SECOND ORDER PHASE TRANSITIONS IN A BOSE FLUID

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A theory is proposed for second order phase transitions in liquid helium. The principal role in the phenomenon under consideration is played by long-wave fluctuations. On approach to the phase transition curve, the mean occupation numbers of low momentum states increase, and as a result the fluctuation correlation increases at large distances. In such a situation, not only the two-particle, but many-particle interactions become important. For this reason, the only small parameter that is introduced in the theory is the deviation from the transition temperature $|T - T_0|/T_0$.

The calculation employs the Green's function technique and the diagram technique. The chief quantities studied are the Green's function G, which determines the fluctuation spectrum, and the total vertex part □, which describes the two-particle scattering. The solution of the set of equations for these quantities is found. It has the form $G \sim [Ap^{3/2}]^{-1}$ (A is a constant) on the phase transition curve; $\square(p_i)$ is a zero-order homogeneous function.

Liquid helium near the phase transition curve can be described thermodynamically as an ideal gas of quasiparticles with a spectrum $\varepsilon = Ap^{3/2}$.

The theory shows that only the width of the phase transition region $\Delta T \sim V^2/T_0$ depends on the magnitude of the potential interaction between the particles V. Within this region, the transition is of a universal nature. With an accuracy to constant factors, the fluctuation spectrum and the particle scattering amplitude are the same for any $V > 0$, and do not depend on the details of the interaction at small distances. In the small momentum region, the effective interaction is determined by a dimensionless charge g. The condition of consistency of the theory uniquely defines g, which is thus independent of V. The complexity of the equations does not permit us to find g or even to demonstrate the existence of the root of the corresponding equation. However, some arguments can be presented to prove that the mathematical scheme proposed here is the only possible one.

The main theoretical conclusions of the theory are: 1) the logarithmic behavior of the specific heat C_p on both sides of the equilibrium curve; 2) the equality of the coefficients before $\ln(|T - T_0|/T_0)$ on both sides of the λ curve; 3) the appearance of a finite jump ΔC_p, superposed on the logarithmic curve. All of these results have been confirmed experimentally.

In conclusion, some simple physical considerations are presented which confirm the law $\varepsilon = Ap^{3/2}$ and the general problem of phase transitions is discussed.

1. INTRODUCTION

IN an ideal Bose gas of given density N, the phenomenon of "condensation" in momentum space takes place at a definite temperature T_0. At temperatures $T < T_0$, a finite fraction of the particles is in a state with momentum equal to zero. The Einstein condensation is a phase transition in which the derivative of the heat capacity C_V undergoes a finite jump. However, C_p has an infinite discontinuity and behaves like $(T - T_0)^{-1/2}$ near T_0; $T > T_0$. For $T < T_0$, the pressure is uniquely determined by the temperature, so that the concept of C_p loses its meaning.

A real system in which an Einstein condensation takes place is not an ideal gas but a liquid (liquid He^4). The strong interaction between the particles of the liquid changes the phase transition picture materially. Evidence of this is given by experimental data on the measurement of the heat capacity close to the critical temperature,[1]; these show a logarithmic increase in the heat

capacity as $T \to T_0$. The reason for the less abrupt behavior of the heat capacity C_p than in an ideal gas at temperatures close to T_0 is the strong repulsion between the helium atoms. This repulsion significantly decreases the compressibility of the material. Attraction would evidently have led to a condensation in real space, that is, to a first order phase transition.

In a Bose liquid, as in an ideal Bose gas the mean occupation numbers of low momentum states increase as T approaches T_0 from above. Particles with long wavelengths are strongly correlated with one another for suitably small interaction. Therefore, neither the weak interaction approximation, nor the "gaseous" approximation,[2] in which only two-particle scattering is taken into account, can be used. As the phase transition point is approached, the role of processes of single particle scattering is increased. Weak interaction or low gas density leads only to the result that the phase transition region is contracted. But inside the transition region the interaction cannot be regarded as weak.

2. STATEMENT OF THE PROBLEM

We shall describe a Bose-liquid by means of the Hamiltonian

$$H = \sum \varepsilon_{\mathbf{p}}^0 a_{\mathbf{p}}^+ a_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4} V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)\, a_{\mathbf{p}_1}^+ a_{\mathbf{p}_2}^+ a_{\mathbf{p}_3} a_{\mathbf{p}_4}, \tag{2.1}$$

where $a_{\mathbf{p}}^+$ and $a_{\mathbf{p}}$ are the creation and annihilation operators and $\varepsilon_{\mathbf{p}}^0 = p^2/2m$ (the volume occupied by the system is set equal to unity). Regarding $V(p_i)$, we shall assume that there exists a finite positive limit $V(0, 0, 0, 0) = V_0$. The interaction has the characteristic dimension p_0 in momentum space. The particle momenta of interest to us are small in comparison with p_0. Therefore, we can assume $V(p_1, p_2, p_3, p_4)$ equal to the constant value V_0, which corresponds to an interaction of the form $V_0\delta(\mathbf{r} - \mathbf{r}')$ in the coordinate space.

