k, 0) = \frac{7}{18}\bar{\varphi}a_0m^2, \quad a_2\,(k, 0) = \frac{7}{9\pi}\bar{\varphi}k\ln\,(190Z^{-1/3}). \quad (25)$$

In this region $a_1 \ll a_2$. Finally, for still higher energies,

$$\frac{k}{m} > \frac{e^2}{16\pi}\frac{L_r}{\Lambda} \approx \frac{1}{7000}\frac{L_r}{\Lambda},$$

where $L_r = 137/4nZ^2r_0^2\ln(190\,Z^{-1/3})$ is the radiation length, and a_2 begins to diminish compared with (25) because of multiple scattering (the Landau–Pomeranchuk effect). Migdal's results [3] enable us to obtain a_2 easily for $k \gg mL_r/16\pi \times 137\,\Lambda \equiv k_s$:

$$a_2\,(k, 0) = \frac{3}{8}\,\xi\,(s_0)\,\varphi\ln\,(190Z^{-1/3})\,(k_sk)^{1/2}. \quad (26)$$

Here

$$\xi\,(s_0) = \min\,\{1 + \ln s_0/\ln s', 2\},$$

$$s_0 = (k_s/k)^{1/2}, \quad s' = Z^{1/3}/190; \; a_1 \ll a_2.$$

The interaction of photons with electrons of the medium induces, to the first order in e^2, an energy change by the real amount

$$\Delta\omega^{(2)} = \omega_p^2/2k, \qquad \omega_p^2 = 4\pi nZe^2/m. \quad (27)$$

A complex term is added in the next order; its real part always appears in succeeding equations as an addition to $\Delta\omega^{(2)}$ and can be omitted because of its smallness. The imaginary part is proportional to the Compton scattering cross section; for not too large k it can be comparable in magnitude to the imaginary part of $\Delta\mathscr{E}_\gamma^{(1)}$ and should therefore be taken into account. For example, for lead the coefficients of absorption due to pair production and the Compton effect are equal at $k \approx 10\,m$, and for aluminum at $k \approx 60\,m$. Thus the total change of photon energy due to interaction with electrons of the medium is

$$\Delta\mathscr{E}_\gamma^{(2)} = \omega_p^2/2k - i\gamma_C/2, \quad (28)$$

where γ_C is the total Compton scattering probability in unit time; for $k \gg m$ we have

$$\gamma_C = \pi nZr_0^2\frac{m}{k}\left(\ln\frac{2k}{m} + \frac{1}{2}\right). \quad (29)$$

From (22) and (28) we have finally for the photon energy $\mathscr{E}_\gamma = \omega - i\gamma/2$ in a medium:

$$\omega = k + \frac{\omega_p^2}{2k} - \frac{2\pi na_1\,(k, 0)}{k},$$

$$\gamma = \gamma_C + \frac{4\pi n}{k}a_2\,(k, 0) = \gamma_C + \gamma_p. \quad (30)$$

The relative contribution of the term proportional to a_1 is maximal for $k \approx 137\,Z^{-1/3}m$, but in this region it is only about one-tenth of the term $\omega_p^2/2k$.

Equation (30) easily yields an expression for the dielectric constant ϵ of the medium with the Compton effect and photon-nucleus interaction taken into account. Using the relation $\mathscr{E}_\gamma = k/\sqrt{\epsilon}$ between the photon energy in the medium and the dielectric constant, we have

$$\varepsilon = 1 - \frac{\omega_p^2}{k^2} + \frac{4\pi a_1\,(k, 0)}{k^2} + \frac{i\gamma}{k}. \quad (31)$$

5. BREMSSTRAHLUNG SPECTRUM

The bremsstrahlung spectrum with account of the interaction between photons and the medium will be calculated from (16) after substituting $i\omega - \gamma/2$ for ik in the exponential. It is convenient to divide the total bremsstrahlung probability into two parts:

$$W\,(p_0, k) = W^0\,(p_0, k) + W'\,(p_0, k). \quad (32)$$

W^0 is represented by the two diagrams of Fig. 5, which describe the emission of "dressed" photons (i.e., interacting with the medium) from a free electron not scattered by atoms. It will be seen subsequently that W^0 is proportional to an imaginary addition to the photon energy, i.e., the ejection of photons from a particle, described by the

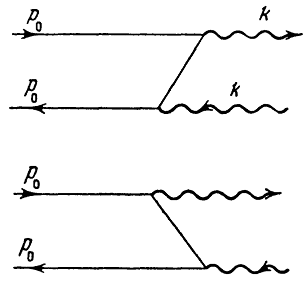

FIG. 5

term W^0, is induced by their absorption. In the optical portion of the spectrum, where absorption is small and $\mathscr{E}_\gamma(k) < k$ due to polarization of the medium, the diagrams of Fig. 5 and the corresponding term in the probability represent Vavilov-Cerenkov emission.

