AVERAGED LASER EQUATIONS AND THEIR STATIONARY SOLUTIONS

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Averaged equations for the electromagnetic field in a two-level active medium are deduced without expanding into eigenfunctions of the unperturbed system. Solutions are derived for stationary nonlinear oscillations in a plane layer. The frequency spectrum of such oscillations is found and the corresponding spatial distributions of the field amplitude and phase are determined.

INTRODUCTION

THE study of processes occurring in optical quantum generators (lasers) requires the solution of a system of nonlinear equations consisting of Maxwell's equations for the electromagnetic fields and the equations for the density matrix of the active medium. The difficulties in solving such a system have been overcome so far only for a certain class of problems involving primarily two-level systems. What has been accomplished is essentially the determination of the conditions for self-excitation of laser oscillation (the instability condition); moreover, these conditions may be found in the linear approximation[1,2]. The same is true of the nonlinear problem where it is usually assumed that the spatial structure of the field is given and determined by the eigenfunctions of the unperturbed system (in the absence of the active medium)[1,3]. As a result one obtains equations describing the oscillations of the nonlinear system with a finite number of degrees of freedom (usually one degree of freedom—the "single mode" approximation). Because of their smallness, the corrections to the field configuration due to the presence of the nonlinear active medium may be found by a perturbation method[4].

Although these idealizations allow one to answer certain important questions, their applicability to real systems is in general quite limited. Actually, since the laser is essentially a distributed system (whose dimensions are large compared to the wavelength of light), even a small nonlinearity may cause significant changes in the spatial distribution of the field, which should not be taken as known. Except for the case of running waves in an unbounded medium[5], and the case already mentioned of small perturbations of the field amplitude caused by the nonlinearities[5], the correct solutions of these nonlinear problems have not been obtained. There are a number of papers (cf. for example, [6,7]), dealing with a determination of the dependence of the field energy on the coordinates for stationary (monochromatic) oscillations in a plane active layer. However, in place of Maxwell's equations for the field these papers make use of a phenomenological equation for the average values of the flux and the radiation energy density [a nonlinear modification of the linear absorption law (Bouger's law)]. As will be shown below, these equations lead to incorrect results.

The present paper will treat several problems in laser theory involving the effect of the nonlinear medium on the structure of the field. It will be shown that the initial equations can (under very general assumptions which nearly coincide with the conditions for the applicability of these equations) be transformed to a system of lower order for quantities which vary slowly in time. As an example of the use of this system we will investigate the steady state processes in a plane slab with end walls of arbitrary reflectivity, which will be taken as a one-dimensional model of a laser. The solutions obtained define a finite number of possible nonlinear oscillations (modes), whose frequencies differ from the eigenfrequencies of the unperturbed system and whose amplitudes and phases depend on the coordinates [1)].

[1)]These stationary vibrations are analogous in the well known way to limit cycles (positions of equilibrium in the "slow" variable space) of fixed parameter systems and, under well known conditions, may be unstable[8]. Since superpositions of modes are not solutions, the question of the possibility of other stationary processes, for example, vibrations with periodically varying amplitude, remains open.

1. CONDENSED LASER EQUATIONS

We consider the interaction of the electromagnetic field with a system of molecules having two energy levels[2]. In this case the initial equations are written as follows

$$\operatorname{rot}\operatorname{rot}\mathbf{H}+\frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}+\frac{4\pi\varepsilon}{c^2}\frac{\partial^2\mathbf{M}}{\partial t^2}=0,$$

$$\operatorname{div}(\mathbf{H}+4\pi\mathbf{M})=0,$$

$$\mathbf{M}=\operatorname{Sp}(\hat{\mu}\hat{\rho})=\mu_{11}\rho_{11}+\mu_{22}\rho_{22}+\mu_{12}\rho_{21}+\mu_{21}\rho_{12},$$

$$\frac{\partial n}{\partial t}+\frac{1}{T_1}(n-n_0)=\frac{2i}{\hbar}\{(\mathbf{H}\mu_{21})\rho_{12}-(\mathbf{H}\mu_{12})\rho_{21})\},$$

