REGGE POLES AND NUCLEAR RESONANCE REACTIONS. I

V. S. POPOV and É. I. DOLINSKIĬ

Institute of Theoretical and Experimental Physics; Institute of Nuclear Physics, Moscow State University

Submitted to JETP editor August 10, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 46, 970-984 (March, 1964)

Resonances in low energy nuclear reactions are considered from the standpoint of the Regge pole philosophy. Formulas for the amplitude of reactions of the type $A + x \rightarrow B + y$ are given which, along with the main Breit-Wigner term, contain corrections as a consequence of the analyticity of the partial amplitudes $f_\lambda(E)$ in λ (effect of the proper asymmetry of the resonance level). The possibility of observing the effect of the proper asymmetry in the angular distribution of the reaction products is discussed. The signature of the resonance level is considered briefly. An attempt is made to extract from the data of the phase shift analysis of the $C^{12}(\alpha, \alpha)C^{12}$ elastic scattering parts of the trajectories of the three Regge poles responsible for the resonances.

1. INTRODUCTION

COMPLEX angular momenta were first introduced by Watson [1] and Sommerfeld [2] in the discussion of the diffraction of radio waves around the surface of the earth. By this method the problem of diffraction could be solved efficiently. Indeed, the usual expansion of the wave function in a series of partial waves f_n is here practically useless, since terms up to the order $n \sim kR$ (R is the radius of the earth, k the wave number) are important. By analytic continuation of f_n into the complex plane of the angular momentum $n = \lambda$, Sommerfeld and Watson represented the wave function in the form of a rapidly converging series in terms of the poles of f_λ. The angular dependence of the wave function, which would seem to be unusually complicated on account of the enormous number of partial waves f_n of the same order of magnitude, was thus represented by a simple analytic function, viz, the Legendre function $P_\lambda(-z)$, where, however, λ does not take on one of the "physical" values ($\lambda = 0, 1, 2, \ldots$), but is a complex number. (Now we would say that this value λ is the "Regge pole" closest to the real axis.)

Some years later, Regge [3] proved the analyticity of the partial amplitudes f_λ as a function of the angular momentum λ for a wide class of potentials in nonrelativistic potential scattering. Here all singularities of f_λ in the right half-plane of the variable λ ($\mathrm{Re}\,\lambda > -1/2$) are simple poles which lie in the upper half-plane ($\mathrm{Im}\,\lambda > 0$) for $E > 0$ and on the real axis for $E < 0$. Thereafter the distribution of the poles and the trajectories described by them as the energy is varied were discussed for different potentials: the square well [4,5] and the Yukawa potential. [6-8]

In a number of papers, [9-12] the analytic properties of the scattering amplitude in λ were used in the investigation of the asymptotic behavior of the cross sections of various processes at high energies (for a review of the work in this direction, see [13]).

Recently Shapiro (private communication) and Rebolia and Viano [14] called attention to the fact that from the standpoint of the theory of complex angular momentum the angular distribution of the products of nuclear resonance reactions must have a forward-backward asymmetry. This asymmetry arises from the circumstance that the angular momentum of the resonance level is a complex number, and not from the interference of the Breit-Wigner amplitude with the scattering amplitude for potential scattering and neighboring resonances. We therefore call it the "proper asymmetry" of the resonance level (p.a.l.). However, the p.a.l. can be distorted if the contribution from one resonance level (Regge pole) is compensated by the contribution from far resonances and also by the known integral term appearing in the "Reggeized" form of the amplitude [see below, formula (2)].

Section 2 of the present paper is devoted to the problem of the compensation. It is shown that the compensation is unimportant for $kR \gg 1$ (R is the nuclear radius) in a wide region of scattering angles, so that the p.a.l. can be observed experimentally. The experimental observation of the p.a.l. is of great interest, since it gives the possibility of establishing the motion of the Regge poles at energies close to resonance. In the same section, we shall discuss the problem of the signature of the resonance level and its effect on the angular distribution.

In Sec. 3, we discuss, from the point of view of the theory of complex angular momenta, the data of the phase shift analysis of the elastic resonance scattering of α particles from C^{12} at α energies up to 5 MeV. It is shown that one can obtain from the energy dependence of the small nonresonant phases the trajectories of the Regge poles corresponding to the first three excited levels of the compound O^{16} nucleus.

2. PROPER ASYMMETRY OF A RESONANCE LEVEL AND POSSIBILITY OF ITS EXPERIMENTAL OBSERVATION

Let us consider a nuclear reaction of the type $A + x \to B + y$ between spinless particles, neglecting the Coulomb interaction.[1)] We expand the reaction amplitude $f(E, z)$ in a series of partial waves:

$$f(E, z) = \frac{1}{k} \sum_{n=0}^{\infty} (2n+1) f_n(E) P_n(z),$$

$$f_n(E) = \frac{k}{2} \int_{-1}^{1} f(E, z) P_n(z)\, dz = e^{i\delta_n(E)} \sin \delta_n(E), \qquad (1)$$

where E is the total kinetic energy of the particles A and x, and $z \equiv \cos\theta$ is the cosine of the angle between the momenta of the particles x and y in the center of mass system.

We assume that the partial amplitudes $f_n(E)$ can be continued analytically from the integer values of the angular momentum $\lambda = n$ into the complex plane of the variable λ, where on the large circle in the right hand plane ($\mathrm{Re}\,\lambda > -\frac{1}{2}$)

$$(2\lambda + 1) f_\lambda(E) P_\lambda(-z)/\sin \pi\lambda \to 0$$

for arbitrary z in the physical region ($-1 \le z \le 1$), and the half-plane $\mathrm{Re}\,\lambda > N$ (N is an arbitrarily large, fixed number) is free of singularities of $f_\lambda(E)$. In this case the analytic continuation from the integer values $\lambda = n$ is unique.[15] We assume that the only singularities of $f_\lambda(E)$ for $\mathrm{Re}\,\lambda > -\frac{1}{2}$ are simple poles in the points $\lambda = \alpha_i(E)$, and denote by $r_{\alpha_i}(E)$ the residue of $f_\lambda(E)$ at the pole α_i. Using the Sommerfeld-Watson transformation [2,3] we represent the reaction amplitude $f(E, z)$ in the form

$$f(E, z) = -\frac{1}{k} \sum_i (2\alpha_i + 1) r_{\alpha_i} \frac{\pi}{\sin \pi\alpha_i} P_{\alpha_i}(-z)$$

$$- \frac{1}{2ik} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{(2\lambda+1) f_\lambda(E) P_\lambda(-z)}{\sin \pi\lambda}\, d\lambda. \qquad (2)$$

It must be emphasized that the above-mentioned properties of the partial amplitudes leading to (2) are rigorously proved in the nonrelativistic theory only for scattering from a potential which consists of a superposition of Yukawa potentials,[3] and in the relativistic theory are derived from the Mandelstam representation with a finite number of subtractions.[9] These proofs are not applicable to the amplitudes of nuclear reactions, since these have anomalous singularities in the invariant variables (the momentum transfer, etc.). We shall, however, assume that the nuclear reaction amplitude $f(E, z)$ can be written in the "Reggeized" form (2) and consider the associated consequences.

