REGGE TRAJECTORIES IN THE BETHE-SALPETER EQUATION

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An equation is derived which defines Regge trajectories for the sum of diagrams of the ladder type (particles of mass m exchange with particles of mass μ). The behavior of the trajectory $l(s)$ as a function of s and of the coupling constant g^2 is investigated. An expansion of $l(s)$ in a series of g^2 is obtained and it is shown that the sum of the senior terms in the asymptotic ladder diagrams yields the first two terms in the expansion, i.e., $l(s) = l_0 + g^2 l_1(s)$. The cause of this situation is ascertained. The equation for $l(s)$ is solved exactly in the particular case of $s = 0$. The analytical form of $l(g^2)$ is presented for $\mu = 0$ and the results of numerical calculations for a number of values of μ^2/m^2 are given.

THE problem of finding the bound states in relativistic theory is of great interest, but is far from solved (only one particular case was actually investigated [1]). On the other hand it is quite important to investigate the asymptotic behavior of the amplitude. Both problems are combined in a single approach, the so-called Regge pole method, in which the bound states can be obtained from the asymptotic behavior of the amplitude in the other channel.

Thus, the problem reduces to finding the Regge trajectories from the asymptotic value of the amplitude. Since the properties of the Regge trajectories have been little investigated in relativistic theory, there is undoubtedly interest in at least using some relativistic model as an example. By way of such a model we use the Bethe-Salpeter equation. The method proposed below can be applied to Bethe-Salpeter equation with any kernel, but for simplicity we confine ourselves to an examination of the sum of ladder diagrams.

Let us consider the sum of ladder diagrams for particles of mass m, which exchange particles of mass μ. We use the usual normalization for the amplitude T:

$$S = \frac{iT(2\pi)^4\delta(p_1+p_2-p_3-p_4)}{\sqrt{16\omega_1\omega_2\omega_3\omega_4}}.$$

Assume for simplicity of all particles are scalar and that the dimensional coupling constant is g. The equation for the amplitude T takes the form

$$T(s, t, p_1^2, p_2^2) = -\frac{ig^2}{(2\pi)^4}\int\frac{d^4q\,T(s, t', p_1'^2, p_2'^2)}{(q^2-\mu^2)(p_1'^2-m^2)(p_1'^2-m^2)} + T_0(s, t). \quad (1)$$

Graphically this equation is shown in Fig. 1. Here

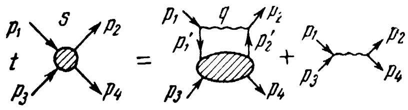

FIG. 1

$$t' = (p_1' + p_3)^2, \quad s = (p_1 - p_2)^2.$$

In equation (1) the external ends with momenta p_3 and p_4 are on the mass surface, whereas the 'masses' of the particles with momenta p_1 and p_2 are equal respectively to p_1^2 and $p_2^2 \neq m^2$.

From (1) we obtain a recurrence relation for the diagrams in perturbation theory.

$$T_{n+1} = -\frac{ig^2}{(2\pi)^4}\int\frac{d^4q\,T_n(s, t', p_1^2, p_2^2)}{(q^2-\mu^2)(p_1^2-m^2)(p_2^2-m^2)}. \quad (2)$$

We change over from integration with respect to d_q^4 to more convenient variables [2]

$$d^4q = \frac{|\mathbf{q}|\,dq^2\,dt'}{2\sqrt{t}}\,\frac{dz_1\,dz_2}{\sqrt{-K(z, z_1, z_2)}}. \quad (3)$$