$$\frac{\partial\rho_{12}}{\partial t}-i\omega_0\rho_{12}+\frac{1}{T_2}\rho_{12}=\frac{i}{\hbar}\{(\mathbf{H},\mu_{12})n+(\mathbf{H},\mu_{11}-\mu_{22})\rho_{12}\}, \tag{1}$$
(1)*

where $\mathbf{H}$ is the magnetic field, $\mathbf{M}$ is the magnetic moment of the molecules per unit volume, ϵ is the dielectric susceptibility of the medium $\hat{\mu}$ is the magnetic dipole moment matrix of the molecule ($\mu_{12}=\mu_{21}^*$), $\hat{\rho}$ is the density matrix ($\rho_{12}=\rho_{21}^*$), $n=\rho_{22}-\rho_{11}$ is the difference between the energy level populations, n_0 is a parameter depending on the temperature and the "illumination," $\hbar\omega_0$ is the energy difference between the upper and lower levels, and T_1 and T_2 are relaxation times. As usual we assume that the molecules are uniformly distributed and uniformly oriented in space[3]. We treat the case of a magnetic dipole interaction between molecules and the field; analogous equations hold for the case of an electric dipole interaction.

For all practically interesting cases the following condition is valid,

$$\hbar/T_1,\ \hbar/T_2,\ |(\mu\mathbf{H})|\ll\hbar\omega_0, \tag{2}$$

i.e., the nonlinear terms and the relaxation terms are small. Hence to a first approximation the nonlinear interaction of the field with the medium is significant only for processes whose frequency is close to ω_0, i.e.,

$$\rho_{12}=\sigma(\mathbf{r},t)e^{i\omega_0 t},\quad \mathbf{H}=\tilde{\mathbf{h}}(\mathbf{r},t)e^{i\omega_0 t}+\tilde{\mathbf{h}}^*(\mathbf{r},t)e^{-i\omega_0 t},$$
$$n=n(\mathbf{r},t),$$

where σ, n and $\tilde{\mathbf{h}}$ are functions which are slowly varying in comparison with $e^{i\omega_0 t}$.

Putting these expressions in (1) and using the proper method of averaging[9], we obtain a system of "condensed" equations for the quantities σ, n and $\tilde{\mathbf{h}}$ in the nonlinear active medium:

$$\operatorname{rot}\operatorname{rot}\tilde{\mathbf{h}}-\frac{\varepsilon\omega_0^2}{c^2}\tilde{\mathbf{h}}+\frac{2i\omega_0\varepsilon}{c^2}\frac{\partial\tilde{\mathbf{h}}}{\partial t}=\frac{4\pi\varepsilon\omega_0^2}{c^2}\sigma\mu_{21},$$
$$\operatorname{div}(\tilde{\mathbf{h}}+4\pi\sigma\mu_{21})=0,$$
$$\frac{\partial n}{\partial t}+\frac{1}{T_1}(n-n_0)=-\frac{2i}{\hbar}\sigma^*(\mu_{12}\tilde{\mathbf{h}})+\text{к. с.},$$
$$\frac{\partial\sigma}{\partial t}+\frac{1}{T_2}\sigma=\frac{i}{\hbar}(\mu_{12}\tilde{\mathbf{h}})n. \tag{3}$$

To second order in infinitesimal quantities, this system (3) is equivalent to the initial system for arbitrary initial and boundary conditions; moreover the present system is considerably simpler (i.e., it is of lower order)[4]. In particular it is important that terms depending on μ_{11} and μ_{22} have completely dropped out of the system (3). This result means that in interacting with an electromagnetic field of frequency close to ω_0 the molecule behaves like an object exhibiting axial symmetry (symmetry axis parallel to μ_{12}). It is easily seen that for the component of the field $\tilde{\mathbf{h}}$ perpendicular to μ_{12}, Eqs. (3) are linear (although they depend on the component $\tilde{\mathbf{h}}_{\parallel}$ parallel to μ_{12}). On the other hand the nonlinear equations for $\tilde{\mathbf{h}}_{\parallel}$ do not depend on $\tilde{\mathbf{h}}_{\perp}$. For the corresponding boundary conditions this allows one to treat the waves with $\tilde{\mathbf{h}}=\tilde{\mathbf{h}}_{\parallel}$ and $\tilde{\mathbf{h}}=\tilde{\mathbf{h}}_{\perp}$ independently.