Let us assume that at some complex energy $W = E_0 - i\Gamma/2$ one of the poles $\alpha(E)$ goes through a positive integer value of the angular momentum $\lambda = l$: $\alpha(W) = l$.[2)] If Γ is not very large, the pole $\alpha(E)$ at the real energy E close to E_0 will be located at a point in the complex plane close to $\lambda = l$:

$$\alpha(E) = l + \nu(E), \quad |\nu(E)| \ll 1, \quad \nu(W) = 0. \qquad (3)$$

If the other poles $\alpha_i(E)$ in the complex λ plane are far away from the real axis, then the term corresponding to the pole $\alpha(E)$ in the sum over poles in (2) will be large (of order $1/\nu$) compared with the remaining terms [of order $\exp(-\pi\,|\,\mathrm{Im}\,\alpha_i\,|)$] and will make the main contribution to the reaction amplitude. We isolate this term and write the amplitude in the form

$$f(E, z) = -\frac{1}{k} (2\alpha + 1) r_\alpha(E)$$

$$\times \left[\frac{\pi}{\sin \pi\alpha} P_\alpha(-z) + g(E, z) \right], \qquad (4)$$

where $g(E, z)$ stands for the remaining terms in (2). Let us consider the pole contribution to the amplitude:

$$f_\alpha^{(p)}(E, z) = -\frac{1}{k} (2\alpha + 1) r_\alpha(E) \frac{\pi}{\sin \pi\alpha} P_\alpha(-z). \qquad (5)$$

[1)]The Coulomb interaction is taken into account in a subsequent paper of the authors.

[2)]This implies that the partial amplitude $f_l(E)$ has a pole in the energy at $E = W$.

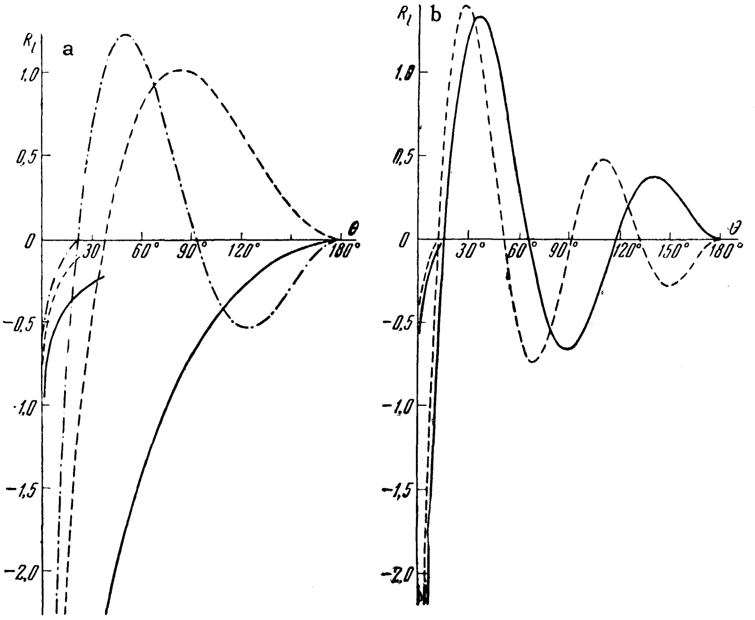

FIG. 1. Behavior of the function $R_l(\cos\theta)$ for various values of l: a — solid curve for $l = 0$, dotted curve for $l = 1$, dash-dotted curve for $l = 2$; b — solid curve for $l = 3$, dotted curve for $l = 4$; for small θ the curves are drawn to the scale 1:10.

According to (3), we can write the amplitude $f_\alpha^{(p)}$ in the form

$$f_\alpha^{(p)}(E, z) = -\frac{1}{k}(2\alpha+1)\, r_\alpha(E) \times \left[\frac{P_l(z)}{\nu} + R_l(z) + O(\nu)\right], \qquad (6)$$

where

$$R_l(z) = (-)^l\, [\partial P_\alpha(-z)/\partial\alpha]_{\alpha=l}. \qquad (7)$$

If Γ is small, then $\nu(E) \approx \alpha'(E_0)(E - E_0 + i\Gamma/2)$ for $E \approx E_0$, and the main term $f_\alpha^{(p)}(E, z)$ takes the form

$$-\frac{1}{k}(2l+1)\frac{r_l/\alpha'(E_0)}{E - E_0 + i\Gamma/2} P_l(z), \quad \frac{r_l}{\alpha'(E_0)} = \frac{\Gamma}{2}, \qquad (8)$$

i.e., it coincides with the Breit-Wigner amplitude for an isolated resonance. We note that singularities in λ of a type different from the simple moving poles do not lead to the Breit-Wigner formula in passing near the integer values $\lambda = l$.

The term $R_l(z)$ in (6) has no natural analog in the usual theory of resonance reactions. It leads to perfectly definite deviations of the angular distribution from the Breit-Wigner formula. Its occurrence is a consequence of the assumed analyticity of the partial reaction amplitudes in the angular momentum. We shall call the quantity $R_l(z)$ the "proper asymmetry" of the resonance level. The explicit form of $R_l(z)$ is given in Appendix A. Figure 1 shows the function $R_l(z)$ for $l = 0, 1, 2, 3$, and 4.