Here

$$z_{1,2} = \frac{p_{1,2}'^2 - p_{1,2}^2 - \mu^2 + 2p_{1,2}^0 q^0}{2p_{1,2}|\mathbf{q}|},$$

$$p_{1,2} = \frac{1}{2\sqrt{t}}\{[t-(m+\sqrt{p_{1,2}^2})^2][t-(m-\sqrt{p_{1,2}^2})^2]\}^{1/2}, \quad (3')$$

$$|\mathbf{q}| = \frac{1}{2\sqrt{t}}[(t+q^2-t')^2 - 4q^2t]^{1/2},$$

$$K(z, z_1, z_2) = z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1.$$

Equation (1) is rewritten in the following form (we leave out the free term T_0, which is no longer of interest):

$$T(s, t) = -\frac{ig^2}{(2\pi)^4}\int\frac{dq^2}{q^2-\mu^2}\int\frac{dt'\,|\mathbf{q}|}{2\sqrt{t}} \times \int\frac{dz_1\,dz_2}{\sqrt{-K(z, z_1, z_2)}}\,\frac{T(s, t', p_1'^2, p_2'^2)}{(p_1'^2-m^2)(p_2'^2-m^2)}. \quad (4)$$

The region of integration with respect to q^2 and t' includes the values $|q| \geq 0$ (see [2]).

It is more convenient to consider in place of (4) the equation for the imaginary part of the amplitude in the t-channel. We assume first that $s < 0$ and put $p_{1,2}^2 = -\kappa_{1,2}$ and $p_{1,2}'^2 = -\kappa_{1,2}'$. Then

$$A(s, t, \varkappa_1, \varkappa_2) = \frac{g^2}{16\pi^3} \int_{4\mu^2}^{(\sqrt{t}-\mu)^2} \frac{|q|\, dt'}{\sqrt{t}}$$

$$\times \int \frac{dz_1\, dz_2}{\sqrt{-K(z, z_1, z_2)}} \frac{A(s, t', \varkappa_1', \varkappa_2')}{(\varkappa_1' + m^2)(\varkappa_2' + m^2)}. \qquad (5)$$

This equation, defined for $s < 0$, can now be analytically continued into the region $s > 0$. Equation (5) enables us to calculate in recurrent fashion A_{n+1}, substituting $A_n(s, t, \kappa_1', \kappa_2')$ in the right side under the integral sign.

In accordance with the statement made in the beginning of the article, we should now investigate Eq. (5) in the asymptotic region $t \to \infty$. We first have to establish some properties of the function A.

First, the region of integration over κ_1' and κ_2' in the integral of (5), with $s < 0$, includes only the positive values, so that the kernel of the equation is positive. Since the first term of the perturbation theory series [the function $A_2(s, t, \kappa_1, \kappa_2)$] is positive for positive κ_1 and κ_2 (this is the imaginary part of the square diagram), A_3 and all the remaining perturbation-theory terms are also positive, from which it follows that the entire function A is positive (at least where the perturbation-theory series converges).

For $s = 0$ the positiveness of A follows from the unitarity relation, which can be continued analytically towards the negative masses $-\kappa_1$ and $-\kappa_2$. From this follows the positiveness of A for $s \leq 4m^2$ by virtue of the property of the imaginary part as proved by Gribov and Pomeranchuk [3].

Analogously, we can prove with the aid of (5) that the function $A(s, t, \kappa_1, \kappa_2)$ decreases with increasing κ_1 and κ_2, in agreement with the fact that the singularities in κ_1 and κ_2 are located at negative values, and as κ_1 and κ_2 increase we go over into the region where $A(s, t, \kappa_1, \kappa_2)$ has no singularities. We prove this in the appendix. It can also be shown that the terms of the perturbation-theory series decrease with increasing κ_1 and κ_2.

Let us investigate the behavior of the right half of (5) as $t \to \infty$ and for $s \leq 4m^2$. We can separate a region $t' < M^2$ where $A(s, t', \kappa_1', \kappa_2')$ does not have a simple asymptotic behavior, and limit this region as a function of t' from above and from below. Replacing the function A in the integral (5) by a constant, we can calculate the integral with respect to $dz_1 dz_2$ (or with respect to $d\kappa_1' d\kappa_2'$). From this we can readily find that the behavior as $t \to \infty$ of this (nonasymptotic) part of the integral takes the form $I_{na} \sim C/t$. By virtue of the positiveness of A it follows therefore that the relation $A(s, t, \kappa_1, \kappa_2) \geq C/t$ is satisfied for the entire amplitude for κ_1 and κ_2 fixed, $s \leq 4m^2$, and $t \to \infty$. We seek the asymptotic behavior of the function A in the form

$$A(s, t, \varkappa_1, \varkappa_2) \gtrsim r(s, \varkappa_1, \varkappa_2)\, t^{l(s)} \quad \text{as } t \to \infty. \qquad (6)$$

