HYDROTHERMOMAGNETIC WAVES IN A WEAKLY INHOMOGENEOUS PLASMA

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Two kinds of instability are possible in an inhomogeneous system: a localized instability, characterized by the development of local fluctuations, and a global instability, characterized by the behavior of the system as a whole. The second develops from the first and is subject to more stringent conditions. In the present work we investigate localized instabilities of a weakly inhomogeneous plasma characterized by a small temperature gradient ∇T, a small density gradient $\nabla\rho$ or, a fixed electric field $\mathbf{E}$. A system of this kind exhibits unique oscillation properties if $v_A \ll s$ (v_A is the Alfvén speed and s is the velocity of sound) i.e., if $p \gg H^2/8\pi$; in this case we can write $\nabla\rho = 0$. The case $\nabla T \neq 0$ and a uniform weak magnetic field $\mathbf{H}$ has been treated earlier.[1] Here we consider the case $\Omega\tau \lesssim 1$ (Ω is the electron Larmor frequency and τ is the electron relaxation time). The magnetic field associated with the thermomagnetic current $\mathbf{j}$ is neglected. The dispersion relation has six branches and an instability can be excited under certain conditions, that is to say, the oscillations can grow. When $\Omega\tau \ll 1$ the instability is convective and manifests itself in the amplification of waves entering the medium from the outside; when $\Omega\tau > 1$ the instability becomes absolute. The growth rate depends on the relative orientation of the vectors $\mathbf{k}$, $\mathbf{H}$ and ∇T and is a maximum when these vectors are parallel. In the presence of the instability an external poloidal field can produce a toroidal field and vice versa. This mechanism may be of importance in the creation of magnetic fields of celestial bodies.

1. WAVE PROPAGATION IN A WEAKLY INHOMOGENEOUS MEDIUM

Two kinds of instability are possible in an inhomogeneous system: a localized instability such as that considered for the case of a collisionless plasma by Rudakov and Sagdeev,[2] and a global instability characterized by the behavior of the system as a whole, which has been treated for the same case by Silin.[3] In this work we investigate an instability of the first kind for hydrothermomagnetic waves. Because of the complexity of the equations, we consider in Sec. 2 the relation between the two kinds of instability in the simplest case of a system with a dielectric constant ϵ that varies in one direction only and is nonvanishing throughout the region being considered.

The electric field equation

$$\nabla^2\mathbf{E}(z,t) - \operatorname{grad}\operatorname{div}\mathbf{E}(z,t) = \frac{1}{c^2}\frac{\partial^2\mathbf{D}(z,t)}{\partial t^2},$$

$$\mathbf{D}(z,t) = \int_{-\infty}^{t}\varepsilon(z,t-t')\,\mathbf{E}(z,t')\,dt' \qquad (1.1)$$

in the case of transverse waves in a bounded inhomogeneous medium in which $\epsilon = \epsilon(z)$ can be reduced to the form

$$\frac{\partial^2}{\partial z^2}E(\omega,z) + \frac{\omega^2}{c^2}\varepsilon(\omega,z)E(\omega,z) = -\frac{1}{c^2}\dot{D}(0,z) + \frac{i\omega}{c^2}D(0,z) - \frac{\omega^2}{c^2}D^{(0)}(\omega,z) \equiv \frac{\omega^2}{c^2}\varepsilon(\omega,z)F(\omega,z) \qquad (1.2)$$

with the boundary conditions $E(\omega, z) = 0$ at $z = \pm L_0$.

Here

$$E(z,t) = \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} E(\omega,z)e^{-i\omega t}\,d\omega \qquad (\sigma \geqslant 0),$$

$$E(\omega,z) = \int_0^\infty E(z,t)\,e^{i\omega t}\,dt, \qquad \varepsilon(\omega,z) = \int_0^\infty \varepsilon(z,t)\,e^{i\omega t}\,dt,$$

$$D^{(0)}(z,t) = \int_{-\infty}^{0}\varepsilon(z,t-t')E(z,t')\,dt'.$$

It is assumed that the field is given for $t < 0$.

In the case of a weak inhomogeneity ($\partial\epsilon(\omega,z)/\partial z \ll \omega\epsilon^{3/2}/c$) the WKB method can be used and the electric field can be expanded in WKB modes ψ_n that satisfy the equation

$$\psi_n''(\omega,z) + \lambda_n^2(\omega)\,k^2(\omega,z)\,\psi_n(\omega,z) = 0, \qquad (1.3)$$

(where $k^2(\omega,z) = \omega^2\epsilon(\omega,z)/c^2$); in the present approximation these modes are of the form

$$\psi_n(\omega, z) = i\left[2k\int_{-L_0}^{L_0} kdz\right]^{-1/2}\left\{\exp\left(i\lambda_n\int_{-L_0}^{z} kdz\right) - \exp\left(-i\lambda_n\int_{-L_0}^{z} kdz\right)\right\}. \quad (1.4)$$

The WKB modes satisfy the boundary condition if

$$\lambda_n\int_{-L_0}^{L_0} k(\omega, z)\,dz = \pi n. \quad (1.5)$$

We then have

$$E(\omega, z) = \sum_n E_n(\omega)\psi_n(\omega, z),$$

$$F(\omega, z) = \sum_n F_n(\omega)\psi_n(\omega, z). \quad (1.6)$$

Because of the orthogonality of the modes for a given frequency (the orthogonality condition is written

$$\int_{-L_0}^{L_0} k^2(\omega, z)\psi_n(\omega, z)\psi_m(\omega, z)\,dz = \delta_{mn}),$$

$$E_n(\omega) = F_n(\omega)/(1-\lambda_n^2(\omega)),$$

$$E(z, t) = \frac{1}{2\pi}\sum_n\int_{-L_0}^{L_0} dz'\int_{-\infty+i\sigma}^{\infty+i\sigma} d\omega e^{-i\omega t} \times \frac{F(z', \omega)k^2(z', \omega)\psi_n(z', \omega)\psi_n(z, \omega)}{1-\lambda_n^2(\omega)}. \quad (1.7)$$

Closing the contour of integration over ω by an infinite semicircle in the lower half-plane we reduce the integral to a sum of residues at poles given by the relation $\lambda_n^2(\omega) = 1$. Then

$$E(z, t) = i\sum_{n,j} e^{-i\omega_{nj}t}\int_{-L_0}^{L_0} dz' \times \frac{F(z', \omega_{nj})k^2(z', \omega_{nj})\psi_n(z', \omega_{nj})\psi_n(z, \omega_{nj})}{2(\lambda_n d\lambda_n/d\omega)_{\omega=\omega_j}}. \quad (1.8)$$

Here j is the number of the pole, that is to say, the number of the branch of the dispersion equation.

The frequencies ω_{nj} are determined by the condition

$$\int_{-L_0}^{L_0} k(z, \omega_{nj})\,dz = \pi n, \quad (1.9)$$

and agree with those given by Silin[3] if, as we have indicated, $k(z, \omega)$ does not vanish at any point within the system (if this condition is not satisfied the integral is taken between points for which $k(z, \omega) = 0$). The appearance of a positive imaginary part in the frequencies ω_{nj} denotes the transition of the system into a state of global instability. In what follows it will be found convenient to replace the quantum number n by the mean wave vector

$$\bar{k} = \frac{1}{2L_0}\int_{-L_0}^{L_0} k(\omega_n, z)\,dz = \frac{\pi n}{2L_0}.$$

Suppose now that the dimensions of the system L_0 increase without limit so that for a given $\bar{k}$ the number n also increases without limit. In the sum

$$\exp\left(i\int_{-L_0}^{z} kdz + i\int_{-L_0}^{z'} kdz\right) + \exp\left(-i\int_{-L_0}^{z} kdz - i\int_{-L_0}^{z'} kdz\right) - \exp\left(i\int_{z}^{z'} kdz\right) - \exp\left(-i\int_{z}^{z'} kdz\right)$$

the first two terms oscillate rapidly so that the summation over n (1.8) can be approximated by an integral which vanishes in the limit. Thus, when $L_0 \to \infty$

$$E(z, t) = -i\sum_{n,j}\int_{-L_0}^{L_0}\frac{dz'}{\pi n}\frac{F(z', \omega_{nj})k^{3/2}(z', \omega_{nj})e^{-i\omega_{nj}t}}{4k^{1/2}(z, \omega_{nj})(\lambda_n d\lambda_n/d\omega)_{\omega_{nj}}} \times \left\{\exp\left(i\int_{z}^{z'} kdz\right) + \exp\left(-i\int_{z}^{z'} kdz\right)\right\}$$

approaches

$$-\frac{i}{4\pi}\sum_j\int_{-\infty}^{\infty}\frac{d\bar{k}}{\bar{k}}\int_{-\infty}^{\infty} dz'\frac{F(z', \omega_j)k^{3/2}(z', \omega_j)}{k^{1/2}(z, \omega_j)(\lambda d\lambda/d\omega)_{\omega_j}} \times \exp\left[-i\omega_j(\bar{k})t - i\int_{z}^{z'} kdz\right]. \quad (1.10)$$

(it should be noted that $k(z, \bar{k}) = -k(z, -\bar{k})$).