W' describes the emission of photons by a particle scattered on atoms. $W'(p_0, k)$ must be calculated from (16) with $v(\eta_0; \eta, \tau)$ replaced by $v' = v - v^0$, where v^0 is the part of the density matrix that is independent of the scattering potential, describes the motion of a free particle, and is included in W^0.

We shall use the expression for v obtained by Gol'dman[10] through a solution of (15) in the Fokker-Planck approximation:

$$v(\eta_0; \eta, \tau)\, e^{i\omega\tau - \gamma\tau/2} = x\pi^{-1}\mathrm{sh}^{-1} z\tau \exp\{-(\eta^2 + \eta_0^2)x\,\mathrm{cth}\, z\tau + 2\eta\eta_0 x\,\mathrm{sh}^{-1} z\tau + i\zeta\tau - \gamma\tau/2\}; \qquad (33)^*$$

$$x = (1 - i)(k/16q)^{1/2},$$
$$z = (1 - i)(kq)^{1/2}, \quad q = 4\pi n Z^2 e^4 p_0^{-2} \ln(190 Z^{-1/3});$$
$$\zeta = \omega - k + m^2 k/2p_0^2, \qquad (34)$$

where ω and γ are defined by (30). In (33) the condition $k \ll p_0$ is used, since the renormalization of photon energy plays an appreciable part only in the soft portion of the spectrum.

After integrating in (16) we obtain

$$W'(p_0, k) = \frac{e^2}{3\pi k}\left\{\frac{\zeta s_1 s_2}{(s_1^2 + s_2^2)^2}\Phi_1(s_1, s_2) + \frac{\gamma(s_1^2 - s_2^2)}{4(s_1^2 + s_2^2)^2}\Phi_2(s_1, s_2)\right\}; \qquad (35)$$

$$\Phi_1(s_1, s_2) = \frac{6(s_1^2 + s_2^2)^2}{s_1 s_2}\int_0^\infty e^{-2s_1 t}\sin 2s_2 t\left(\mathrm{cth}\, t - \frac{1}{t}\right)dt,$$

$$\Phi_2(s_1, s_2) = \frac{12(s_1^2 + s_2^2)^2}{s_1^2 - s_2^2}\int_0^\infty e^{-2s_1 t}\cos 2s_2 t\left(\mathrm{cth}\, t - \frac{1}{t}\right)dt,$$

$$s_1 = (\zeta + \gamma/4)/4(kq)^{1/2}, \qquad s_2 = (\zeta - \gamma/2)/4(kq)^{1/2}. \qquad (36)$$

*sh = sinh, cth = coth.

The use of the Fokker-Planck approximation has made the value of W' somewhat too high. To obtain the correct result we introduce, following Migdal,[3] the interpolation factor ξ: $\xi = 1$ for $|s_i| > 1$, $\xi = 1 + \ln|s_i|/\ln s'$ for $1 \geq |s_i| \geq s'$, and $\xi = 2$ for $|s_i| < s'$, where s_i is the largest of the quantities s_1, s_2, and $s' = Z^{1/3}/190$. The correct value of W' is obtained when in (35) we substitute

$$B = qp_0^2 = 2\pi n Z^2 e^4 \xi \ln(190 Z^{-1/3}). \qquad (37)$$

For $s_1 \ll 1$, $|s_2| \ll 1$ we have

$$\Phi_1(s_1, s_2) = 3(s_1^2 + s_2^2)/s_1,$$
$$\Phi_2(s_1, s_2) = 6(s_1^2 + s_2^2)\, s_1/(s_1^2 - s_2^2). \qquad (38)$$