This form agrees with the assumed existence of an extreme-right pole in the l plane for the partial amplitude in the s-channel. Within the framework of the model considered here (sum of ladder diagrams), this assumption was proved by Lee and Sawyer [4].

The function $l(s)$ cannot depend on κ_1 or κ_2, since the Regge poles of the amplitude $A(s, t, \kappa_1, \kappa_2)$ (not on the mass surface) coincide with the Regge poles of the physical amplitude $A(s, t, m^2, m^2)$. It can also be shown that (5) is not satisfied when $l(s)$ depends on κ_1 and κ_2.

We now consider the asymptotic part of the integral of (5), where the lower limit with respect to t' is larger than M^2, and we transform the kernel using the fact that $t \to \infty$. We neglect in this case terms of the type s/t, $\kappa_{1,2}/t$ and $\kappa_{1,2}'/t$, and also μ^2/t and m^2/t.

Inasmuch as the function A decreases with increasing κ_1' and κ_2', the neglect of $\kappa_{1,2}'/t$ is justified if the resultant integrals converge as $\kappa_{1,2}' \to \infty$.

Making the substitution $x = t'/t$ and letting $t \to \infty$, we obtain from (5) the following equation[1)] relating the Regge trajectory with the residue at the pole:

$$r(s, \varkappa_1, \varkappa_2)$$

$$= \lambda \int_0^1 dx\, x^{l(s)} \frac{1}{\pi} \int \frac{d\varkappa_1'\, d\varkappa_2'\, r(s, \varkappa_1', \varkappa_2')}{\sqrt{L(x, u_1, u_2, s)}\,(\varkappa_1' + m^2)(\varkappa_2' + m^2)},$$

$$\lambda = \frac{g^2}{16\pi^2}. \qquad (7)$$

We have neglected the asymptotic part of the integral, assuming that $l(s) > -1$. In fact, $l(s)$ is always ≥ -1 and becomes equal to -1 when $s \to \pm\infty$ [4].

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1)An equation of type (7) was obtained independently by Fubini et al.[5] for the case of $\pi\pi$ interaction.

The function $L(x, u_1, u_2, s)$ has the following form:

$$L(x, u_1, u_2, s) = 4s\mu^2 x - s^2(1-x)^2 - (u_1 - u_2)^2$$
$$- 2s(1-x)(u_1 + u_2);$$
$$u_{1,2} = \varkappa'_{1,2} - x\varkappa_{1,2}. \tag{8}$$

The region of integration with respect to κ'_1 and κ'_2 in (7) extends over those values where $L \geq 0$. The boundary of the region is the parabola shown in Fig. 2. When $\kappa_{1,2} > 0$ the region of integration encloses only positive values of $\kappa'_{1,2}$ (with $s < 0$). The integral with respect to $\kappa'_{1,2}$ converges by virtue of the decrease of $r(s, \kappa'_1, \kappa'_2)$ for large $\kappa'_{1,2}$.

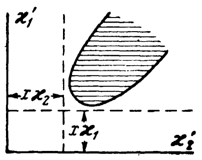

FIG. 2

Equation (7) is a linear integral equation in $r(s, \kappa_1, \kappa_2)$, with a kernal that depends on $l(s)$; s is a parameter of the equation. Therefore for each l (7) has a solution only for some value λ (or a set of values of λ), which defines λ as a function of l, and, conversely, $l = l(\lambda, s)$. The function $l(\lambda, s)$ is, generally speaking, multiple-valued, but Eq. (7) has been written out for the extreme-right Regge trajectory, so that we should choose the branch with the largest value of l.