In cases where the function $g(E, z)$ in (4) can be neglected, the differential reaction cross section is near the resonance given by the square of the pole contribution to the amplitude and has the form

$$\frac{d\sigma}{d\Omega} = \left|\frac{(2\alpha+1)\, r_\alpha(E)}{k\nu(E)}\right|^2 \times [P_l^2(z) + 2\mathrm{Re}\,\nu\cdot P_l(z)\, R_l(z) + O(\nu^2)]. \qquad (9)$$

The deviation of the angular distribution from the simple form $d\sigma/d\Omega \sim [P_l(z)]^2$ predicted by (9) would, in the language of the usual resonance theory of nuclear reactions, require the introduction of interference terms between the given resonance and a large number of neighboring resonances. A comparison of (9) with experiment would allow a direct determination of the energy variation of the real part of the Regge pole across the resonance.[3)]

However, the pole contribution to the amplitude $f_\alpha^{(p)}(E)$ cannot represent the amplitude of a real physical process in the entire range of angles $(-1 \le z \le 1)$, for the following reasons. Since $P_\alpha(-z)$ has a logarithmic singularity at $z = 1$, $f_\alpha^{(p)}(E, z)$ goes to infinity for $\theta = 0$ (forward scattering). This is reflected in the circumstance that the partial amplitudes $f_n^{(p)}$ corresponding to $f_\alpha^{(p)}(E, z)$ fall of too slowly for $n \to \infty$:

$$f_{\alpha n}^{(p)}(E) = r_\alpha \frac{2\alpha+1}{2n+1}\left(\frac{1}{n-\alpha} + \frac{1}{n+\alpha+1}\right) \sim \frac{1}{n^2}. \qquad (10)$$

The correct reaction amplitude is finite for all physical values of z and its partial amplitudes

[3)]In this section we assumed for simplicity that at a given energy only one pole $\alpha(E)$ comes close to an integer value. The results are easily generalized to the case of several such poles (cf. the analysis of the experimental data, Sec. 3).

decrease exponentially for $n \to \infty$ owing to the finite range of the nuclear forces. Therefore the term $g(E, z)$ in (4) must guarantee the correct behavior of $f(E, z)$ for $z = 1$ and compensate the pole-type partial amplitudes $f^{(p)}_{\alpha n}(z)$ for large n. We call this the effect of the compensation of the distant pole phases.

It is easy to show that the function $g(E, z)$ does indeed compensate the logarithmic infinity of the pole contribution to the amplitude, by substituting in (2) the expansion of $P_\alpha(-z)$ for $z \to 1$,

$$P_\alpha(-z) = \frac{\sin \pi\alpha}{\pi} \ln \frac{1-z}{2} + \text{const}$$

$$+ O[(1-z)\ln(1-z)],$$

and taking into account that, by Cauchy's theorem, the coefficient of $\ln[(1-z)/2]$ in the expression

$$f(E, z) = -\left\{\sum_i (2\alpha_i + 1) r_{\alpha_i} + \frac{1}{2\pi i} \int_{-1/2 - i\infty}^{-1/2 + i\infty} (2\lambda + 1) f_\lambda(E)\, d\lambda\right\}$$

$$\times \ln \frac{1-z}{2} + \varphi(E, z) \qquad (11)$$

goes to zero, while the remainder is finite for $z = 1$.

For an estimate of the effect of compensation we write the partial amplitude $f_n(E)$ in the form

$$f_n(E) = f^{(p)}_{\alpha n}(E)\, \xi_n(E). \qquad (12)$$

The quantity ξ_n, considered for complex values $n = \lambda$, is an analytic function of λ with the following properties: ξ_n is exponentially small on the real axis for sufficiently large n ($n > L$), and $\xi_l \approx 1$ for $n = l$ (l is the angular momentum of the resonance level). It will be seen from the following that $L \gg l$ in the cases of interest.

We shall assume that ξ_λ is a smooth function of λ in the interval from zero to L. This is in line with the hypothesis of the analyticity of f_λ.

We add some comments on the magnitude of the cut-off parameter L. The nuclear reaction amplitude receives contributions from resonance and direct processes. The amplitudes of the direct processes are given by Feynman graphs with nuclear form factors in the vertices.[16,17] The partial amplitudes of the direct processes $f^{(d)}_\lambda(E)$ become exponentially small for $\lambda = n > L_d$, where L_d is determined by the analytic properties of the direct reaction amplitude in the momentum transfer. These properties depend on the form of the vertex functions and the position of the closest singularities of the Feynman graphs in z ($z = z_0$): $L_d = L_d(kR, z_0)$. Since the amplitude of the direct reaction $f^{(d)}(E, z)$ is finite for $z = 1$, it cannot compensate the logarithmic singularity of the pole contribution to the amplitude. Therefore the cut-off parameter L in (12) is not connected with L_d.

As shown in Appendix B, the partial amplitudes $f^{(d)}_\lambda(E)$ have no singularities for $\operatorname{Re}\lambda > -1/2$ and therefore the direct processes are entirely contained in the integral along the line $\operatorname{Re}\lambda = -1/2$ in (2). However, it follows from (11) that this integral contains, besides the direct processes, also a part leading to the compensation of the distant pole phases. This part of the integral has no simple physical interpretation, and in estimating L we must therefore take recourse to qualitative model-type considerations. For potential scattering [18] and scattering from a black nucleus [19] it is known that a sharp cut-down of the phases begins for $n > kR$, where R is the radius of the nucleus. We assume therefore that the compensation of the pole phases in well-defined resonance reactions begins at $n > L \sim kR$ (with $kR > 1$).

For an estimate of the effect of compensation in line with our assumption about the behavior of ξ_n we consider a crude model:

$$\xi_n = \begin{cases} 1 & 0 \leqslant n \leqslant L-1 \\ 0 & n \geqslant L \end{cases}. \qquad (13)$$

Then

$$g(E, z) = \frac{1}{(2\alpha+1) r_\alpha(E)} \sum_{n=L}^{\infty} (2n+1) f^{(p)}_{\alpha n}(E) P_n(z). \quad (14)$$

Substituting the known asymptotic form of the Legendre polynomial $P_n(z)$ for $n \gg 1$,

$$P_n(z) = \left(\frac{2}{\pi n \sin\theta}\right)^{1/2} \left\{\cos\left[\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + O\left(\frac{1}{n}\right)\right\},$$

which is valid for $0 < \varepsilon \le \theta \le \pi - \varepsilon$, $\varepsilon \gg 1/n$, and retaining the main terms in the expansion in L^{-1}, we obtain

$$g(E, z) = 2\left(\frac{2}{\pi \sin\theta}\right)^{1/2} \sum_{n=L}^{\infty} \frac{\cos[(n + 1/2)\theta - \pi/4]}{n^{3/2}}$$

$$= 2\left(\frac{2}{\pi \sin\theta}\right)^{1/2} \operatorname{Re}\{e^{i[(L+1/2)\theta - \pi/4]}\, \Phi(e^{i\theta}, 3/2, L)\}, \qquad (15)$$

$$\Phi(z, s, L) = \sum_{n=0}^{\infty} \frac{z^n}{(n+L)^s}. \qquad (16)$$

Using the integral representation for the function $\Phi(z, s, L)$,[20] we easily obtain its asymptotic expansion for $L \gg 1$, which has the form ($z = e^{i\theta}$)

$$\Phi(e^{i\theta}, s, L) = \frac{ie^{-i\theta/2}}{2L^s \sin(\theta/2)}\left[1 + O\left(\frac{s}{2L \sin(\theta/2)}\right)\right]. \quad (17)$$