Now assume that at $t = 0$ at the point $z = z_0$ a fluctuation arises that can be written in the form $H_x(z, 0) = Ac^{-1}\delta(z - z_0)$ whence $\dot{D}_y(z, 0) = A\delta'(z - z_0)$, $E(z, t) = 0$ when $t < 0$.

Then,

$$E(z, t) = -\frac{A}{4\pi c^2}\sum_j\int_{-\infty}^{\infty}\frac{d\bar{k}}{\bar{k}}\left[\frac{k(z_0, \omega_j)}{k(z, \omega_j)}\right]^{1/2}\left(\lambda\frac{d\lambda}{d\omega}\right)^{-1}_{\omega_j} \times \exp\left[-i\omega_j(\bar{k})t - i\int_{z}^{z_0} kdz\right]. \quad (1.11)$$

In each of the terms of the summation over j we introduce the quantity $\varkappa = k(z_0\,\omega_j)$ as the independent variable. We then have

$$E(z, t) = -\frac{A}{4\pi c^2}\sum_j\int\frac{d\varkappa}{\bar{k}}\frac{d\bar{k}}{d\varkappa}\sqrt{\frac{\varkappa}{k(z, \omega_j)}} \times \exp\left[-i\omega_j t - i\int_{z}^{z_0} kdz\right]\left(\lambda\frac{d\lambda}{d\omega}\right)^{-1}_{\omega_j}$$

$$= \frac{A}{4\pi c^2}\sum_j\int d\omega_j\sqrt{\frac{k(z_0, \omega_j)}{k(z, \omega_j)}}\exp\left[-i\omega_j t - i\int_{z}^{z_0} kdz\right] \quad (1.12)$$

by virtue of (1.5) and the conditions $\lambda^2(\omega) = 1$ at the poles.

The integration is taken over a contour which coincides with the real axis in both integrals (1.12) at $\pm\infty$ since $k(z_0, \omega) \to \omega/c$ and $\bar{k} \to \omega/c$ when $\omega \to \pm\infty$. If there are no singularities between the contour and the real axis (i.e., $d\omega/dk \neq 0$) we can integrate over real values of ω in Eq. (1.12). However, the condition $d\omega/dk \neq 0$ also means that we integrate over real κ in (1.12). We then arrive at the notion of a field with frequencies depending on the coordinates of the points at which the fluctuations arise. These frequencies, which are analogous to those used in [2], are obtained from a dispersion equation which is written formally in the same way as for a uniform medium but with coefficients depending on coordinates.

When $|z - z_0| \ll L$, where $L = \epsilon/|\nabla\epsilon|$ is the scale length of the inhomogeneity, then

$$\exp\left[i\int_{z_0}^{z} kdz - i\omega t\right] \approx \exp\left[ik(z - z_0) - i\omega t\right],$$

and we obtain the usual plane wave.

Thus, for wavelengths satisfying the relation $kL \gg 1$ we can formally write the dispersion equation for an inhomogeneous medium in the same way as for a uniform medium but with coefficients that depend on coordinates; this equation then yields frequencies that will also depend on coordinates. These frequencies $\omega_j(z_0)$ characterize the time development of fluctuations arising at a point z_0 as long as its dimensions are much smaller than the dimensions of the inhomogeneity. The appearance of a positive imaginary part in the frequency $\omega_j(k, z_0)$ implies a localized instability which, as is well known, can be convective or absolute.[4] The characteristic frequencies of an inhomogeneous medium bounded in the z-direction are determined from the quantization condition (1.9); in this case the frequencies are obviously independent of coordinates and characterize the entire volume. The fact that (1.9) is an integral expression means that the localized instability appears before the global instability and that the second develops from the first. While the first can be either convective or absolute, the second is always an absolute instability.

2. BASIC EQUATIONS FOR HYDROMAGNETIC AND THERMOMAGNETIC WAVES

For the case of a magnetic field of arbitrary magnitude and hydrodynamic motions of $\mathbf{v}(\mathbf{r}, t)$ the electric current density is given by the expression

$$= \sigma\mathbf{E}^* + \sigma_1[\mathbf{E}^*\mathbf{H}] + \sigma_2(\mathbf{E}^*\mathbf{H})\,\mathbf{H} - \alpha\nabla T - \alpha_1[\nabla T\mathbf{H}] - \alpha_2\mathbf{H}(\mathbf{H}\nabla T),$$

$$\mathbf{E}^* = \mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{H}] + \frac{T}{e}\frac{\nabla\rho}{\rho}. \qquad (2.1)^*$$

whence

$$\mathbf{E} = -\frac{1}{c}[\mathbf{v}\mathbf{H}] - \frac{T}{e}\frac{\nabla\rho}{\rho} + \eta\mathbf{j} + \eta_1[\mathbf{j}\mathbf{H}] + \eta_2\mathbf{H}(\mathbf{j}\mathbf{H}) + \Lambda\nabla T + \Lambda_1[\nabla T\mathbf{H}] + \Lambda_2\mathbf{H}(\mathbf{H}\nabla T), \qquad (2.2)$$

where

$$\eta = \frac{\sigma}{\sigma^2 + (\sigma_1 H)^2}, \qquad \eta_1 = -\frac{\sigma_1}{\sigma^2 + (\sigma_1 H)^2},$$

$$\Lambda = \frac{\alpha\sigma + \alpha_1\sigma_1 H^2}{\sigma^2 + (\sigma_1 H)^2}, \qquad \Lambda_1 = \frac{\alpha_1\sigma - \sigma_1\alpha}{\sigma^2 + (\sigma_1 H)^2},$$

$$\eta_2 = \frac{\sigma_1^2 - \sigma\sigma_2}{[\sigma^2 + (\sigma_1 H)^2](\sigma + \sigma_2 H^2)},$$

$$\Lambda_2 = \frac{\alpha(\sigma_1^2 - \sigma\sigma_2) - \alpha_1\sigma_1(\sigma + \sigma_2 H^2) + \alpha_2(\sigma^2 + (\sigma_1 H)^2)}{[\sigma^2 + (\sigma_1 H)^2](\sigma + \sigma_2 H^2)}.$$

The energy flow expression which we shall need below is of the form

$$\mathbf{q} = (\Lambda - 3/2e)\,T\mathbf{j} + \Lambda_1 T[\mathbf{j}\mathbf{H}] + \Lambda_2 T\mathbf{H}(\mathbf{j}\mathbf{H}) - \rho C_p[\varkappa\nabla T + \varkappa_1[\nabla T\mathbf{H}] + \varkappa_2\mathbf{H}(\mathbf{H}\nabla T)], \qquad (2.3)$$

where κ is the thermal conductivity and C_p is the specific heat. We have made use of the Onsager relations; the thermal conductivity is assumed to be due to the electrons ($\kappa = \kappa_e$). However, if radiative thermal effects predominate ($\kappa_r \gg \kappa_e$) then $\kappa_1 = \kappa_2 = 0$.

We consider waves of frequency $\omega \ll ck$, $\omega \ll 1/\tau$ in which the electric field is not purely longitudinal; the displacement current $(4\pi)^{-1}\partial\mathbf{E}/\partial t$ can be neglected. In this case the equations assume the form

$$\partial\mathbf{H}/\partial t = -c\,\mathrm{rot}\,\mathbf{E}, \qquad (2.4)\dagger$$

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{c}[\mathbf{j}\mathbf{H}] + \rho\nu\left(\nabla^2\mathbf{v} + \frac{1}{3}\,\mathrm{grad\,div}\,\mathbf{v}\right), \qquad (2.5)$$

$$\frac{1}{\gamma}\frac{dT}{dt} + \left(\frac{1}{\gamma} - 1\right)\frac{T}{\rho}\frac{d\rho}{dt} = -\frac{1}{\rho C_p}\,\mathrm{div}\,\mathbf{q}, \qquad (2.6)$$

$$\mathrm{rot}\,\mathbf{H} = 4\pi\mathbf{j}/c, \qquad \mathrm{div}\,\mathbf{H} = 0, \qquad \partial\rho/\partial t + \mathrm{div}(\rho\mathbf{v}) = 0, \qquad (2.7)$$

where $\gamma = C_p/C_v$; ν is the kinematic viscosity.