Using (3) it is not difficult, for example, to obtain the well known results of the single mode approximation, which are usually obtained by expanding the field in (1) in terms of the eigenfunctions of the unperturbed resonator[1,3]. To do this it is necessary to fix the spatial structure of the field, i.e., to put $\tilde{\mathbf{h}}=\mathbf{h}_1(\mathbf{r})q(t)$ where $\mathbf{h}_1(\mathbf{r})$ is a known function.

2. THE SOLUTION FOR STEADY-STATE OSCILLATIONS

Using the equations derived, we now consider the stationary states of the laser, for which the amplitudes of the oscillations do not depend on time. Using the fact that the frequency of the process ω need not coincide with ω_0, we insert in (3)

$$\tilde{\mathbf{h}}=\mathbf{h}(\mathbf{r})e^{i\Delta\omega t},\quad \sigma=\sigma(\mathbf{r})e^{i\Delta\omega t},\quad n=n(\mathbf{r}), \tag{4}$$

where $\Delta\omega=\omega-\omega_0$ is a constant frequency offset. Then after some simple transformations we obtain the equations for the amplitude of the magnetic field:

[2]For a treatment of the usual two-level idealization cf.[1].

*rot = curl.

[3]In the opposite case one must average the system (1) over the distribution function of the active molecules.

[4]Note that for a solid T_2 is much less than any other relaxation time in the system ($T_2<10^{-11}$ sec). In a number of problems this allows one to neglect the derivative $\partial\sigma/\partial t$ in the third equation of (3) even in the nonstationary case; this again lowers the order of the system.

$$\operatorname{rot}\operatorname{rot}\mathbf{h} - k^2\mathbf{h} = \left[i\alpha k^2 \frac{\mu^*}{\mu^*}\left(\frac{\mu}{\mu}\mathbf{h}\right)\right] \Big/ \left[1 + \beta^2 \left|\left(\frac{\mu}{\mu}\mathbf{h}\right)\right|^2\right]. \tag{5}$$

Here

$$k^2 = \frac{\varepsilon\omega^2}{c^2}, \quad \alpha = \frac{4\pi T_2 n_0 |\mu|^2 (1 - iT_2\Delta\omega)}{\hbar\, [1 + T_2^2 (\Delta\omega)^2]},$$

$$\beta^2 = \frac{4T_1T_2|\mu|^2}{\hbar^2\, [1 + T_2^2(\Delta\omega)^2]};$$

where here and elsewhere we write μ in place of μ_{12}.

We will now consider the one dimensional problem. For simplicity we will consider μ to be real [5]. Let the magnetic field be parallel to μ. We choose the coordinate system so that the y-axis coincides with the direction μ. Furthermore let $\mathbf{h} \sim hy^0$ depend only on the coordinate x. Then from (5) we obtain

$$\frac{d^2\mathcal{H}}{dx^2} + k^2\mathcal{H} = -i\frac{\alpha k^2\mathcal{H}}{1 + \mathcal{H}\mathcal{H}^*}, \tag{6}$$

where we use $\mathcal{H}$ to indicate the dimensionless function βh. Since according to (3) the right hand side of (6) is small (for a solid, one in fact has $\alpha \lesssim 10^{-4}$) we can average over x.