For $\theta \gg 1/L$ we finally obtain from (17) and (15) the asymptotic expansion of $g(E, z)$ for $L \gg 1$:

$$g(E, z) = \left(\frac{2}{\pi \sin \theta}\right)^{1/2} \frac{\cos(L\theta + \pi/4)}{L^{3/2}\sin(\theta/2)} \times \left[1 + O\left(\frac{1}{L}, \frac{\alpha}{L}, \frac{1}{L\theta}\right)\right]. \quad (18)$$

From (18) we find the upper estimate

$$|g(E, z)| \leqslant \left(\frac{2}{\pi \sin \theta}\right)^{1/2} \Big/ \left(L^{3/2} \sin \frac{\theta}{2}\right). \quad (19)$$

It follows from this that $|g(E, z)/R_l(E, z)| \leq \delta$. in the region of angles for which

$$F_l(\theta) = (\sin \theta)^{1/2} \sin \frac{\theta}{2} |R_l(\cos \theta)| \geqslant \frac{0.8}{L^{3/2}\delta}, \quad (20)$$

If we take $\delta \ll 1$, we can neglect the effect of compensation in this region of angles and approximate the exact amplitude $f(E, z)$ by the pole contribution to the amplitude (5).

Figure 2 shows the function $F_l(\theta)$ for $l = 0$, 1, 2, 3, and 4. We see, for example, that for $l = 0$ the range of angles θ in which the proper asymmetry of the level is distorted by no more than 10% due to the compensation is $35° > \theta < 85°$ for $L = 6$ and $15° < \theta < 120°$ for $L = 10$. On the whole, we can for $L \gg 1$ and excluding the intervals of angles near $\theta = 0$ and the zeros of $R_l(\cos \theta)$, use to a good approximation formula (9) for the description of the angular distribution.

We note that the effect of the p.a.l. is clearly not observable for small energies ($kR \ll 1$). In this case the phases $\delta_n \sim (kR)^{2n+1}$ and decrease very rapidly as n is changed by unity, beginning with $n = 0$ (this is a trivial consequence of the centrifugal barrier). On the other had it is known that for $k \to 0$ the Regge poles condense near $\lambda = -1/2$ [8,10,21] and it becomes understandable that no single Regge pole can predominate in the reaction amplitude for $kR \ll 1$.

Since $L \gtrsim kR$, the effect of the proper asymmetry of the level is best observed in its undistorted form at a large energy E. However, one must stay in a region of energies E where there are still isolated resonances, since otherwise the angular distribution will be determined by several Regge poles. Therefore it is most convenient to study light nuclei close to the magic nuclei at resonance energies of the order of a few MeV.

In connection with the form (2) of the amplitude we remark the following. In the analytic continuation from the physical values $\lambda = n$ into the complex λ plane we require of the function $f_\lambda(E)$ that it be bounded on the large circle for Re Re $\lambda > -1/2$. This is a necessary condition for the

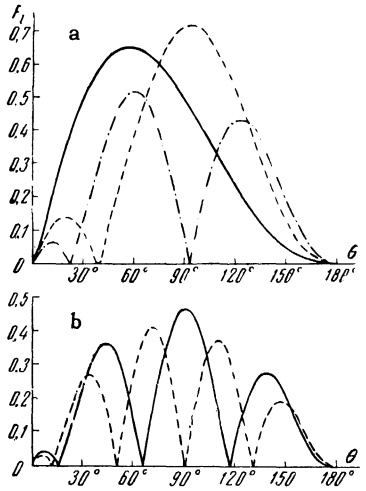

FIG. 2. Behavior of the function $F_l(\theta)$ for different values of l: a—solid curve for $l = 0$, dotted curve for $l = 1$, dash-dotted curve for $l = 2$; b—solid curve for $l = 3$, dotted curve for $l = 4$.

uniqueness of the analytic continuation from the discrete set $\lambda = n$.[15] In the relativistic theory, this requirement forces one to make a separate analytic continuation from even and odd values of $\lambda = n$, introducing two different analytic functions f_λ^+ and f_λ^-.[9] The Regge poles for f_λ^+ and f_λ^- describe, in general, different trajectories as the energy is varied, and each trajectory is characterized by a new quantum number σ ($\sigma = \pm 1$), called signature.

In the low energy nuclear resonance reactions under consideration we have cases where the Regge pole definitely has a signature.[4)] If the resonance reaction goes through a compound level with at least one decay channel into two identical particles (for example, $Be^{8*} \to 2\alpha$), the expansion of the reaction amplitude contains only partial amplitudes of a given parity. In this case the compound level lies on a Regge trajectory with a definite signature, and for spinless particles the resonance occurs only if the signature of the trajectory coincides with the parity of the level $[\sigma = (-)^l]$. Then the function $R_l(z)$ in formula (9) for the differential cross section must be replaced by

$$R_l^\sigma(z) = \frac{1}{2}[R_l(z) + \sigma R_l(-z)] = P_l(\cos\theta)\ln\frac{\sin\theta}{2} + \frac{1}{2}[V_l(\cos\theta + \sigma V_l(-\cos\theta)], \quad \sigma = (-)^l, \quad (21)$$

and the angular distribution becomes symmetric

4)This circumstance was pointed out to us by I. S. Shapiro.

about 90°. An analogous situation obtains also in the case of nonrelativistic scattering, where the signature of the Regge pole comes in if the potential has an exchange part. If the resonance level has no decay channel into identical particles, then the corresponding Regge trajectory has evidently no signature.

3. COMPARISON WITH EXPERIMENT

The most direct test of the assumption of Sec. 2 about the smooth compensation of the pole phases and hence the existence of the p.a.l. $R_l(z)$ [formula (9)] is the exact measurement of the angular distribution of the reaction products at various energies within the resonance peak. In particular, we have for a narrow resonance $\nu(E) \approx \alpha'(E_0)(E - E_0 + i\Gamma/2)$, and (9) predicts a definite energy dependence for the p.a.l. effect. As is seen from the analysis of the elastic scattering of α particles from C^{12}, typical values of ν in the neighborhood of the resonance are of the order 0.01 to 0.1. Therefore, according to (9), the interference of the main term in the amplitude with the p.a.l. term can lead to energy dependent corrections in the angular distribution from a few percent to about 20%. Unfortunately, we had no data on precisely measured angular distributions at the resonances.