From (7) we can draw certain qualitative conclusions concerning the behavior of the trajectory $l(s)$. Since $0 \leq x \leq 1$, we have $\mathrm{Re}\, l(s) < N$, otherwise $xl(s) \to 0$ as $\mathrm{Re}\, l \to \infty$, and (7) has no solution. The singularity of $l(s)$ as a function of s occurs first at such values of s that the boundary of the integration region (7) is tangent to the lines $\kappa'_1 = \kappa'_2 = -m^2$, and in this case x is also on the boundary of the contour, i.e., $x = 0$ or $x = 1$. With the aid of (8) we find that a singularity occurs when $s = 4m^2$.

We can analogously investigate with the aid of (7) the threshold behavior of $l(s)$ near $s = 4m^2$.

As $s \to -\infty$, the integration region of κ'_1 and κ'_2 shifts to $+\infty$, and the integral in the right side tends to zero. Therefore the equation can have a solution only when $l(s) \to -1$ as $s \to -\infty$. This agrees with the results of Lee and Sawyer [4]. The form of the trajectory obtained from the qualitative considerations is shown in Fig. 3.

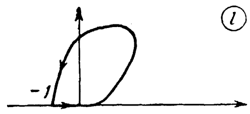

FIG. 3

Let us consider now the expression obtained for $l(s)$ by using perturbation theory. The first term in the expansion of $l(s)$ in powers of g^2 is particularly easy to find. It is first necessary to investigate the function A in the lower order of perturbation theory—the imaginary part of the square diagram A_2.

As $t \to \infty$ the function A_2 can be written as follows:

$$A_2 = \frac{g^4}{16\pi}\frac{f(s)}{t}, \tag{9}$$

$$f(s) = \frac{1}{\sqrt{s(s-4m^2)}} \ln \frac{2m^2 - s + \sqrt{s(s-4m^2)}}{2m^2 - s - \sqrt{s(s-4m^2)}}. \tag{10}$$

A characteristic property of (9) is that A_2 does not depend on κ_1 or κ_2, nor on the mass μ (in the limit as $t \to \infty$). In the next order the function A_3 depends essentially on these variables.

The quantities $r(s, \kappa_1, \kappa_2)$ and $l(s)$ can be expanded in powers of the coupling constant g^2:

$$r(s, \varkappa_1, \varkappa_2) = r_0(s) + g^2 r_1(s, \varkappa_1, \varkappa_2), \tag{11}$$

$$l(s) = l_0(s) + g^2 l_1(s). \tag{12}$$

We then conclude from (9) that $r_0(s) = g^4 f(s)/16\pi$, and $l_0 = -1$. In order to find $l_1(s)$, we use (7), in which we put $l(s) = -1 + g^2 l_1(s)$, $r(s, \kappa_1, \kappa_2) = r_0(s)$:

$$1 = \lambda \int_0^1 dx\, x^{-1+g^2 l_1(s)} \frac{1}{\pi} \int \frac{d\varkappa'_1\, d\varkappa'_2}{[L(x, u_1, u_2, s)(\varkappa'_1 + m^2)(\varkappa'_2 + m^2)]^{1/2}}. \tag{13}$$

The integral with respect to $d\kappa'_1 d\kappa'_2$ can be evaluated and is equal to $\Phi(x, s)$, where

$$\Phi(x, s) = \{-s[4x\mu^2 + (4m^2 - s)(1-x)^2]\}^{-1/2} \ln(F_+/F_-),$$
$$F_\pm = (2m^2 - s)(1-x)^2 + 2x\mu^2 \pm \{-s(4m^2 - s)(1-x)^2(1 - 2bx + x^2)\}^{1/2},$$
$$b = 1 - 2\mu^2/(4m^2 - s). \tag{14}$$

When $x = 0$ we have $\Phi(0, s) = f(s)$. Thus, (13) is rewritten in the form

$$1 = \lambda \int_0^1 dx\, x^{-1+g^2 l_1(s)} \Phi(x, s). \tag{15}$$

We integrate the integral in (15) by parts, and then let $g^2 \to 0$. This yields

$$l_1(s) = \Phi(0, s)/16\pi^2 = f(s)/16\pi^2.$$

Consequently in first approximation of perturbation theory $l(s)$ takes the form

$$l(s) = -1 + \lambda f(s), \tag{16}$$

where $f(s)$ is defined in (10).