We now separate the quantities ρ, T, $\mathbf{v}$, $\mathbf{H}$ into fixed and varying parts, for example $\rho = \rho_0 + \rho' e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ ($v_0 = 0$) (the subscript zero will henceforth be omitted) and linearize the equations

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*$[\mathbf{E}^*\mathbf{H}] = \mathbf{E}^* \times \mathbf{H}$, $(\mathbf{E}^*\mathbf{H}) = \mathbf{E}^* \cdot \mathbf{H}$.

†rot = curl.

in the primed quantities; in this case it is necessary to take account of the dependence of the kinetic coefficients on ρ, T, and H^2. We limit ourselves to the case $kL \gg 1$, where $L = T/|\nabla T|$. Under these conditions $v\nabla\rho \ll \rho$ and these and similar terms can be omitted.

Hydrothermomagnetic waves will exist only if the thermomagnetic terms in Eq. (2.4) [taking account of Eq. (2.2)] are larger than the galvanomagnetic terms. When $\Omega\tau \ll 1$ the largest thermomagnetic term $c\Lambda_1\mathbf{H}'(\mathbf{k}\nabla T) = -\omega_T H'$, while the largest galvanomagnetic terms $\nu_m k^2 \mathbf{H}'$, where $\nu_m = c^2/4\pi\sigma$ is the magnetic viscosity. The criterion for predominance of the thermomagnetic terms is then $\omega_T \gg \nu_m k^2$. Carrying out an estimate for a fully ionized plasma $\omega_T \sim kT\tau/mL$, $\sigma \sim ne^2\tau/m$ we find $kL \ll (\omega_0 l/c)^2$ where l is the electron mean free path and $\omega_0 = \sqrt{4\pi ne^2/m}$ is the electron plasma frequency. When $\Omega\tau \gg 1$ the largest thermomagnetic terms in Eq. (2.4) are of order $c\Lambda_2(\mathbf{H}\nabla T)\,\mathbf{k}\times\mathbf{H}'$ while the largest galvanomagnetic terms are of order $c^2\eta_1(\mathbf{k}\cdot\mathbf{H})\,\mathbf{k}\times\mathbf{H}'$ and for estimates $\Lambda_2 H^2 \sim e^{-1}$, $\eta_1 H \sim \Omega\tau/\sigma_0$ (σ_0 is the electrical conductivity for $\mathbf{H} = 0$) our condition assumes the form $kL \ll s^2/v_A^2$. In arbitrary fields

$$kL \ll \frac{s^2}{v_A^2}\frac{(\Omega\tau)^2}{1+(\Omega\tau)^2}. \tag{2.8}$$

If these inequalities are reversed, in which case the thermomagnetic terms can be neglected, we obtain the dispersion relations considered by Ginzburg[6] and Piddington.[7]

It has been shown in [1] that when the plasma thermal conductivity is neglected the thermomagnetic frequency in a weak magnetic field is ω_T. We now show that taking account of the thermal conductivity does not change this result. Because of the inequality $kL \gg 1$, the frequency $\omega_T \ll \kappa k^2$. It then follows from Eqs. (2.3), (2.5) and (2.7) that

$$\frac{T'}{T} \sim \frac{H'}{H}\frac{\Omega\tau}{kL} \ll \frac{H'}{HkL}.$$

Thus, in Eq. (2.4) it is permissible to neglect terms containing T', that is to say, the thermomagnetic oscillations are isothermal.

3. INVESTIGATIONS OF STABILITY WHEN k, H AND ∇T ARE PARALLEL

Assume that at time $t = 0$ a fluctuation arises for which the magnetic field

$$\mathbf{H}'(z, 0) = \sum_{j=1}^{n}\int \mathbf{H}_j'(k)\,e^{ikz}dk, \qquad \sum_{j=1}^{n}\mathbf{H}_j'(k) = \mathbf{H}'(k), \tag{3.1}$$

where j is the number of the branch; then, at time t

$$\mathbf{H}'(z, t) = \sum_{j=1}^{n}\int \mathbf{H}_j'(k)\,e^{ikz - i\omega_j(k)t}dk. \tag{3.2}$$

The initial fluctuation is bounded in space and hence the integral in Eq. (3.2) converges when $z \to \pm\infty$. If the frequency is complex $\omega = \omega_r + i\omega_i$ an instability arises when $\omega_i > 0$. In Eq. (3.2) we go from an integration over real k to an integration in the plane of the complex variable $k = k_r + ik_i$ for which ω is real. If this is possible then

$$\mathbf{H}'(z, t) = \sum_{j=1}^{n}\int \mathbf{H}_j'(k)\,e^{ik(\omega_j)z - i\omega_j t}\frac{dk}{d\omega_j}\,d\omega_j \tag{3.3}$$

and $\mathbf{H}'(z, t) \to 0$ as $t \to \infty$ for finite z. In this case the instability is of the convective type; if the transition in the opposite case from (3.2) to (3.3) cannot be made the instability is absolute.[4,5]

We now consider the case in which the vectors **k**, **H** and ∇T are parallel (or antiparallel). We show below that the growth rate is a maximum in this case. We take the vector $\mathbf{u}_1 = c\Lambda_1\nabla T$ along the z axis. The dispersion relation for the two pairs of branches is then of the form

$$\omega_{1,2} = -\tfrac{1}{2}k\{i(\nu+\nu_m)k + u_1 + \varepsilon(iu_2 + \nu_m k\sigma_1 H/\sigma) \pm [(i(\nu_m - \nu)k + u_1 + \varepsilon(iu_2 + \nu_m k\sigma_1 H/\sigma))^2 + 4v_A^2]^{1/2}\}, \tag{3.4}$$

where $u_2 = c\Lambda_2(\mathbf{H}\cdot\nabla T) \sim u_1\Omega\tau$, $\epsilon = \pm 1$. When $|k| \to \infty$ we get branches of two kinds:

$$\omega = -i\nu k^2, \qquad \omega = -\nu_m k^2(i + \varepsilon\sigma_1 H/\sigma). \tag{3.5}$$

In the branch of the first kind when $\omega_i = 0$, $k_r = \pm k_i$, that is to say the curve showing k_i as a function of k_r or $\omega_i = 0$ is approximated by two lines that divide the quadrants in half. For the branch of the second kind with $\omega_i = 0$

$$k_i = k_r\,[\varepsilon\sigma_1 H/\sigma \pm \sqrt{1 + (\sigma_1 H/\sigma)^2}],$$

i.e., the asymptotes are also lines.

An instability arises at small values of k; it is convenient to consider two limiting cases in this region:

$$1)\ v_A \ll \max(u_1, u_2), \quad 2)\ v_A \gg \max(u_1, u_2).$$

In the first case, expanding the roots in powers of v_A^2 we obtain two pair of branches

$$\omega_1(k) = -k\,[u_1 + i\nu_m k + \varepsilon(iu_2 + \sigma_1 H\nu_m k/\sigma)], \tag{3.6}$$

$$\omega_2(k) = -i\nu k^2 + \frac{kv_A^2}{u_1 + i(\nu_m - \nu)k + \varepsilon(iu_2 + \nu_m k\sigma_1 H/\sigma)}. \tag{3.7}$$

For the first pair of branches (3.6) the contour equation is of the form

$$k_i = \frac{1}{2\nu_m}\Big\{u_1 + 2\varepsilon\nu_m k_r \frac{\sigma_1 H}{\sigma} \pm \Big[u_1^2 + 4\varepsilon\nu_m k_r\Big(u_2 + \frac{\sigma_1 H}{\sigma} u_1\Big) + 4\nu_m^2 k_r^2\Big(1 + \Big(\frac{\sigma_1 H}{\sigma}\Big)^2\Big)\Big]^{1/2}\Big\}. \quad (3.8)$$

When

$$\mu = 1 - \frac{u_1^2}{u_2^2} - 2\frac{u_2}{u_1}\frac{\sigma_1 H}{\sigma} > 0,$$

which is satisfied when

$$\Omega\tau \leqslant \Omega_{cr}\tau \sim 0.2 \div 0.5 \ [8],$$

this is the equation of a hyperbola for which the asymptotes of one branch are located in the first and second quadrants; the other asymptotes are in the third and fourth quadrants. The ± signs in front of radical correspond to the upper and lower branches. In the integral in Eq. (3.2) we can convert from the real axis to the upper or the lower branch depending on the sign of z.