In this limit renormalization of the photon energy has no effect and (35) goes over into the Migdal equation corresponding to $s = (m^4 k/64B)^{1/2} \ll 1$, $k \ll p_0$:

$$W'(p_0, k) = (2e^2/\pi p_0)(B/k)^{1/2}. \qquad (39)$$

For $s_1 \gg 1$, $|s_2| \gg 1$, Eq. (36) yields $\Phi_1 = \Phi_2 = 1$, and (35) gives

$$W'(p_0, k) = 4e^2 B\zeta/3\pi p_0^2(\zeta^2 + \gamma^2/4). \qquad (40)$$

The effect of absorption, i.e., the imaginary addition $i\gamma/2$ to the photon energy, will affect W' appreciably in the spectral region

$$\omega_p^2/\gamma \lesssim k \lesssim 10^{-4}\gamma(p_0/m)^2. \qquad (41)$$

It is easily verified that in this region $\gamma_C \ll \gamma_p$; therefore Compton scattering does not affect W'. The region (41) will exist only for electron energies $\mathscr{E}_0 > 100\, m\omega_p/\gamma$. However, we shall see [Eq. (46)] that at these energies $W' \ll W^0$, i.e., the term W^0 is predominant in the total probability.

At photon energies $k \ll \omega_p^2/\gamma$ and high electron energies $\mathscr{E}_0 \gg mk/\omega_p$, W' is determined mainly by the polarization of the medium. In this region (40) gives the result first obtained by Ter-Mikaélyan:[2]

$$W'(p_0, k) = \frac{4Ze^4 mk}{3\pi p_0^2}\ln(190 Z^{-1/3}). \qquad (42)$$

It is convenient to calculate W^0, i.e., the contribution of the diagrams of Fig. 5, classically in order to facilitate the physical interpretation of the results. This is similar to the procedure of Fermi[11] and Landau[12] in the theory of the ionization loss of fast particles.

The electromagnetic field of a particle moving in a medium at a constant velocity v is obtained from Maxwell's equations. In cylindrical coordinates with the axis along the particle trajectory the nonvanishing monochromatic field components are

$$E_{\omega z}(\mathbf{x}, t) = \frac{ie\omega}{\pi c^2}\left(1 - \frac{1}{\beta^2\varepsilon}\right) K_0(sr)\, e^{i\omega\,(z/v-t)},$$

$$E_{\omega r}(\mathbf{x}, t) = \frac{es}{v\pi\varepsilon} K_1(sr)\, e^{i\omega\,(z/v-t)},$$

$$H_{\omega\varphi}(\mathbf{x},\ t) = \frac{es}{\pi c} K_1(sr) e^{i\omega(z/v-t)}, \qquad (43)$$

where

$$s^2 = \omega^2/v^2 - \omega^2 c^{-2}\,\varepsilon(\omega), \qquad \mathrm{Re}\, s > 0, \qquad (44)$$

K_0 and K_1 are modified Bessel functions, and $\epsilon(\omega)$ is the dielectric constant of the medium.

The energy lost by a particle in unit time equals the Poynting flux through a cylindrical surface of length $v \approx c$ and radius b surrounding the particle trajectory:

$$-\frac{d\mathscr{E}_0}{dt} = 2\pi cb \int_{-\infty}^{\infty} \frac{c}{4\pi}[\mathbf{EH}]_r\, dt = -2\pi c^2 b\, \mathrm{Re}\int_0^{\infty} H^{\bullet}_{\omega\varphi} E_{\omega z}\, d\omega. \qquad (45)^*$$

Because of the uncertainty principle b must be of the order of the Compton wavelength Λ; here $|bs| \ll 1$. Substituting (43), (44), and (31) into (45), and noting that the frequency and wave vector of the field components of an extremely relativistic particle are related by $\omega \approx k$ ($\hbar = c = 1$), we obtain the number of pseudophotons with momentum k that are absorbed in the medium in unit time:

$$W^0(p_0, k) = -\frac{1}{k}\left(\frac{d\mathscr{E}_0}{dk}\right)_k = \frac{e^2\gamma}{2\pi k}\ln\frac{m^4}{k^2(4\zeta^2+\gamma^2)}. \qquad (46)$$

In deriving this formula it was assumed that the logarithm greatly exceeds unity; the argument of the logarithm was determined to within a factor of the order of unity.