Substituting in the right half of (5) the expression (9) for A_2, we obtain

$$A_3 = r_0(s)\lambda f t^{-1}\ln t + O(1/t) \tag{17}$$

and we get for the n-th term of the perturbation-theory series

$$A_n = r_0(s)\frac{(\lambda f \ln t)^{n-2}}{(n-2)!} + \frac{c_n(\ln t)^{n-3}}{t}. \tag{18}$$

Summing the principal terms in the asymptotic values of A_n, we get

$$A'(t, s) = \sum_{n=2}^{\infty} A'_n = r_0(s)\, t^{-1+\lambda f(s)}, \tag{19}$$

which gives expression (16) for the trajectory $l(s)$.

Thus, summation of the principal terms in the asymptotics of the diagrams leads to the first perturbation-theory approximation for $l(s)$. This explains why the methods of Polkinghorne [6] and Arbusov et al.[7] do not give an exact expression for the trajectory $l(s)$. We know that the properties of $l(s)$ determined in (16) do not correspond to the ordinary physical notions regarding the trajectory, viz., $l(s)$ does not depend on μ ('radius' of the potential) and as $s \to 4m^2 \pm 0$ we have $l(s) \to \pm\infty$. These Coulomb properties of the trajectory (16) are only the consequence of the expansion in g^2 and do not appear, as we have already seen, in the exact trajectory $l(s)$.

In order to explain the meaning of the result (19) in greater detail, we first neglect in (5) the dependence of $A(s, t, \kappa_1, \kappa_2)$ on κ_1 and κ_2. Without writing out explicitly the argument s of the functions A and $\Phi(x, s)$, we arrive at a simplified mathematical model of (5):

$$A_{n+1}(t) = \lambda \int_{1/t}^{1} dx\, A_n(tx)\,\Phi(x), \quad A_0 = \frac{1}{t}. \tag{20}$$

For $A_{n+1}(t)$ we get from (20) the expansion

$$A_n(t) = \sum_{k=0}^{n} a_n^k (\ln t)^k t^{-1}, \tag{21}$$

$$a_n^n = (\lambda f)^n / n!, \quad a_n^{n-1} = -\lambda L_1 (\lambda f)^{n-1} n/(n-1)!$$

etc., where

$$L_1 = \int_0^1 \ln x\, \Phi'(x)\, dx.$$

The sum of the principal terms yields

$$J_0(t) = \sum_{n=0}^{\infty} a_n^n t^{-1} (\ln t)^n = t^{-1+\lambda f}. \tag{22}$$

The sum of the next terms in the asymptotic expression is

$$J_1(t) = \sum_{n=1}^{\infty} a_n^{n-1} t^{-1} (\ln t)^{n-1} = -\lambda L_1 t^{-1+\lambda f}(1 + \lambda f \ln t). \tag{23}$$

It is seen from (22) and (23) that $J_1(t)$ becomes comparable with $J_0(t)$ at sufficiently large t, i.e., it gives in principle an additional term in the trajectory $l(s)$. At the same time $J_1(t)$ contains an extra λ compared with $J_0(t)$, so that in the first approximation in λ the trajectory is determined by the contribution of the principal asymptotic terms of $J_0(t)$.

In the exact equation (5), the next terms in the asymptotic value of the ladder diagrams have a more complicated form, but even these include terms of the form (21), which lead to $J_1(t)$ upon summation, so that the situation for ladder diagrams is qualitatively similar to that considered above.

Thus, the summation of the principal terms in the asymptotic value of the diagrams does not give the principal term in the asymptotic expression for the entire sum, and can claim only to yield the Regge trajectory in the lower order in g^2.