When $\Omega = \Omega_{cr}$ the approaching vertices come together and the hyperbola is transformed into two intersecting lines (in which case $\mu = 0$); it then separates again [shifting to the right or the left depending on the sign of u_2 in Eq. (3.8)] becoming a new hyperbola for which the asymptotes of one branch lie in the first and fourth quadrants and for the other in the second and third quadrants (in this case $\mu < 0$). It is impossible to go from integration over this branch to the integration over the real axis for any value of z; hence when $\Omega > \Omega_{cr}$ the instability is absolute. The transition point satisfies the condition $d\omega/dk = 0$[5] and is independent of ∇T.

If at t = 0 a fluctuation arises corresponding to the first pair of branches

$$H'(z, 0) = A\delta(z) = \frac{A}{2\pi}\int e^{ikz}dk, \quad (3.9)$$

then at the time t

$$H'(z, t) = \frac{A}{2\pi}\int e^{ikz - i\omega(k)t}dk.$$

Using Eq. (3.6) we have

$$H'(z,t) = \frac{\text{const}}{\sqrt{t}} \exp\Big\{-\frac{t}{2\nu_m\big[1+(\sigma_1 H/\sigma)^2\big]}\Big[\mu u_1^2 + 2\frac{z}{t}\Big(u_1 - u_2\frac{\sigma_1 H}{\sigma}\Big)\Big]\Big\}$$
$$\times \exp\Big\{-\frac{t}{2\nu_m[1+(\sigma_1 H/\sigma)^2]}\Big[\Big(\frac{z}{t}\Big)^2 + i\varepsilon\Big(2u_1u_2 + \frac{\sigma_1 H}{\sigma} \times \Big(u_1^2 - u_2^2 + 2u_1\frac{z}{t} + \frac{z^2}{t^2}\Big)\Big)\Big]\Big\}. \quad (3.10)$$

When $z \to \pm\infty$, $H'(z, t) \to 0$ for finite t; when $t \to \infty$

$$|H'(z,t)| = \frac{\text{const}}{\sqrt{t}}\exp\Big\{-\frac{t}{2\nu_m[1+(\sigma_1 H/\sigma)^2]} \times \Big[\mu u_1^2 + 2\frac{z}{t}\Big(u_1 - u_2\frac{\sigma_1 H}{\sigma}\Big)\Big]\Big\}, \quad (3.11)$$

that is to say, in the region of the convective instability ($\mu > 0$) the quantity $H'(z, t) \to 0$ when $t \to \infty$ and in the region of the absolute instability ($\mu < 0$) $H'(z, t)$ increases without limit. Under conditions of the absolute instability the field $H'(z, t)$ increases with time in the region between two points z_1 and z_2 located on opposite sides of z = 0 with coordinates given by the relation

$$z_{1,2} = -\Big[u_1 - u_2\frac{\sigma_1 H}{\sigma} \pm u_2\sqrt{1 - \Big(\frac{\sigma_1 H}{\sigma}\Big)^2}\Big]t, \quad (3.12)$$

and decays in time outside of this region. As time increases the growth region of the field expands while the damping region moves away from the point at which the fluctuation grows (z = 0). For a convective instability the coordinates of the points are negative and the region of growth located between them moves out to infinity in the course of time.

This same result can be obtained from the form of the $\omega_i = 0$ curves in the $k = k_r + ik_i$ plane. In the region of the convective instability with $z > 0$ the transition from the real axis in Eq. (3.2) is possible only for the upper branch of the hyperbola which is located entirely in the upper half plane ($k_i > 0$). Hence the integrand, which is proportional to $e^{-k_i z}$, approaches zero when $z \to \infty$ and consequently the integral in Eq. (3.2) approaches zero when $z \to \infty$ with finite t in the same way as $t \to \infty$ with finite z.

On the other hand, when $z < 0$ the transition from the real axis in Eq. (3.2) is possible only for the lower branch of the hyperbola so that the integrand is proportional to $e^{k_i|z|}$. But the maximum of the lower branch is located in the upper half-plane in the region $|k_r| < |u_2|/\nu_m$, $k_i > 0$. In this region the integrand, which is proportional to $e^{|k_i z|}$, approaches infinity when $z \to -\infty$. Consequently the convective instability grows and moves in the direction $z < 0$. The instability will occur if the smallest wave vector k satisfying the condition $kL \gg 1$ lies in the region in which $k_i > 0$; thus, the existence of a convective instability requires that the parameters of the medium satisfy the inequality $u_2/\nu_m \gg 2\pi/L$, i.e.,

$$2\pi\Big(\frac{c}{\omega_0 l}\Big)^2\Big(\Omega\tau + \frac{1}{\Omega\tau}\Big) \ll 1,$$

where $\Omega < \Omega_{cr}$.

The maximum of the lower branch of the hyperbola lies at

$$k_{rm} = -\frac{\varepsilon}{2\nu_m[1+(\sigma_1 H/\sigma)^2]}\left[u_2 + u_1\frac{\sigma_1 H}{\sigma}(1-\sqrt{\mu})\right],$$

$$\omega_m = \varepsilon\frac{u_1u_2}{2\nu_m},$$

$$k_{im} = \frac{1}{2\nu_m[1+(\sigma_1 H/\sigma)^2]}\left[u_1(1-\sqrt{\mu}) - u_2\frac{\sigma_1 H}{\sigma}\right]. \quad (3.13)$$

If the wave packet given below is incident on the boundary ($z = 0$) of a convectively unstable medium

$$H'(0, t) = \int H'(\omega)e^{-i\omega t}d\omega, \quad (3.14)$$

inside the medium (at $z < 0$) at a distance z the field will be

$$H'(z, t) = \int H'(\omega)e^{ik_r z - k_i z - i\omega t}d\omega. \quad (3.15)$$

Expanding in powers of $\omega - \omega_m$ around the maximum of the lower branch of the hyperbola at large distances we have

$$H'(z, t) = H'(\omega_m)\exp\{i(k_{rm}z - \omega_m t) - k_{im}z\}$$
$$\times \exp\left[\frac{\sqrt{\mu}u_1}{4\nu_m z}(z + tu_1\sqrt{\mu})^2\right]$$
$$\times \int_{-\infty}^{\infty}\exp\left[-\frac{(\omega-\omega_m)^2}{2\overline{(\Delta\omega)^2}}\right]d\omega. \quad (3.16)$$

This formula holds only at distances such that the width of the packet in the medium

$$\sqrt{\overline{(\Delta\omega)^2}} = u_1^{3/2}\mu^{3/4}/\sqrt{2\nu_m|z|}$$

becomes smaller than the width of the packet entering the medium $H'(\omega)$. The relative width of the transmission poles

$$\frac{\sqrt{\overline{(\Delta\omega)^2}}}{\omega_m} = \sqrt{\frac{2\nu_m u_1}{u_2^2|z|}}\,\mu^{3/4}$$

approach zero as the transition point from the convective instability to the absolute instability is approached. The maximum wave growth occurs at $z \sim L$ [Eq. (3.16) holds when $z \ll L$] at a time $t \sim L/u_1\sqrt{\mu}$ and is given by the factor

$$\exp\left[\frac{L}{2\nu_m[1+(\sigma_1 H/\sigma)^2]}\left(u_1 - u_1\sqrt{\mu} - u_2\frac{\sigma_1 H}{\sigma}\right)\right]$$
$$\sim \exp\left[\frac{1}{2}\left(\frac{\omega_0 l}{c}\right)^2\right].$$

The amplification can be appreciable when $(\omega_0 l/c)^2 \gg 1$.

For the second pair of branches of (3.7) the curve k_i as a function of k_r with $\omega_i = 0$ has two branches similar to a hyperbola which are located in approximately the same way as in the preceding case; however, in the region of the convective instability the minimum of the upper branch and the maximum of the lower branch lie at $k_i < 0$. Hence the upper branch corresponds to a wave that is amplified in the direction $z > 0$.