Another possible test of the hypothesis of the smooth compensation of the pole phases $\delta_{\alpha n}^{(p)}(E)$ for values of n close to the angular momentum of the resonance level l is the determination of the Regge trajectory from the experimental data on the small nonresonant scattering phases. In the case of an isolated resonance caused by a single Regge pole $\alpha(E)$ we obtain from (10) and (12), taking into account that $\xi_l \approx 1$,

$$f_n(E)/f_l(E) = \xi_n \nu(E)(2l+1)/[(l-n)(l+n+1)]. \quad (22)$$

Here $\alpha(E) = l + \nu(E)$, $|\nu(E)| \ll 1$, $n \neq l$; l is the spin of the resonance level. In the approximation of the crude model (13) for ξ_n, we have

$$\nu(E) = \frac{(l-n)(l+n+1)}{2l+1}\frac{\sin\delta_n}{\sin\delta_l}e^{i(\delta_n-\delta_l)}, \quad n \neq l. \quad (23)$$

With the help of this formula one can from each nonresonant phase $\delta_n(E)$, establish a section of the Regge trajectory in the neighborhood of the resonance.[5)]

If the quantity ξ_n is indeed weakly dependent on n for values of n close to l, then the values of $\nu(E)$ calculated from different phases $\delta_n(E)$ must be close to one another. We note, however, that the deviations of the true values of $\xi_n(E)$ from unity manifest themselves more in the values of $\nu(E)$ obtained from (23) than in the estimates of Sec. 2 of the range of angles in which formula (9) holds for the angular distribution.

Formulas analogous to (23) can easily be obtained for the case of several poles close to integer values (overlapping resonances). We quote the formulas for the case of two poles corresponding to resonances with the spins l_1 and l_2:

$$\nu_{l_1}(E) = C_1[(l_2-m)(l_2+m+1)f_m(E) - (l_2-n)(l_2+n+1)f_n(E)]/f_{l_1}(E), \quad (24)$$

$$\nu_{l_2}(E) = -C_2[(l_1-m)(l_1+m+1)f_m(E) - (l_1-n)(l_1+n+1)f_n(E)]/f_{l_2}(E).$$

Here $m, n \neq l_1, l_2$; $f_m(E)$, $f_n(E)$ are the nonresonant partial amplitudes for the angular momenta m, n:

$$f_n(E) = e^{i\delta_n(E)}\sin\delta_n(E);$$

$$C_i = -(l_i-m)(l_i-n)(l_i+m+1)(l_i+n+1) \times [(2l_i+1)(l_1-l_2)(l_1+l_2+1)(m-n)(m+n+1)]^{-1}$$

$(i = 1, 2)$.

Table I

E_r, MeV	Γ, MeV	J^π	η
3.205	0.792	1^-	2.10
3.556	~0.001	2^+	2.00
4.241	0.033	4^+	1.83

E_r is the resonance energy of the α particles in the lab system, Γ is the width of the resonance in the lab system, J^π is the spin and parity of the level, $\eta = Z_1Z_2e^2m_{12}/\hbar^2k$ is the Coulomb parameter, where m_{12} is the reduced mass of the colliding nuclei, Z_1 and Z_2 are their charges, and k is the momentum in the c.m.s.

5)Formulas (23) and (24) were obtained without account of the Coulomb interaction. As seen from Table I (cf. below), $\eta > 1$ for the resonance levels under consideration. Therefore, the Coulomb interaction has an important effect on the angular distribution of the scattered particles, which is different from (9). However, as will be shown in a subsequent paper of the authors, (23) and (24) retain their form even when the Coulomb interaction is taken into account, if the phases δ_n are understood to represent the so-called nuclear scattering phases. The numerical values quoted below refer to the nuclear phases without any further specific mention.

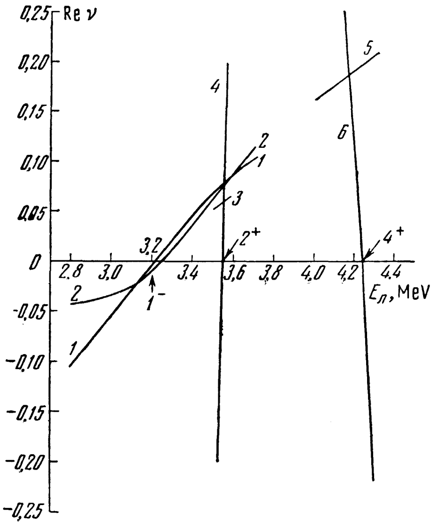

FIG. 3

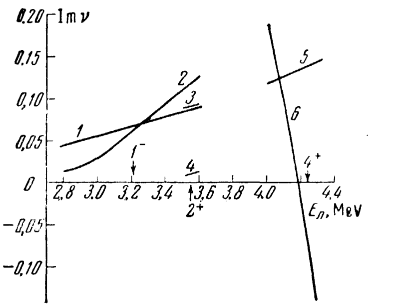

FIG. 4

Using (24), one can from each pair of nonresonant phases δ_m, δ_n, determine the trajectories of the two poles $\alpha l_1(E) = l_1 + \nu l_1(E)$ and $\alpha l_2(E) = l_2 + \nu l_2(E)$.

Specific calculations of the pole trajectories according to (23) and (24) were carried out for the resonant elastic scattering of α particles on C^{12}, using the experimental data of [22]. In the energy interval from 0 to 5 MeV (in the lab system) there are three resonance levels of the compound nucleus O^{16} whose parameters [23] are given in Table I. Since $\delta_5 \approx 0.1°$ for hard sphere scattering (at E = 4.8 MeV and R = 5.4 f), the authors of [22] set $\delta_n = 0$ for $n \geq 5$ in carrying out the phase analysis. In the energy region considered, δ_0 and δ_3 are the only nonresonant phases.[6)]

Figures 3 and 4 show the behavior of Re ν and Im ν in the region of the three resonances. In the region of the 1^- resonance it was assumed that the phases are determined by the single Regge pole $\alpha_1(E) = 1 + \nu_1(E)$, whose trajectory passes near the point $l = 1$. The values of Re $\nu_1(E)$ and Im $\nu_1(E)$, calculated with the help of (23) with $l = 1$, are shown in Figs. 3 and 4, using the phase $\delta_0(E)$ (curves 1) and the phase $\delta_3(E)$ (curves 2). The curves 1 and 2 run very close to one another near the resonance energy, but across the half-width of the resonance $\Gamma/2 \approx 0.4$ MeV there is some discrepancy.