We will seek a solution of the form

$$\mathcal{H} = c_1(x)\, e^{-ikx} + c_2(x)\, e^{ikx},$$

where c_1 and c_2 are slowly varying complex functions giving the amplitude and phase of the waves running in the positive and negative directions respectively. Then to first approximation in the small parameter we find the following equations

$$\frac{dc_{1,2}}{dx} = \pm\frac{\alpha k}{2} \times \left\langle\!\!\left\langle \frac{c_{1,2} + c_{2,1}\exp(\pm 2ikx)}{1 + |c_1|^2 + |c_2|^2 + c_1c_2^*\exp(-2ikx) + c_1^*c_2\exp(2ikx)} \right\rangle\!\!\right\rangle. \tag{7}$$

Here the double angle brackets designate averaging with respect to x over the period $2\pi/k$. Finally we obtain [6]

$$\frac{dc_{1,2}}{dx} = \pm\frac{\alpha k}{4} c_{1,2} \times \frac{[1 + 2(|c_1|^2 + |c_2|^2) + (|c_1|^2 - |c_2|^2)^2]^{1/2} \pm |c_2|^2 \mp |c_1|^2 - 1}{|c_{1,2}|^2\, [1 + 2(|c_1|^2 + |c_2|^2) + (|c_1|^2 - |c_2|^2)^2]^{1/2}}. \tag{8}$$

We put $c_{1,2}$ in the form $|c_{1,2}|e^{i\varphi_{1,2}}$; then the equations for the moduli ($|c_{1,2}|^2 = m_{1,2}$) separate:

$$\frac{dm_{1,2}}{dx} = \mp\frac{k\operatorname{Re}\alpha}{2}\frac{[1 + 2(m_1 + m_2) + (m_1 - m_2)^2]^{1/2} \pm m_2 \mp m_1 - 1}{[1 + 2(m_1 + m_2) + (m_1 - m_2)^2]^{1/2}}. \tag{9}$$

As can easily be shown from (8), the phases $\varphi_{1,2}$ are simply related to $m_{1,2}$ by

$$\frac{d\varphi_{1,2}}{dx} = \frac{\operatorname{Im}\alpha}{2\operatorname{Re}\alpha}\frac{d\ln m_{1,2}}{dx}. \tag{10}$$

Putting $\operatorname{Im}\alpha/\operatorname{Re}\alpha = -T_2\Delta\omega$, we obtain from (10)

$$\varphi_{1,2} = -\tfrac{1}{2}T_2\Delta\omega \ln m_{1,2} + D_{1,2}, \tag{11}$$

where $D_{1,2}$ are arbitrary constants.

We treat the amplitude variation first. With the following change of variables

$$u = m_1 - m_2,$$

$$v = \sqrt{1 + 2(m_1 + m_2) + (m_1 - m_2)^2} - 1 \tag{12}$$

Eqs. (9) are transformed to the system

$$u' = -\frac{kv\operatorname{Re}\alpha}{v + 1}, \quad v' = -\frac{ku\operatorname{Re}\alpha}{v + 1}, \tag{13}$$

where the integrals are easily put in the form

$$v^2 - u^2 = 4A,$$

$$u + \ln|u + \sqrt{u^2 + 4A}| = -kx\operatorname{Re}\alpha + B \tag{14}$$

(A, B are arbitrary constants).

Transforming back to the variables $n_{1,2}$ we obtain

$$(m_1 - A)(m_2 - A) = A,$$

$$m_{1,2} - A - \frac{A}{m_{1,2} - A} + \ln(m_{1,2} - A) - \frac{\ln A}{2} = \pm kx\operatorname{Re}\alpha, \tag{15}$$

where $m_{1,2} > A$. The origin is taken at the point $(B + \ln 2 + \ln\sqrt{A})/k\operatorname{Re}\alpha$, around which the whole pattern is symmetric. At this point $m_1 = m_2 = A + \sqrt{A}$, and the sum $m_1 + m_2$, proportional to the average (over the period $2\pi/k$) of the energy density of the field, is a minimum.