At small k, including the region $k_i < 0$ for the upper branch, the equation for the contour can be written

$$k_i = \frac{1}{2\nu(u_1^2+u_2^2)}\{-u_1v_A^2 \pm [u_1^2v_A^4 + 4\varepsilon\nu k_r v_A^2(u_1^2+u_2^2)$$
$$+ 4\nu^2k_r^2(u_1^2+u_2^2)^2]^{1/2}\}. \quad (3.17)$$

A convective instability arises when

$$u_1^2 > u_2^2 \qquad (\Omega < \Omega_{cr} \sim (0.5 \div 1)\tau^{-1}),$$

and an absolute instability arises when

$$u_2^2 > u_1^2 \qquad (\Omega > \Omega_{cr})$$

and the transition is again independent of the temperature gradient. At the transition point $d\omega/dk = 0$. We find $k_i < 0$ for the upper branch when

$$|k_r| < v_A^2|u_2|/\nu(u_1^2+u_2^2) \qquad (\text{for } \Omega < \Omega_{cr}). \quad (3.18)$$

The convective instability will occur when $\Omega < \Omega_{cr}$ if the parameters of the medium satisfy the inequality

$$v_A^2|u_2|/\nu(u_1^2+u_2^2) \gg 2\pi/L,$$

which is equivalent to $\sqrt{m/M}\,(v_A/s)^2(L/l)^2\Omega\tau \ll 1$. The minimum of the upper branch occurs at

$$k_{im} = \frac{v_A^2u_1}{2\nu(u_1^2+u_2^2)}\left(-1+\sqrt{1-\frac{u_2^2}{u_1^2}}\right),$$

$$k_{rm} = -\frac{\varepsilon u_2v_A^2}{2\nu(u_1^2+u_2^2)}, \qquad \omega_m = \frac{v_A^2u_1k_{rm}}{u_1^2+u_2^2}.$$

As for the case of the first branch (3.6) the relative width of the transmission poles

$$\frac{\sqrt{\overline{(\Delta\omega)^2}}}{\omega_m} = \left(1-\frac{u_2^2}{u_1^2}\right)^{3/4}\sqrt{\frac{\nu u_1(u_1^2+u_2^2)}{zv_A^2u_2^2}}$$

approaches zero as the transition point from the convective instability to the absolute instability is approached.

The maximum growth of a wave packet incident on the medium ($z > 0$) at $z \sim L$ is given by the factor

$$e^{-k_{im}L} \approx \exp\left[\frac{v_A^2u_1L}{2\nu(u_1^2+u_2^2)}\left(1-\sqrt{1-\frac{u_2^2}{u_1^2}}\right)\right].$$

The growth can be large when

$$\frac{v_A^2}{s^2}\left(\frac{L}{l}\right)^2\sqrt{\frac{m}{M}} \ll 1$$

(in this case $\Omega < \Omega_{cr} \sim 1/\tau$, $v_A/u_1 \sim (v_AL/sl)\sqrt{m/M} \ll 1$).

We have investigated the case in which the thermomagnetic terms dominate the Alfvén terms. We now consider the other limiting case $v_A \gg \max(u_1, u_2)$. For small values of k, in which case the $4v_A^2$ term is larger than the other terms under the radical in Eq. (3.4), the radical can be expanded

$$\omega = \pm kv_A - \tfrac{1}{2} k\, [u_1 + i(\nu + \nu_m) k + \varepsilon (iu_2 + \sigma_1 H \nu_m k/\sigma)]. \qquad (3.19)$$

It will be shown below that the instability arises at precisely those values of k for which the radical can be expanded. From Eq. (3.19) with $\omega_i = 0$ we have

$$k_i = \pm (k_r/2v_A)\, [\varepsilon u_2 + (\nu + \nu_m) k]. \qquad (3.20)$$

The general analytic expression $k_i(k_r)$ is complicated and is not needed to investigate the kind of instability. The asymptote of the curve $\omega_i = 0$ is known (3.5). We also know the behavior of the curve at small k (3.20). It then is easy to get an idea of the behavior of the k_i curve as a function of k_r for arbitrary k. The curve comprises two branches similar to the branches of a hyperbola. The asymptotes of the upper branch lie in the first and second quadrants while the asymptotes of the lower branch lie in the third and fourth quadrants. The upper sign in Eq. (3.19) and (3.20) corresponds to the case in which the minimum of the upper branch lies at $k_i < 0$ while the lower branch is located entirely in the lower half-plane. Consequently, in this case the wave propagates in the direction $z > 0$. For the lower sign ($-$) the maximum of the lower branch is located at $k_i > 0$, and the upper branch lies in the upper half-plane. The wave propagates in the direction $z < 0$.

When $v_A \gg \max(u_1, u_2)$ only the convective instability is possible. This instability will occur if the smallest wave vector $k \gg 2\pi/L$ lies in the region $|k_r| < |u_2|/(\nu + \nu_m)$; thus, for an instability to occur we require

$$|u_2|/(\nu + \nu_m) \gg 2\pi/L,$$

which is equivalent to

$$2\pi \left(1 + \frac{\nu_m}{\nu}\right)\left(\Omega\tau + \frac{1}{\Omega\tau}\right)\sqrt{\frac{m}{M}} \ll 1.$$

The minimum of the upper branch of $k_i(k_r)$ for $\omega_i = 0$ with the (+) sign in Eq. (3.19) and (3.20) is located at

$$k_{rm} = -\frac{\varepsilon u_2}{2(\nu + \nu_m)}, \qquad k_{im} = -\frac{u_2^2}{4v_A(\nu + \nu_m)},$$

$$\omega_m = k_{rm} v_A. \qquad (3.21)$$

The maximum of the lower branch for the ($-$) sign in Eqs. (3.19) and (3.20) is located at

$$k_{rm} = -\frac{\varepsilon u_2}{2(\nu + \nu_m)}, \qquad k_{im} = \frac{u_2^2}{4v_A(\nu + \nu_m)},$$

$$\omega_m = -k_{rm} v_A.$$

The relative width of the transmission poles

$$\frac{\sqrt{\overline{(\Delta\omega)^2}}}{|\omega_m|} = \sqrt{\frac{(\nu + \nu_m) v_A}{2u_2^2 z}}$$

can be smaller than unity if

$$\frac{L}{l} \ll \sqrt{\frac{M}{m}} \frac{s}{v_A}\left(1 + \frac{\nu_m}{\nu}\right)^{-1}\left(\Omega\tau + \frac{1}{\Omega\tau}\right)^{-2}.$$

The maximum wave growth is given by the factor

$$\exp\left[\frac{u_2^2 L}{4v_A(\nu + \nu_m)}\right] \sim \exp\left[\frac{u_2}{v_A}\sqrt{\frac{M}{m}}\, \frac{L}{l(1 + \nu_m/\nu)(\Omega\tau + 1/\Omega\tau)}\right].$$

Up to this point we have investigated various limiting cases of the parameters of the medium. We have shown that in the case $v_A \gg \max(u_1, u_2)$ there can only be an instability of the convective type; in the case $v_A \ll \max(u_1, u_2)$ with fields $\Omega < \Omega_{cr} \sim \tau^{-1}$[8] there can only be a convective instability and for fields $\Omega > \Omega_{cr}$ only an absolute instability. The transition from the convective to the absolute instabilities is independent of ∇T so long as the limiting condition $v_A \ll \max(u_1, u_2)$ is satisfied.