The next resonance 2^+ is very narrow and the values of $\nu_2(E)$ calculated with (23) from the phases δ_0 and δ_3 with $l = 2$ differ by a sign. This difference is explained by the fact that the 2^+ resonance lies within the limits of the half-width of the neighboring 1^- resonance whose pole $\alpha_1(E)$ has not yet receded far into the complex plane $[|\gamma_1(E) \ll 1]$. In calculating the trajectory of the pole $\alpha_2(E) = 2 + \gamma_2(E)$, corresponding to the 2^+ resonance, one must therefore take the pole $\alpha_1(E)$ into account and use the two-pole formulas (24). The curves 3 and 4 in Figs. 3 and 4 show the motion of the poles $\alpha_1(E)$ and $\alpha_2(E)$ in the region of the 2^+ resonance as obtained by a two-pole treatment of the phases δ_0 and δ_3.

From the curves for $\nu_1(E)$ and $\nu_2(E)$ in Figs. 3 and 4 we have obtained the derivatives[7)] $\alpha_i' = d\alpha_i(E)/dE$ and the residues $r\alpha_i(E)$ at the corresponding resonance energies (see Table II). The residue at the resonance is given by the relation $r_\alpha(E_0) = -i\nu(E_0)$, since we have for the resonant partial amplitude $f_l(E_0) = -r_\alpha(E_0)/\times \nu(E_0) = i$. According to (3), $\nu(E_0 - i\Gamma/2) = 0$. Therefore we obtain for a narrow resonance $(\Gamma/E_0 \ll 1)$ in the region $|E - E_0| \lesssim \Gamma$

$$\alpha'(E_0) = 2\Gamma^{-1}[\mathrm{Im}\,\nu(E_0) - i\,\mathrm{Re}\,\nu(E_0)]. \qquad (25)$$

This relation provides a second independent method of calculating $\alpha'(E_0)$.

In Table II we give the values of Im $\nu(E_0)$ obtained from the curves of Fig. 4 and the values of Re $\alpha'(E_0)$, calculated from the former with the

Table II

Resonance	α (E_0) calculated from Fig. 3 (MeV^{-1})	Im ν (E_0)	Re α' (E_0) from (25) (MeV^{-1})	$\lvert r_\alpha \rvert$
1^-	$0.2+i0.1$	0.06	0.16	~0.06
2^+	$9+i0.05$	0.01	10—20	~0.01
4^+	$-4-i$	—0.06	—3.6	~0.06

[6)]The phase δ_4 is small in the region of the 1^- resonance ($\delta_4 \sim 0.06$) and one cannot rely on its stability as the number of phases in the phase analysis is increased. The phase δ_2 is obtained from [22] with poor accuracy, but the order of magnitude of δ_2 is not in contradiction with (23).

[7)]For δ_1' we give the average value of the derivative calculated from the curves 1 and 2.

help of (25). Owing to the steepness of the curves for Re $\nu(E)$ and the change of sign of Re $\nu(E)$ in the neighborhood of $E = E_0$, the calculation of Im $\alpha'(E_0)$ according to (25) leads to widely scattered values and is rather unreliable. As seen from Table II, both methods for the determination of Re $\alpha'(E_0)$ are in satisfactory agreement.

Let us now turn to the 4^+ resonance. In the neighborhood of this resonance $|\nu_1(E)| \ll 1$, and the pole $\alpha_1(E)$ must be taken into account. Since Re $\alpha_2' \sim 10$, Re $\alpha_2(E)$ rises to unity as the energy is increased by $\Delta E \sim 0.1$ MeV. If the pole α_2 to the right of the 2^+ resonance continued to move near the real axis, then the 3^- resonance would be observed at $E \sim 3.7$ MeV [even if we assume that the trajectory of the pole $\alpha_2(E)$ has a signature, there would be the 4^+ resonance at $E \sim 3.8$ MeV]. Apparently, the pole $\alpha_2(E)$ moves away sharply from the real axis into the complex plane to the right of the 2^+ resonance, and it may be neglected in the consideration of the 4^+ resonance. The parts of the trajectories of the poles $\alpha_1(E) = 1 + \nu_1(E)$ and $\alpha_4(E) = 4 + \nu_4(E)$ calculated with the help of (24) in the neighborhood of the 4^+ resonance are shown in Figs. 3 and 4 (curve 5: ν_1, curve 6: ν_4). The values of α_4' and r_{α_4} for the 4^+ resonance are given in Table II.

It is seen from Figs. 3 and 4 that the parts of the $\alpha_1(E)$ trajectory constructed independently in the neighborhoods of all three resonances are in satisfactory agreement with one another. It should be emphasized that the $\alpha_4(E)$ trajectory moves in the opposite direction [Re $\alpha_4'(E) < 0$] and intersects the real axis, with Im $\alpha_4(E) < 0$ in the region of the 4^+ resonance itself. Such a behavior of the Regge pole is not possible for scattering from a real [3] or complex potential with absorption (optical model). However, since these three resonances cannot be described by any reasonable potential, the problem of the theoretical interpretation of this trajectory remains open.

We also make a general remark on the phase analysis carried out in [22]. According to (22), the small nonresonant phases in the region of the 1^- resonance are given by the formula $\delta_n \approx 3\xi_n$ Im $\nu_1/[(n-1)(n+2)]$. It follows from the closeness of the curves 1 and 2 of Fig. 4 that up to $n = 3$ the compensation of the pole phases is small and $\xi_n \approx 1$. Since $kR \sim 3$, it may turn out that $\xi_n \approx 1$ even for $n = 4$, 5, and 6. Taking into account that Im $\nu_1 \sim 0.06$, one may then expect the following values of the phases: $\delta_4 \sim 0.6°$, $\delta_5 \sim 0.4°$, $\delta_6 \sim 0.3°$. These considerations show that the precise determination of the Regge trajectories evidently requires a phase analysis with a larger number of phases then indicated by the estimate from the hard sphere scattering.