The constant A is determined from the boundary conditions (the quantity B effects only the origin of x). Let the region $-L_1 < x < L_2$ define an active layer in the oscillating regime, i.e., outside this layer there are outgoing waves only. Then $m_1/m_2 = r_1$ for $x = -L_1$ and $m_2/m_1 = r_2$ for $x = L_2$ where $r_{1,2}$ are the power reflectivities of the boundaries of the slab. Putting these relations in (15) it is not difficult to express A in terms of r_1, r_2 and the thickness of the layer $L = L_1 + L_2$.

Equations (15) define positive $m_{1,2}$ only for $A > 0$. From this it is not difficult to show that stationary oscillations are possible under the condition

[5]Note that if μ is complex the waves under consideration are elliptically polarized.

[6]Of course the oscillating terms in the right hand side of (7), proportional to $\exp(\pm 2ikx)$, make a significant contribution to the result of the averaging.

$$L > L_{cr} = \ln [(r_1 r_2)^{-1/2}]/k \operatorname{Re} \alpha. \qquad (16)$$

Since for $L \to L_{cr}$ we have $m_{1,2} \to 0$, expression (16) of course coincides with the condition for self-excited oscillations obtained in the linear theory[1,2].

The spatial variation of the phase of the wave depends, according to (11), on $m_{1,2}$. If the medium inside the layer is lossless then the boundary conditions for $\varphi_{1,2}$ have the form

$$(\varphi_1 - \varphi_2 - 2kx)_{x=-L_1, L_2} = \pi p_{1,2}, \qquad (17)$$

where $p_{1,2} = 0, \pm 2, \pm 4, \ldots$ if the dielectric constant of the external medium $\epsilon_0 > \epsilon$, and $p_{1,2} = \pm 1, \pm 3, \ldots$ if $\epsilon_0 < \epsilon$. Using (16) it is easy to determine the frequency deviation $\Delta\omega$ in (11) which corresponds to a given p. Finally we find for the oscillation frequency $\omega_p = \omega_0 + \Delta\omega$

$$\omega_p = \omega_0 \frac{\pi p + \omega_0 T_2 \ln [(r_1 r_2)^{-1/2}]}{2k_0 L + \omega_0 T_2 \ln [(r_1 r_2)^{-1/2}]}, \qquad k_0 = \frac{\omega_0 \sqrt{\varepsilon}}{c},$$

$$p = p_1 - p_2, \qquad (18)$$

and then the solution is completely defined by (15) and (11).

3. DISCUSSION OF RESULTS

The averaged equations allow one to find the spectrum of the steady state nonlinear oscillations (laser modes). Superpositions of these modes are, of course, not solutions. It is clear from (18) that the presence of the active medium leads to a condensation of the mode spectrum, leaving the modes equidistant. The values of ω_p do not depend on n_0 or T_1 [7]. If the quantity $2k_0L$ is a multiple of π then one of the frequencies ω_p (corresponding to $2k_0L = \pi p$) is equal to ω_0. It must be kept in mind that the number of stationary modes in the layer is finite. In fact, with increasing $\Delta\omega$ the quantity $\operatorname{Re}\alpha$ decreases, until finally (16) can not be satisfied.

According to (15) the difference between the amplitudes $m_{1,2}$ for the various modes is determined primarily by the quantity $T_2\Delta\omega$ which occurs in $\operatorname{Re}\alpha$ and β^2. The number of modes having significant amplitude depends on $\delta = T_2(\Delta\omega)_{min}$, where $(\Delta\omega)_{min}$ is the frequency interval between neighboring modes. If $2k_0L + T_2\Delta\omega \ln[(r_1r_2)^{-1/2}] \gg \omega_0 T_2$ then $\delta \ll 1$ and there is a large number of possible stationary states with similar distributions $m(x)$. If on the other hand $2k_0L + T_2\Delta\omega \ln[(r_1r_2)^{-1/2}] \ll \omega_0 T_2$, then $\delta \gg 1$ and the amplitudes of all modes except the fundamental are small; the frequency of the primary mode is equal to or close to ω_0. For a solid $\omega_0 T_2 \sim 10^4 - 10^5$, $2k_0L \sim 10^5$, i.e., the number of possible modes having appreciable amplitude is not very large (1—10).