In the general case, however, the transition points depend on the magnetic field as well as ∇T. The transition points can be found by using the general dispersion relation (3.4) rather than the asymptotic expressions. The transition point must satisfy two conditions $d\omega/dk = 0$, $\omega_i = 0$. Furthermore, beyond the transition point the asymptote of the curve $\omega_i = 0$ must change to make the transition from Eq. (3.2) to Eq. (3.3) impossible. Using Eq. (3.4) we find two transition points from convective to absolute instabilities corresponding to the two signs in front of the radical in Eq. (3.4). For the upper sign (+) with

$$u_1^2 - u_2^2 - 2u_1 u_2 \sigma_1 H/\sigma + A v_A^2 > 0, \qquad (3.22)$$

where $A > 0$ is a function of the temperature, density, and magnetic field, the instability is convective; in the opposite case the instability is absolute

$$A = 4 \text{ for } \nu_m \gg \nu,$$

$$A = 4\frac{\nu_m}{\nu}\left[1 + \left(\frac{\sigma_1 H}{\sigma}\right)^2\right] \text{ for } \nu \gg \nu_m. \qquad (3.23)$$

For the lower sign (−) in front of the radical with

$$u_1^2 - u_2^2 + Bv_A^2 > 0, \quad (3.24)$$

where B > 0, the instability is of the convective type; if the quantity in (3.24) is negative, the instability is absolute. The quantity B assumes the values 1.75 > B > 1.66 when $\nu \gg \nu_m$,

$$B = \frac{27}{32}\frac{\nu}{\nu_m}\left[1 + \left(\frac{\sigma_1 H}{\sigma}\right)^2\right] \text{ for } \nu_m \gg \nu. \quad (3.25)$$

The actual transition from the convective to the absolute instability arises at the lowest values of the magnetic field and ∇T given by Eqs. (3.22) and (3.24).

4. INSTABILITY OF HYDRO- AND THERMO-MAGNETIC WAVES IN A WEAK MAGNETIC FIELD

In [1] the oscillation frequencies were calculated for various relative values of the characteristic frequencies (the acoustic frequency $\omega_s = ks$, the Alfvén frequency ω_A, the thermomagnetic frequency $\omega_T = -c\Lambda_1 \mathbf{k}\nabla T$) neglecting a number of terms that are small in weak fields. We shall see that the neglected terms can lead to an instability that leads to growth of hydro- and thermomagnetic waves.

First of all we take $\omega_T \ll \omega_\kappa = \kappa k^2$, $\omega_A \ll \omega_s$. The relative values of ω_s and ω_κ are unimportant for the excitation of the unstable Alfvén wave (when $\omega_A \gg \omega_T$) or the thermomagnetic wave (when $\omega_T \gg \omega_A$). We now consider the remaining possible cases.

1. When $\omega_A \gg \omega_\kappa$ the Alfvén wave is stable;

2. When $\omega_\kappa \gg \omega_A \gg \omega_T$ we write $\omega = \omega_A + \Delta\omega_A$ where $\Delta\omega_A \ll \omega_A$. Eliminating **v** and T′ from Eqs. (2.4)—(2.7) and making use of (2.8) we have

$$[2\Delta\omega_A - \omega_T + i(\nu + \nu_m)k^2]\,\mathbf{H}' - 2c\frac{\partial\Lambda}{\partial H^2}(\mathbf{HH}')(\mathbf{k}\nabla T)$$
$$- c\Lambda_2\{[\mathbf{kH}'](\mathbf{H}\nabla T) + [\mathbf{kH}](\mathbf{H}'\nabla T)\}$$
$$-[\mathbf{k}(\mathbf{kH}) - k^2\mathbf{H}]\frac{c\Lambda_1\varkappa_1}{\varkappa k^2}([\mathbf{kH}']\nabla T) + \omega_A\left[\frac{1}{k^2}\mathbf{k}(\mathbf{kH}) - \mathbf{H}\right]$$
$$\times\left\{\frac{i\varkappa_1}{\varkappa k^2 T}([\mathbf{kH}']\nabla T) - \frac{\gamma}{4\pi\rho s^2}(\mathbf{H}'\mathbf{H})\right\} = 0. \quad (4.1)$$

In the case in which the electron thermal conductivity $\kappa_1 H/\kappa \sim \Omega\tau \ll 1$, using (4.1) we have

$$2\Delta\omega_A = 2(\Delta\omega_1 \pm \Delta\omega_2) = \omega_T - i(\nu + \nu_m)k^2$$
$$- c\left(\frac{\partial\Lambda}{\partial H^2} - \Lambda_2 + \frac{\Lambda_1\varkappa_1}{\varkappa}\right)(\mathbf{k}\,[\mathbf{H}\nabla T])$$
$$\pm c\left\{\frac{1}{4}\left(\Lambda_2 + \frac{\Lambda_1\varkappa_1}{\varkappa} + 2\frac{\partial\Lambda}{\partial H^2}\right)^2(\mathbf{k}\,[\mathbf{H}\nabla T])^2\right.$$
$$-\left[2\frac{\partial\Lambda}{\partial H^2}(k^2\mathbf{H}\nabla T - (\mathbf{kH})(\mathbf{H}\nabla T))\right.$$
$$\left.+ \Lambda_2(2k^2\mathbf{H}\nabla T - (\mathbf{kH})(\mathbf{k}\nabla T))\right]$$
$$\left.\times\left[\Lambda_2\mathbf{H}\nabla T - \frac{\Lambda_1\varkappa_1}{\varkappa k^2}(k^2\mathbf{H}\nabla T - (\mathbf{kH})(\mathbf{k}\nabla T))\right]\right\}^{1/2}. \quad (4.2)$$

An instability arises when the expression in front of the radical is negative. The growth rate is a maximum when the vectors **k**, **H** and ∇T lie in one plane and the first term under the radical in Eq. (4.2) vanishes. We denote the angle between the vectors **k** and **H** by α and the angle between **H** and **∇T** by β and take account of the fact, as follows from the relation $\Lambda + \Lambda_2 H^2 = \Lambda_0$ (where Λ_0 is the value of Λ for **H** = 0) that $\partial\Lambda/\partial H^2 = -\Lambda_2$ (when $\Omega\tau \ll 1$). Then

$$(\Delta\omega_2)^2 = -\frac{1}{4}(c\Lambda_2\nabla T)^2 k^2 H^2 \cos\alpha\cos(\beta - \alpha)\{\cos\beta$$
$$+ (\Lambda_1\varkappa_1/\Lambda_2\varkappa)[\cos\alpha\cos(\beta-\alpha) - \cos\beta]\}. \quad (4.3)$$

According to [8] $\Lambda_1 > 0$, $\Lambda_2 < 0$, $\kappa > 0$, and $\kappa_1 < 0$ and for singly charged ions $\kappa_1\Lambda_1/\kappa\Lambda_2 = 1.1$. For a Lorentz gas $\kappa_1\Lambda_1/\kappa\Lambda_2 = 1.95$, that is to say, it is always true that $1 < \kappa_1\Lambda_1/\kappa\Lambda_2 < 2$. For a fixed β the maximum growth rate obtains when $2\alpha - \beta = 0, \pi, 2\pi, 2\pi$. When the angle β varies the maximum growth rate, given by $(1/2)\gamma_{max} = (1/4)\, ckH\,|\Lambda_2\nabla T|$, is obtained when $\beta = 0$ if $2\alpha - \beta = 0$, and 2π and $\beta = \pi$, if $2\alpha - \beta = \pi$ and 3π. Growth occurs if $\gamma_{max} > (\nu + \nu_m)k^2$, which is equivalent to

$$\sqrt{m/M}\,kL\,(1 + \nu_m/\nu) < \Omega\tau.$$

In the other limiting case $\kappa_r \gg \kappa_e(\Omega\tau/kL)(s^2/v_A^2)$

$$2\Delta\omega_A = \omega_T - i(\nu + \nu_m)k^2$$
$$- \frac{1}{2}\left\{\frac{\gamma\omega_A}{\omega_s^2}[\mathbf{kv}_A]^2 + c\left(2\frac{\partial\Lambda}{\partial H^2} - \Lambda_2\right)\right.$$
$$\left.\times(\mathbf{k}\,[\mathbf{H}\nabla T])\right\} \pm \left\{\frac{1}{4}\left[\frac{\gamma\omega_A}{\omega_s^2}[\mathbf{kv}_A]^2 + c\left(2\frac{\partial\Lambda}{\partial H^2} + \Lambda_2\right)\right.\right.$$
$$\left.\left.\times(\mathbf{k}\,[\mathbf{H}\nabla T])\right]^2 - (c\Lambda_2)^2(\mathbf{kH})(\mathbf{k}\nabla T)(\mathbf{H}\nabla T)\right\}^{1/2}. \quad (4.4)$$

An instability arises if

$$ck\,|\Lambda_2\nabla T|\,H \gg |\omega_A|\omega_s^{-2}[\mathbf{kv}_A]^2,$$
$$ck\,|\Lambda_2\nabla T|\,H > (\nu + \nu_m)k^2. \quad (4.5)$$

If Eq. (4.5) is satisfied, the growth rate is a maximum when **kH** × ∇T = 0, $2\alpha - \beta = 0, \pi, 2\pi, 3\pi$ for a fixed value of β. If the angle β is changed the maximum growth rate $(1/2)\gamma_{max}$ obtains at $\beta = 0$ if $2\alpha - \beta = 0, 2\pi$, and at $\beta = \pi$, if $2\alpha - \beta = \pi, 3\pi$.