In conclusion we remark on the work of Rebolia and Viano.[14] In this paper it was attempted to describe the angular distribution in the elastic scattering of α particles on C^{12} in the neighborhood of the 4^+ resonance with the help of the single-pole amplitude (5). The fit to the experimental angular distribution led to the trajectory $\alpha_4(E)$, which runs, as in our case, in the opposite direction and intersects the real axis; however, the quantity $|\alpha_4(E) - 4|$ attains, starting from zero, very large values (of order two) within the width of the resonance. This sharp increase of $\alpha_4(E)$ within the width of the resonance appears strange to us. We note further that, when the distance of the pole from the real axis becomes of order unity, its contribution to the angular distribution is exponentially small and the inclusion of only one pole in the scattering amplitude is unjustified. In the light of our results on the angular distribution for this resonance, one must include the pole α_1 besides the pole α_4. Moreover, in describing the angular distribution in this energy region it is necessary to take the Coulomb interaction into account.

The authors express their deep gratitude to I. S. Shapiro for numerous fruitful criticisms and a number of useful remarks, L. D. Blokhintsev and B. V. Geshkenbeĭn for a discussion of the results of this paper, and A. S. Kronrod and L. M. Voronina for the numerical computations.

APPENDIX A

We give several formulas for the function $R_l(z)$:

$$R_l(z) = (-)^l \left[\partial P_\alpha(-z)/\partial\alpha\right]_{\alpha=l}. \qquad \text{(A.1)}$$

The function $R_l(z)$ satisfies the relations

$$(l+1)R_{l+1}(z) - (2l+1)zR_l(z) + lR_{l-1}(z) = \frac{1}{2l+1}\left[P_{l+1}(z) - P_{l-1}(z)\right], \qquad \text{(A.2)}$$

$$(l+1)R_{l+1}(z) + (1-z^2)\frac{dR_l(z)}{dz} - (l+1)zR_l(z) = -\frac{l}{2l+1}\left[P_{l+1}(z) - P_{l-1}(z)\right].$$

Knowing $R_0(z)$, one can use (A.2) for a consecutive calculation of the functions $R_l(z)$ for $l \geq 1$. According to (8.762) in [20], we have

$$R_0(z) = \ln\frac{1-z}{2}. \qquad \text{(A.3)}$$

Equations (A.2) are satisfied if we set

$$R_l(z) = P_l(z) \ln \frac{1-z}{2} + V_l(z), \qquad (A.4)$$

where $P_l(z)$ is the Legendre polynomial and $V_l(z)$ is a polynomial of degree l defined by the following recurrence relation:

$$(l+1) V_{l+1}(z) - (2l+1) z V_l(z) + l V_{l-1}(z) = \frac{1}{2l+1} [P_{l+1}(z) - P_l(z)],$$

$$V_0(z) = 0, \qquad V_1(z) = 1 + z. \qquad (A.5)$$

We give the explicit expressions for the polynomials $V_l(z)$ for $l = 2, 3$, and 4:

$$V_2(z) = \frac{7}{4} z^2 + \frac{3}{2} z - \frac{1}{4},$$

$$V_3(z) = \frac{37}{12} z^3 + \frac{5}{2} z^2 - \frac{5}{4} z - \frac{2}{3},$$

$$V_4(z) = \frac{533}{96} z^4 + \frac{35}{8} z^3 - \frac{59}{16} z^2 - \frac{55}{24} z + \frac{7}{32}. \qquad (A.6)$$

It follows from (A.4) and (A.5) that

$$R_l(-1) = 0 \quad \text{for arbitrary } l. \qquad (A.7)$$

APPENDIX B

Let us consider the simplest graphs for a direct reaction of the type $A + x \to B + y$, corresponding to the singularities of the amplitude in the momentum transfer closest to the physicsl region (the nuclear vertices are considered constant for simplicity). The kinematics of the reaction is given by the formulas

$$z = a + bt, \qquad a = \frac{1}{2}(s_1 + s_2)(s_1 s_2)^{-1/2}, \qquad b = \frac{1}{2}(s_1 s_2)^{-1/2}, \qquad (B.1)$$

where

$$s_1 = M_A M_y M^{-2} s_{Ax}, \quad s_2 = M_B M_x M^{-2} s_{By},$$

$$M = \frac{1}{2}(M_A + M_x + M_B + M_y),$$

and s_{Ax}, s_{By}, and $t = t_{xy}$ are nonrelativistic invariants introduced in [24]. From this we have

$$z = 1 + \frac{1}{2}[(s_1^{1/2} - s_2^{1/2})^2 + t](s_1 s_2)^{-1/2}. \qquad (B.2)$$

The quantity $(s_1^{1/2} - s_2^{1/2})^2$ depends on the kinetic energy of the colliding particles. Its minimum value is

$$(s_1^{1/2} - s_2^{1/2})^2_{min} = \begin{cases} 0, & \text{if } (M_x - M_y) Q < 0 \\ 2(M_x - M_y) Q, & \text{if } (M_x - M_y) Q > 0, \end{cases} \qquad (B.3)$$

where $Q = M_A + M_x - M_B - M_y$ is the Q value of the reaction.

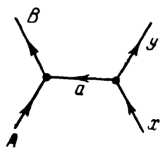

FIG. 5

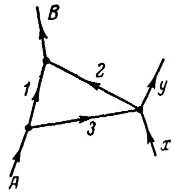

FIG. 6

The pole graph of Fig. 5 corresponds to the amplitude

$$f^{(p)}(E, z) = C_p(E)/(z_0 - z),$$

$$z_0 = 1 + \frac{1}{2}[(s_1^{1/2} - s_2^{1/2})^2 + 2m_a \varepsilon](s_1 s_2)^{-1/2},$$

$$\varepsilon = m_a + M_y - M_x. \qquad (B.4)$$

The partial amplitudes have the form

$$f_n^{(p)}(E) = C_p(E) Q_n(z_0).$$

The only singularities of $f_\lambda^{(p)}(E)$ are simple poles in the points $\lambda = -1, -2, -3, \ldots$ Therefore the amplitude of the pole graph has the following representation:

$$f^{(p)}(E, z) = -\frac{1}{2i} \int_{-1/2 - i\infty}^{-1/2 + i\infty} \frac{(2\lambda + 1) f_\lambda^{(p)}(E) P_\lambda(-z)}{\sin \pi\lambda} d\lambda. \qquad (B.5)$$