Equation (7) can be greatly simplified in one very interesting case—for forward scattering ($s = 0$). In this case the region of integration with respect to κ'_1 and κ'_2 in (7) degenerates into a straight line, if the values of κ_1 and κ_2 are set equal to each other. Introducing the notation $r(\kappa, \kappa, 0) \equiv r(\kappa)$, we reduce (7) to the form

$$r(\varkappa) = \lambda \int_0^1 x^l dx \int_{\varkappa_0}^{\infty} \frac{d\varkappa' r(\varkappa')}{(\varkappa' + m^2)^2}, \quad \varkappa_0 = \varkappa x + \mu^2 \frac{x}{1-x}. \tag{24}$$

Integrating with respect to x and introducing the dimensionless variables $y = \kappa/m^2$, and $\lambda_0 = \lambda/m^2$, we obtain the integral equation

$$r(y) = \lambda_0 \int_0^{\infty} dy' r(y')\, K_l(y', y), \tag{25}$$

$$K_l(y', y) = \frac{1}{(l+1)(y'+1)^2} \times \left[y' + y + \frac{\mu^2}{m^2} - \sqrt{\left(y' + y + \frac{\mu^2}{m^2}\right)^2 - 4yy'}\right]^{l+1} (2y)^{-l-1}. \tag{26}$$

Equation (25) is homogeneous, of the Fredholm type, with a kernel that can be reduced to a symmetrical form. Consequently there exists an enumerable and real set of values λ_0 for each l [conversely, (25) defines a multiply-valued function $l(\lambda_0)$]. In addition, from the positiveness of $r(y)$ and K_l follows the positiveness of λ_0. Further, since (25) has been written for the extreme-right Regge pole, we should choose the branch of $l(\lambda_0)$ giving the largest l.

From (25) and from the condition $\partial K_l/\partial\mu^2 < 0$ we find that with increasing μ^2 the same l is obtained from ever increasing λ_0, i.e., the coupling weakens with increasing particle mass (with decreasing effective interaction radius).

Let us consider the limiting case $\mu^2/m^2 \to 0$. Then

$$K_l(y', y) = \frac{1}{(l+1)(y'+1)^2}\begin{cases}(y'/y)^{l+1}, & y' < y\\ 1, & y' > y\end{cases}. \qquad (27)$$

Differentiating (25) with the kernel (27) twice, we arrive at the following differential equation:

$$yr'' + (l+2)\,r' + \lambda r/(1+y)^2 = 0. \qquad (28)$$

The function $r(y)$ satisfies the following conditions: regularity at $y = 0$, and $r(y) \sim 1/y^{l+1}$ as $y \to \infty$. The solution of (28) under these conditions takes the form

$$r(y) = CF\,(l+2;\, -l-1;\, l+2;\, y/(1+y)), \qquad (29)$$

where $F(a, b, c, z)$ is the hypergeometric function, and the connection between l and λ_0 is given by

$$l + 2 = (1 + \sqrt{1 + 4\lambda_0})/2. \qquad (30)$$

For small λ_0 we have $l = -1 + \lambda_0 + O(\lambda_0^2)$, which coincides with (16) for $s = 0$. For $\mu^2/m^2 \neq 0$ Eq. (25) was solved numerically.

The table lists the values of λ_0, with the values of $l + 1$ indicated on the upper line and the values of μ^2/m^2 to which the given λ_0 pertains on the right.

Figure 4 shows the variation of $l(\lambda_0)$ for some μ^2/m^2. We note that l increases slowly with λ_0; whereas l increases like $\sqrt{\lambda_0}$ for $\mu = 0$, for the growth becomes logarithmic for $\mu^2/m^2 \neq 0$. Nonetheless, $l(\lambda_0)$ has no upper bound.

From the table and from Fig. 4 we can draw two conclusions. First, $l(\lambda_0)$ can become larger than unity, and this leads to violation of the Froissart theorem [8]. Second, once $l(\lambda_0)$ becomes positive for $s = 0$, a 'ghost' state arises—a bound state with integer l and $s < 0$. Both these features are connected apparently with the character of the model: the sum of ladder diagrams does not satisfy the unitarity conditions in the t channel, nor does it satisfy the crossing symmetry condition.