3. We now consider unstable thermomagnetic waves. When $\omega_s \gg \omega_T \gg \omega_A$ the stability does

not arise if the terms containing $\tilde{\Lambda} = \rho\partial\Lambda/\partial\rho + e^{-1}$ predominate in Eq. (2.4), [taking account of Eq. (2.2)] as is possible when $\omega_T^2 \gg \omega_s^2 kL\Omega\tau$, in which case terms containing $\tilde{\Lambda}$ are negligible, the maximum growth rate $\gamma_{max} = ckH\,|\Lambda_2\nabla T|$ obtains for the same directions of the vectors **k**, **H** and ∇T as in case (2) in which the electro thermal conductivity predominates. Growth will occur when $\gamma_{max} > \nu_m k^2$.

In the case of radiative thermal conductivity $\kappa_r \gg \kappa_e(s^2/v_A^2)(\Omega\tau/kL)$ the maximum growth rate is the same as in the analogous case in (2) with the same orientation of the vectors.

4. When $\omega_\kappa \gg \omega_T \gg \omega_s \gg \omega_A$ terms containing $\tilde{\Lambda}$ are negligible and the results are the same as in the preceding case.

5. HYDROTHERMOMAGNETIC WAVES IN A STRONG MAGNETIC FIELD

For strong magnetic fields characterized by $v_A \ll s$, we need consider only the kinetic coefficients Λ, Λ_2, and κ_2 and their derivatives with respect to H^2, ρ and T in Eqs. (2.4) and (2.6) in the case of electron thermal conductivity. If the radiative thermal conductivity predominates in Eq. (2.6) only κ will remain.

In the first case the dispersion equation is

$$\omega^7 + i\gamma\omega_{\varkappa 2}\omega^6 - (\omega_s^2 + \omega_{T2}^2)\,\omega^5 - i\omega_{\varkappa 2}\omega^4(\omega_s^2 + \gamma\omega_{T1}^2) + \omega_s^2\omega^3(2\omega_A^2 - \omega_{T3}^2) + i\omega_{\varkappa 2}\omega_s^2\omega^2(2\omega_A^2 + \omega_{T1}^2) - \omega\omega_s^2\omega_A^4 - i\omega_{\varkappa 2}\omega_s^2\omega_A^4 = 0. \quad (5.1)$$

Here we have introduced the following characteristic frequencies:

$$\omega_{\varkappa 2} = \varkappa_2(\mathbf{kH})^2,$$

$$\omega_{T1}^2 = -(ck\Lambda_2)^2(\mathbf{H}\nabla T)^2\left[1 + 2\frac{[\mathbf{kH}]^2}{k^2}\frac{\partial}{\partial H^2}\ln\frac{\Lambda_2}{\varkappa_2}\right],$$

$$\omega_{T2} = -2\frac{c\gamma\tilde{\Lambda}}{T}\frac{\partial\varkappa_2}{\partial H^2}(\mathbf{kH})(\mathbf{H}\nabla T)(\mathbf{k}[\mathbf{H}\nabla T]),$$

$$\omega_{T3}^2 = -2\frac{c\gamma\Lambda_2\varkappa_2}{k^2T}[\mathbf{kH}]^2 \times (\mathbf{H}\nabla T)(\mathbf{k}[\mathbf{H}\nabla T])(\mathbf{kH})\frac{\partial}{\partial H^2}\ln\frac{\Lambda_2}{\varkappa_2};$$

ω_{T1}^2, ω_{T2}^2, ω_{T3}^2 can be negative. For frequencies satisfying the relation $\omega\tau \ll 1$ the following ordering applies:

$$\omega_{T2}^2 \sim \omega_{T3}^2 \sim \Omega\tau\omega_{T1}^2, \qquad \omega_{\varkappa 2} \sim \omega_{T1}kL\Omega\tau.$$

The frequency ω_{T1} appears only in the form of a product $\omega_{T1}\omega_{\kappa 2}$; evidently the frequencies ω_{T2} and ω_{T3} can be neglected in all cases of interest here. Inasmuch as the relations $\omega_s \gg \omega_A$, $\omega_{\kappa 2} \gg \omega_{T1}$ hold the effect of ∇T on the spectrum is determined primarily by the ratio ω_{T1}/ω_A.

We first consider the case in which $\omega_{\kappa 2} \gg \omega_s \gg \max(\omega_A, \omega_{T1})$. Then, retaining the $\omega^7 + i\gamma\,\omega_{\kappa 2}\omega^6$ terms in Eq. (5.1) we obtain the damped branch $\omega = -i\gamma\omega_{\kappa 2}$. To find the other roots we must consider terms

$$i\gamma\omega_{\varkappa 2}\omega^6 - i\omega_{\varkappa 2}\omega_s^2\omega^4,$$

which give isothermal sound waves $\omega^2 = \omega_s^2/\gamma$. For the remaining terms, by virtue of the relation $\omega_{\kappa 2}\omega_{T1} \gg \omega_{T2}^2 \sim \omega_{T3}^2$ we neglect terms containing ω and ω^3 and obtain an equation for the four hydro- or thermomagnetic branches:

$$(\omega^2 - \omega_A^2)^2 - \omega^2\omega_{T1}^2 = 0, \quad (5.2)$$

whence

$$\omega = \tfrac{1}{2}\left(\pm\omega_{T1} \pm \sqrt{\omega_{T1}^2 + 4\omega_A^2}\right). \quad (5.3)$$

When $\omega_A \gg \omega_{T1}$ we obtain the Alfvén wave. When

$$\frac{\omega_A}{\omega_{T1}} \ll \min\left(\frac{\omega_s}{\omega_A}, \frac{\omega_{\varkappa 2}}{\omega_A}\right)$$

the most important correction to the Alfvén wave is given by Eq. (5.3) and

$$\omega = \pm\omega_A \pm \tfrac{1}{2}\omega_{T1}. \quad (5.4)$$

When $\Omega\tau \gg 1$ we have $\partial\ln(\Lambda_2/\kappa_2)/\partial H^2 \sim 0$. Thus the maximum growth rate for the Alfvén wave $ckH\,|\Lambda_2\nabla T|$, obtains when $\mathbf{H} \parallel \nabla T$.

When $\omega_{T1} \gg \omega_A$, Eq. (5.2) gives four thermomagnetic branches

$$\omega_{1,2}^2 = \omega_{T1}^2, \qquad \omega_{3,4}^2 = \omega_A^4/\omega_{T1}^2.$$

These frequencies correspond to exponentially growing and damped waves. The maximum growth obtains when $\nabla T \parallel \mathbf{H}$.

We consider briefly other possible relations between the characteristic frequencies. If $\omega_{\kappa 2} \gg \omega_{T1} \gg \omega_s \gg \omega_A$, we obtain the damped branch with frequency $\omega = -i\gamma\omega_{\kappa 2}$ as before, two thermomagnetic branches $\omega^2 = \omega_{T1}^2$, two isothermal acoustic branches $\omega^2 = \omega_s^2/\gamma$ and two branches $\omega^2 = \omega_A^4/\omega_{T1}^2$. For another configuration of the frequencies $\omega_s \gg \omega_{\kappa 2} \gg \max(\omega_A, \omega_{T1})$ the spectrum separates into the two branches $\omega^2 = \omega_s^2$, $\omega = -i\omega_{\kappa 2}$; and four hydro- and thermomagnetic branches as in the first case. Finally, when $\omega_s \gg \omega_A \gg \omega_{\kappa 2} \gg \omega_{T1}$ the spectrum does not contain thermomagnetic waves (terms containing ∇T only cause splitting) and the spectrum exhibits the usual form.[6,7]