Using the explicit form of the amplitude for the triangular graph [24] (Fig. 6), we can write it in the form of a dispersion integral,

$$f^{(\Delta)}(E, z) = \frac{1}{\pi} \int_{t_\Delta}^{\infty} \frac{A(E, t')}{t' - t} dt' \qquad (B.6)$$

$$A(E, t) = \frac{C_\Delta(E)}{(t - t_0)^{1/2}} \theta(t - t_\Delta),$$

where

$$t_0 = -2(M_x - M_y) Q,$$

$$t_\Delta = t_0 + M_A M_B m_1^{-2} (\varkappa_A + \varkappa_B)^2,$$

$$\varkappa_A = (2m_{13}\varepsilon_A)^{1/2}, \qquad \varkappa_B = (2m_{12}\varepsilon_B)^{1/2},$$

$$m_{ik} = m_i m_k (m_i + m_k)^{-1},$$

$$\varepsilon_A = m_1 + m_3 - M_A, \quad \varepsilon_B = m_1 + m_2 - M_B.$$

From (B.6) we obtain

$$f_\lambda^{(\Delta)}(E) = \frac{1}{\pi} \int_{z_\Delta}^{\infty} A(E, a + bz) Q_\lambda(z)\, dz,$$

$$z_\Delta = a + bt_\Delta. \qquad (B.7)$$

It follows from (B.2) and (B.3) that $z_0 = a + bt_0 \geq 1$. To find the singularities of $f_\lambda^{(\Delta)}$ we go over to the variable $\xi = z/z_0$ in (B.7) and take into account that [20]

$$\int_1^\infty Q_\lambda(z_0\xi)(\xi-1)^{-1/2}\,d\xi$$

$$= \pi(2z_0)^{-1/2}\left(\lambda+\frac{1}{2}\right)^{-1}[z_0+(z_0^2-1)^{1/2}]^{-\lambda-1/2}$$

(with $z_0 \geq 1$). As a result we obtain

$$f_\lambda^{(\Delta)}(E) = C_\Delta(E)(bz_0)^{1/2}\Big\{(2z_0)^{-1/2}\left(\lambda+\frac{1}{2}\right)^{-1} \times [z_0+(z_0^2-1)^{1/2}]^{-\lambda-1/2} - \frac{1}{\pi}\int_1^{z_\Delta/z_0} Q_\lambda(z_0\xi)(\xi-1)^{-1/2}\,d\xi\Big\}, \qquad \text{(B.8)}$$

from where it is clear that the only singularities of $f_\lambda^{(\Delta)}$ in the finite part of the λ plane are simple poles in the points $\lambda = -\frac{1}{2}, -1, -2, \ldots$

The amplitude $f^{(\Delta)}(E, z)$ has the representation

$$f^{(\Delta)}(E, z) = -\frac{1}{2i}\int_C \frac{(2\lambda+1)\,f_\lambda^{(\Delta)}(E)\,P_\lambda(-z)}{\sin\pi\lambda}\,d\lambda, \qquad \text{(B.9)}$$

where the contour C runs from $-\frac{1}{2} - i\infty$ to $-\frac{1}{2} + i\infty$ parallel to the imaginary axis, passing the pole $\lambda = -\frac{1}{2}$ on the right. One might think that taking account of the dependence of the vertex functions on the momentum transfer leads to a more rapid decrease of $f(E, z)$ for $z \to \infty$. Therefore, the result that the right half-plane $\mathrm{Re}\,\lambda > -\frac{1}{2}$ is free of singularities of $f_\lambda(E)$ is preserved also for nonrelativistic pole and triangular graphs with nuclear vertices. Thus it is seen on the example of the simplest graphs that the direct processes enter only in the integral along the line $\mathrm{Re}\,\lambda = -\frac{1}{2}$ in the representation (2) of the total reaction amplitude $f(E, z)$.

[1] G. N. Watson, Proc. Roy. Soc. 95, (1918).

[2] A. Sommerfeld, Partial Differential Equations in Physics, N.Y., 1954.

[3] T. Regge, Nuovo cimento 14, 951 (1959) and 18, 947 (1960).

[4] C. G. Bollini and J. J. Giambiagi, Nuovo cimento 26, 619 (1962).

[5] Patashinskiĭ, Pokrovskiĭ, and Khalatnikov, JETP 44, 2062 (1963), Soviet Phys. JETP 17, 1387 (1963).

[6] C. Lovelace and D. Masson, Proc. of the 1962 Int. Conf. on High Energy Physics at CERN, p. 510.

[7] C. Lovelace and D. Masson, Nuovo cimento 26, 472 (1962).

[8] Azimov, Ansel'm, and Shekhter, JETP 44, 361 (1963), Soviet Phys. JETP 17, 246 (1963).

[9] V. N. Gribov, JETP 41, 1962 (1961) and 42, 1260 (1962), Soviet Phys. JETP 14, 1395 (1962) and 15, 873 (1962).

[10] V. N. Gribov and I. Ya. Pomeranchuk, JETP 43, 308 (1962), Soviet Phys. JETP 16, 220 (1963). Phys. Rev. Lett. 9, 238 (1962).

[11] G. F. Chew and S. C. Frautschi, Phys. Rev. Lett. 7, 394 (1961) and 8, 41 (1962).

[12] Frautschi, Gell-Mann, and Zachariasen, Phys. Rev. 126, 2204 (1962).

[13] V. N. Gribov, In Voprosy fiziki elementarnykh chastits (Problems in Elementary Particle Physics), Academy of Sciences, Armenian SSR, 1962, pp. 178 and 204.

[14] L. Rebolia and G. A. Viano, Nuovo cimento 26, 1426 (1962).

[15] E. J. Squires, Nuovo cimento 25, 242 (1962).

[16] I. S. Shapiro, JETP 41, 1616 (1961) and 43, 1727 (1962), Soviet Phys. JETP 14, 1148 (1962) and 16, 1219 (1963).

[17] R. D. Amado, Phys. Rev. Lett. 2, 399 (1959).

[18] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Gostekhizdat, 1948. Engl. Transl., Quantum Mechanics, Pergamon Press, London, 1958.

[19] J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics, John Wiley & Sons, N.Y., 1952 (Russ. Transl., IIL, 1954).

[20] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1962.

[21] B. R. Desai and R. G. Newton, Phys. Rev. 130, 2109 (1963).

[22] Miller Jones, Phillips, Harris, and Beckner, Nucl. Phys. 37, 1 (1962).

[23] F. Ajzenberg-Selove and T. Lauritsen, Energy Levels of Light Nuclei, Technical Report, August, 1960.

[24] Blokhintsev, Dolinskiĭ, and Popov, Nucl. Phys. 40, 117 (1963).

Translated by R. Lipperheide
138