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APPENDIX

BEHAVIOR OF $r(\kappa_1, \kappa_2, s)$ AS $\kappa_1 \to \infty$

Equation (7) has the following form:

$$r(\varkappa_1, \varkappa_2, s) = \lambda \int_0^1 \frac{dx\, x^{l(s)}}{\sqrt{L(u_1, u_2, x, s)}} \frac{1}{\pi} \int \frac{d\varkappa_1' d\varkappa_2' r(\varkappa_1', \varkappa_2', s)}{(\varkappa_1' + m^2)(\varkappa_2' + m^2)}, \qquad (A.1)$$

with the integration region given by the condition $L \geq 0$ and shown in Fig. 2. Since the integral with respect to κ_1' converges for fixed κ_1 and the region of integration with respect to κ_1' shifts towards larger κ_1' with increasing κ_1, $r(\kappa_1, \kappa_2, s)$ should decrease with increasing κ_1. Let us explain the character of this decrease. We change over in (A.1) to the variable u_1: $\kappa_1' = x\kappa_1 + u_1$; for fixed κ_2 the integration with respect to u_1 occurs over the finite region between the roots of the equation $L = 0$.

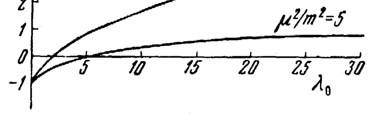

FIG. 4

μ^2/m^2 \ $l+1$	0.1	0.2	0.5	0.8	1	1,2	1,5	2	3	5
0	0,11	0.24	0.75	1.44	2.0	2.64	3.75	6	12	30
0.1	0.111	0.246	0.80	1.61	2.30	3.13	4,66	8.1	19	64.2
0.5	0.114	0.258	0.90	1.94	2.90	4.12	6.50	12.4	34	147
1	0,116	0.27	0.988	2.22	3.42	5,0	8.17	16.4	49.1	238
5	0,125	0.31	1.37	3.59	6.04	9.57	17.5	40.7	131	906

The integral with respect to du_1 is of the form

$$\int_{u^{(1)}(\varkappa_2')}^{u^{(2)}(\varkappa_2')} \frac{du_1 r(u_1 + x\varkappa_1, \varkappa_2')}{\sqrt{(u_1 - u^{(1)})(u^{(2)} - u_1)(u_1 + x\varkappa_1 + m^2)}}. \quad (A.2)$$

Let $r(\kappa_1', \kappa_2') \sim f(\kappa_2')(\kappa_1')^{-a}$ as $\kappa_1' \to \infty$. We break up the region x into parts: $0 \leq x < C/\kappa_1$, and the region where $x\kappa_1 \gg m^2$. In the second region the integral of (A.2) is $\sim f(\kappa_2', s)/(\tilde{u}_1 + x\kappa_1)^{a+1}$, where $\tilde{u}_1 \sim \kappa_2'$. Then

$$\int \frac{d\varkappa_2' f(\varkappa_2', s)}{(\tilde{u}_1 + x\varkappa_1)^{a+1}(\varkappa_2' + m^2)}$$

converges and behaves like $1/(\text{const} + x\kappa_1)^{a+1}$. Therefore the entire integral in the right half can be written in the form

$$\int_0^1 \frac{dx x^{l(s)}}{(C + x\varkappa_1)^{a+1}} \sim F\left(a+1, l+1, l+2; -\frac{\varkappa_1}{C}\right). \quad (A.3)$$

The asymptotic expansion (A.3) contains the terms κ_1^{-l-1} and κ_1^{-a-1}, and since the left half of (A.1) behaves like κ_1^{-a} we conclude that $a = l + 1$.

Thus, $r(\kappa_1, \kappa_2, s) \sim \kappa_1^{-l-1}$ as $\kappa_1 \to \infty$.

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