The probability (46) has the simple structure $W^0 = n(k)(\gamma_p + \gamma_C)$, where

$$n(k)\, dk = \frac{e^2 dk}{2\pi k}\ln\frac{m^4}{k^2(4\zeta^2+\gamma^2)} \qquad (47)$$

is the number of photons equivalent to the self-field of the particle and belonging to the momentum interval dk. The latter statement is easily confirmed by calculating the Poynting flux through a plane perpendicular to the particle trajectory.

We thus find that $n(k)\gamma_p$ is the number of photons absorbed from the self-field of a particle because of pair production, or, equivalently, the number of pairs with combined particle energy k produced by the field of the fast particle. It should be noted that W' has the same meaning in the spectral region where the effective length required for radiation is at least of the order of the radiation length. Photon emission in the usual sense does not occur in this region.

*$[\mathbf{EH}] = \mathbf{E}\times\mathbf{H}$.

The number of photons absorbed through the Compton effect is given by $n(k)\gamma_C$. Other lower-energy photons are produced, i.e., bremsstrahlung from electrons of the medium. The spectrum W^C of this radiation equals the integral $\int n(k')\gamma_C(k', k)dk'$, where

$$\gamma_C(k', k) = \pi n Z r_0^2 \frac{m}{k'^2}\left[\frac{k'}{k} + \frac{k}{k'} + \left(\frac{m}{k} - \frac{m}{k'}\right)^2 - 2m\left(\frac{1}{k} - \frac{1}{k'}\right)\right] \qquad (48)$$

is the probability of Compton scattering of a photon having the initial and final energies k' and k, respectively. Assuming $k \gg m$ and removing the logarithm from the integrand, when $k' = k$ we obtain

$$W^C(p_0,\ k) = \frac{2nZe^2 r_0^2 m}{3k^2}\ln\frac{m^4}{k^2(4\zeta^2+\gamma^2)}. \qquad (49)$$

In the spectral region where $\gamma_C \gg \gamma_p$, the logarithmic factor is simplified. Taking into account all terms in (48), we obtain, correct up to $k = m/2$,

$$W^C(p_0,\ k) = \frac{4nZe^2 r_0^2 m}{3k^2}\left(1 - \frac{m}{4k} + \frac{m^2}{16k^2}\right)\ln\frac{m^3}{4\pi nZe^2}. \qquad (50)$$

For $k < m/2$ we have

$$W^C(p_0,\ k) = \frac{8nZe^2 r_0^2}{3k}\left(1 - \frac{k}{m} + \frac{k^2}{m^2}\right)\ln\frac{m^3}{4\pi nZe^2}. \qquad (51)$$

The last equation is correct as long as the electrons of the medium can be regarded as free, i.e., as long as the mean energy transferred to a recoil electron in the Compton effect exceeds the binding energy of the electron in an atom. This condition leads to the inequality $k \gg \sqrt{m\mathscr{E}_b}$ with the binding energy $\mathscr{E}_b \approx 21\, Z^{4/3}$ eV computed according to the Thomas-Fermi model.

In the region where (50) and (51) are applicable W' is given by the Ter-Mikaélyan equation (42). The ratio W^C/W' exceeds unity for $\mathscr{E}_0 \gtrsim k/\sqrt{10\pi \times 137 n r_0^2 \Lambda}$. In a condensed medium ($n = 3\times 10^{12}/\mathrm{cm}^3$) this denotes $\mathscr{E}_0 \gtrsim 10^5$ k. For small $\mathscr{E}_0$ the region of (42) is bounded by the requirement $\mathscr{E}_0 \gtrsim mk/\omega_p$, which for the same value of n gives $\mathscr{E}_0 \gtrsim 8\times 10^4 k/\sqrt{Z}$; this region is thus very narrow: $8\times 10^4 k/\sqrt{Z} \lesssim \mathscr{E}_0 \leq 10^5$ k. For $k > m$ the inequality $W^C/W' > 1$ is satisfied when $\mathscr{E}_0 \gtrsim k^{3/2}/\sqrt{10\pi \times 137 n r_0^2}$.

Equations (49)–(51) describe bremsstrahlung emitted by electrons of a medium as a result of collisions with a passing particle. Photon emission by a fast particle colliding with the electrons is highly inhibited by multiple scattering and by polarization of the medium; the corresponding contribution is indicated by substituting $Z(Z+1)$ for Z^2 in W'.

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126