For each mode, m_1 increases and m_2 decreases with increasing x. We note that the minimum averaged energy density, which is proportional to $w = m_1 + m_2$, occurs close to the less transparent boundary; the largest value of w occurs at the more transparent boundary [8]

In Fig. 1 we show the dependence of the dimensionless variables m_1 and w on the dimensionless coordinate $x' = kx \operatorname{Re}\alpha$. The left-hand boundary of the layer is assumed to be ideally reflecting (and the minimum of ω lies at this boundary); that is, $r_1 = 1$. A is a parameter determined by the quantity $r_2 = r$ and the dimensionless thickness of the layer $L' = kL \operatorname{Re}\alpha$. For fixed A the curves constructed are the same for all modes. However in a given layer the amplitudes of the modes are not all the same since they have different L′ values (hence different A values).

The efficiency of the laser is characterized by

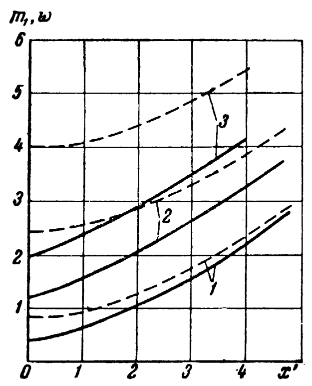

FIG. 1

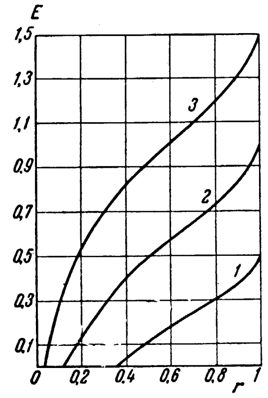

FIG. 2

FIG. 1. The dependence of m_1 (solid lines) and w (dotted lines) on $x' = kx\operatorname{Re}\alpha$ for 1 – A = 0.1; 2 – A = 0.5; and 3 – A = 1. For L′ = 1 the curves 1 – 3 correspond to values of r of 0.4, 0.62, and 0.77 respectively.

FIG. 2. The dependence of the dimensionless output energy flux density $E = m_1(1 - r)$ on r for 1 – L′ = 0.5, 2 – L′ = 1, and 3 – L′ = 1.5.

[7]Note that the frequencies (18) do not coincide with the frequencies for self-excitation of the corresponding laser modes (except for the "threshold" mode for which $L = L_{cr}$ and $m_{1,2} \equiv 0$).

[8]If $r_1, r_2 \to 1$, then, as expected, the energy of the field in the slab grows without limit. It is simple to show that for $(1 - r_{1,2})\, kL\operatorname{Re}\alpha \ll 1$, the functions $m_{1,2}(x)$ differ very little from constants. The solutions in this case agree with the results of Kuznetsova and Rautian[4] obtained by a perturbation method.

the output power. For a layer with $r_1 = 1$ the flux density from the righthand boundary is proportional to $E = m_1(1 - r)$. In this case eliminating A for $x = L_2$ from (15), we get the relation between r and E:

$$r = \frac{E + 1 - \exp[-2(L' - E)]}{E - 1 + \exp[2(L' - E)]}. \tag{19}$$

It follows that the function $E(r)$ increases monotonically from zero at $r = r_{cr}$ [corresponding to (16)] up to the value L' for $r = 1$. The function (19) is shown in Fig. 2 for various L'.

As has already been pointed out there are a number of papers dealing with the question of the stationary states of a plane active layer (cf. [6,7]) which, in place of Maxwell's equations for the field, use phenomenological equations for the time and spatial averages of the flux and the radiation energy density (the question of the spectrum of the oscillations and their phases was in general not treated). Corresponding expressions may be obtained formally from (7) if, in carrying out the averaging in these equations one omits the oscillating terms proportional to $\exp(\pm 2ikx)$ (see footnote 5). It is easy to see that in the nonlinear case these solutions are valid only for running waves in an unbounded medium when one of the quantities $c_{1,2}$ is equal to zero.

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