We now consider in detail the case in which the magnetic field is perpendicular to ∇T and $\omega_{T1} = 0$. In this case, if the thermomagnetic waves are to predominate over the galvanomagnetic terms Eq. (2.8) must be replaced by the more stringent requirement

$$(v_A/s)^2 kL\Omega\tau \ll 1. \quad (5.5)$$

If this requirement is satisfied we obtain the equation

$$\omega^7 + i\gamma\omega_{\varkappa 2}\omega^6 - (\omega_s^2 + i\gamma\omega_{T5}^2)\omega^5 - i\omega_{\varkappa 2}\omega^4(\omega_s^2 + \gamma\omega_{T4}^2) + \omega_s^2\omega^3(2\omega_A^2 + i\omega_{T5}^2) + i\omega_s^2\omega^2\omega_{\varkappa 2}(2\omega_A^2 + \omega_{T4}^2) - \omega_s^2\omega_A^2\omega(\omega_A^2 + i\omega_{T5}^2) - i\omega_{\varkappa 2}\omega_A^4\omega_s^2 = 0, \quad (5.6)$$

where

$$\omega_{T4}^2 = -c^2\Lambda_1\Lambda_2(\mathbf{k}\nabla T)(\mathbf{k}[\mathbf{H}\nabla T])\left(1 + 2H^2\frac{\partial}{\partial H^2}\ln\Lambda_1\right),$$

$$\omega_{T5}^2 = \gamma c(\mathbf{kH})^2(\mathbf{k}\nabla T)\left[\varkappa_1\Lambda_2 - 3\Lambda_1\varkappa_2 - 2\varkappa_2 H^2\frac{\partial\Lambda_1}{\partial H^2}\right];$$

ω_{T4}^2 and ω_{T5}^2 can be negative and of order

$$\omega_{T5}^2 \sim kL\Omega\tau\omega_{T4}^2, \quad \omega_{T5}^2 \sim \omega_{\varkappa 2}^2/kL(\Omega\tau)^2.$$

Considering the same case as in the Eq. (5.1) and making use of the inequality $\omega_{T5}^2 \ll \omega_{T1}\omega_{K2}$, we again obtain the acoustic branches and the four branches which are damped by thermal conductivity; when $(\omega_S, \omega_{K2}) \gg \max(\omega_{T4}, \omega_A)$ these satisfy the equation

$$(\omega^2 - \omega_A^2)^2 - \omega_i^2\omega_{T4}^2 = 0. \quad (5.7)$$

Whence

$$\omega = \tfrac{1}{2}\left(\pm\omega_A \pm \sqrt{\omega_{T4}^2 + 4\omega_A^2}\right). \quad (5.8)$$

When $\omega_A \gg \omega_{T4}$, we can limit ourselves to Eq. (5.7) if

$$\frac{\omega_A}{\omega_{T4}} \ll \min\left(\frac{\omega_s}{\omega_A}, \frac{\omega_{\varkappa 2}}{\omega_A}\right).$$

Then we have

$$\omega = \pm\omega_A \pm \tfrac{1}{2}\omega_{T4}. \quad (5.9)$$

Since $\Lambda_1 > 0$, $\Lambda_2 < 0$ and if one of the angles $(\widehat{\mathbf{k}, \nabla T})$, $(\widehat{\mathbf{k}, [\mathbf{H} \times \nabla T]})$ is acute and the other is obtuse the frequency in Eq. (5.9) is real; however, if both angles are acute or obtuse an instability arises. The maximum growth rate occurs when the plane containing the vectors **k** and ∇T is perpendicular to **H** and the angle $|(\widehat{\mathbf{k}, \nabla T})| = \pi/4$.

In the other limiting case the frequencies are given by

$$\omega_{1,2}^2 = \omega_{T4}^2, \quad \omega_{3,4}^2 = \omega_A^4/\omega_{T4}^2. \quad (5.10)$$

The conditions for instability and maximum growth rate are the same as before. When $\omega_{T4} \gg \omega_S$ we obtain the same four frequencies (5.10).

6. CASE IN WHICH RADIATIVE THERMAL CONDUCTIVITY PREDOMINATES

If Eq. (2.8) is satisfied we obtain the dispersion relation

$$\omega^7 + i\gamma\omega_{\varkappa 2}\omega^6 - (\omega_s^2 - i\gamma\omega_\varkappa\omega_{T6})\omega^5 - i\omega_\varkappa\omega^4(\omega_s^2 - \gamma\omega_{T7}^2) + \omega_s^2\omega^3(2\omega_A^2 - i\omega_\varkappa\omega_{T6}) + i\omega_\varkappa\omega_s^2\omega^2(2\omega_A^2 - \omega_{T7}^2) - \omega_s^2\omega_A^2\omega(\omega_A^2 - i\omega_\varkappa\omega_{T6}) - i\omega_\varkappa\omega_s^2\omega_A^4 = 0, \quad (6.1)$$

where

$$\omega_{T6} = -c\Lambda_2(\mathbf{k}[\mathbf{H}\nabla T]),$$

$$\omega_{T7}^2 = (ck\Lambda_2)^2(\mathbf{H}\nabla T)[2(\mathbf{H}\nabla T) - k^{-2}(\mathbf{kH})(\mathbf{k}\nabla T) + 2k^{-2}[\mathbf{H}\nabla T]^2(\mathbf{H}\nabla T)\,\partial\ln\Lambda_2/\partial H^2];$$

$\omega_{T6} \sim \omega_{T7} \sim \omega_K/kL\Omega\tau$ in order of magnitude and ω_{T7}^2 can be negative.

If $(\omega_S, \omega_K) \gg \max(\omega_A, \omega_{T6})$ the hydrothermomagnetic branches can be separated

$$\omega^4 + \omega^3\omega_{T6} - (2\omega_A^2 - \omega_{T7}^2)\omega^2 - \omega\omega_A^2\omega_{T6} + \omega_A^4 = 0. \quad (6.2)$$

In the limiting case $\omega_A \gg \omega_{T6}$ we obtain the Alfvén waves. The correction to the frequency $\omega = \omega_A$ can be computed from Eq. (6.2) when

$$\min\left(\frac{\omega_s}{\omega_A}, \frac{\omega_\varkappa}{\omega_A}\right) \gg \frac{\omega_A}{\omega_{T6}}.$$

The frequency is given by

$$\omega = \pm\omega_A - \tfrac{1}{4}\left(\omega_{T6} \pm \sqrt{\omega_{T6}^2 - 4\omega_{T7}^2}\right). \quad (6.3)$$

Since $\mathrm{Im}\,\omega_{T6} = 0$, the case most favorable for instability is that in which ∇T is in the same plane as the vectors **k** and **H**. Introducing the same angles as in Sec. 4 we have

$$\omega = \pm\omega_A \pm \tfrac{1}{2}\gamma_{max}\sqrt{-\cos\alpha\cos\beta\cos(\beta+\alpha)}, \quad (6.4)$$

where

$$\gamma_{max} = ckH\,|\Lambda_2\nabla T|.$$

For a given β the growth rate is a maximum when $\beta + 2\alpha = 0, 2\pi$ for $|\beta| < \pi/2$ and for $\beta + 2\alpha = \pi, 3\pi$ when $|\beta| > \pi/2$. If the angle β is changed the maximum growth rate, $(1/2)\gamma_{max}$, is obtained when $\beta = 0, \pi$ (in this case we also have $\alpha = 0, \pi$). If $\omega_{T6} \sim \omega_{T7} \gg \omega_A$ then

$$\omega_{1,2} = -\tfrac{1}{2}\left(\omega_{T6} \pm \sqrt{\omega_{T6}^2 - 4\omega_{T7}^2}\right),$$

$$\omega_{3,4} = (\omega_A^2/2\omega_{T7}^2)\left(\omega_{T6} \pm \sqrt{\omega_{T6}^2 - 4\omega_{T7}^2}\right). \quad (6.5)$$

Growth occurs under the same conditions as in the preceding case, differing from that considered in Sec. 5 in that growth occurs for the traveling wave when $\omega_{T6} \neq 0$

$$\mathbf{H}' = \mathbf{H}_0' e^{\gamma t}\sin(\mathbf{kr} - \omega t),$$

rather than

$$\mathbf{H}' = \mathbf{H}_0' e^{\gamma t} \sin \mathbf{kr}.$$

Finally, when $\omega_{T6} \gg \omega_S$ the hydrothermomagnetic waves are again obtained in the form in Eq. (6.5).

The general result is that when $\Omega\tau \gg 1$ thermomagnetic waves exist when $\omega_{T1} \sim \omega_{T6} \sim \omega_{T7} \gg \omega_A$; this condition is equivalent to

$$L \ll \frac{s^2}{v_A \Omega} \frac{M}{m}.$$

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