We shall apply the method of temperature Green's functions. The fundamental quantities subject to investigation are the single-particle Green's functions $G(p, \omega_n)$ and the vertex part $\square(p_1, p_2, p_3, p_4)$, the determination of which is given in the book of Abrikosov, Gor'kov and Dzyaloshinskiĭ.[3] The equations for these quantities are different for different sides of the phase transition curve. We begin with the description of the region lying above the phase transition curve (no condensation). The region of the existence of condensation will be considered in Sec. 6.

The total Green's function $G(p, \omega_n)$ has the form

$$G(p, \omega_n) = [i\omega_n + \mu - \varepsilon_p^0 - \Sigma_{\mu T}(p, \omega_n)]^{-1},$$

$$\omega_n = 2\pi n T \quad (n = 0, \pm 1, \pm 2 \ldots), \tag{2.2}$$

where $\Sigma_{\mu,T}(p, \omega_n)$ is the total part of the characteristic energy. The number of particles N is expressed most simply in terms of $G(p, \omega_n)$:

$$N = -T \sum_{\omega_n} \int d^3p\, G(p, \omega_n). \tag{2.3}$$

A singularity in N as a function of T and μ arises for those values of the variables for which

$$G^{-1}(0, \omega_n) = 0 \tag{2.4}$$

for any value of n. If Eq. (2.4) is satisfied for $n \neq 0$, then it is equivalent to two real equations and determines an isolated singularity. We shall not consider this case, but turn our attention to the situation in which there is a phase transition curve. Such a situation corresponds to satisfaction of Eq. (2.4) for $\omega_n = 0$ (since $\Sigma(p, 0)$ is real):

$$-\Sigma_{\mu, T}(0, 0) + \mu = 0. \tag{2.5}$$

Equation (2.5) determines the phase transition curve

$$\mu = \mu_0(T) \quad \text{or} \quad T = T_0(\mu).$$

We introduce the notation

$$\varphi_{\mu, T}(p) = \Sigma_{\mu, T}(p, 0) - \Sigma_{\mu, T}(0, 0), \tag{2.6}$$

by means of which one can write the Green's function in the form

$$G(p, 0) = -[\eta + \varepsilon^0(p) + \varphi_{\mu, T}(p)]^{-1}, \tag{2.7}$$

where

$$\eta = -\mu + \Sigma_{\mu, T}(0, 0). \tag{2.8}$$

For fixed T,

$$\eta = -[\mu - \mu_0 - \Sigma_{\mu, T}(0, 0) + \Sigma_{\mu_0, T}(0, 0)]. \tag{2.8'}$$

For fixed μ,

$$\eta = \Sigma_{\mu, T}(0, 0) - \Sigma_{\mu, T_0}(0, 0). \tag{2.8''}$$

It must be expected that as $p \to 0$ the value of $\varphi_{\mu,T}(p)$ falls off more slowly than ε_p^0. If this were not the case, then the phase transition would take place in the same way as in an ideal Bose gas. Further calculation confirms this supposition. The momentum for which ε_p^0 is equal to $\varphi_{\mu,T}(p)$ we shall denote by p_0. In a real Bose liquid, this quantity is identical in order of mag-

nitude with the characteristic momentum for the interaction, and also with the value of $N^{1/3}$. In the models, it is customary to take the smallest of the momenta described as the value of p_0. For $p \ll p_0$, ε_p will be neglected in Eq. (2.7). Then $G(p, 0)$ takes the form

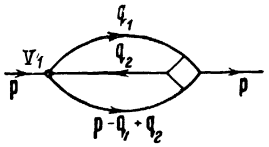
$$G(p, 0) = -\frac{1}{\eta + \varphi_{\mu, T}(p)}. \qquad (2.7')$$

In the construction of various quantities as the sum of the contribution of graphs, it is necessary to carry out summation over the frequencies of the internal lines. Close to the phase transition,

$$|G(p, 0)| \approx |(\eta + \varphi_{\mu, T}(p))|^{-1} \gg |G(p, \omega_n)| \approx |(i\omega_n)^{-1}| \quad (n \neq 0).$$

If only the region of small momenta $p \ll p_0$ makes a real contribution to the graph under consideration, then components with $\omega_n \neq 0$ can generally be neglected, producing an error in this case of the order of $\sim\eta/T$. If there is a contribution from the region $\sim p_0$, then one must also take into consideration the components with $\omega_n \neq 0$. But for graphs whose external momenta p_i are small in comparison with p_0, the contribution from the region $p \sim p_0$ does not depend on p_i. Therefore, in all the sums, we have left only components with $\omega_n = 0$, which results in this case in an error small relative to the error $\sim\eta/T$, or adds a constant quantity in other cases. The diagram technique becomes three dimensional.[1)] In what follows, we shall assume that all $\omega_n = 0$.

Let us consider the set of graphs for the self energy part $\Sigma(p)$. In each graph we separate its first section on the left by three lines with $\omega_n = 0$, $p \ll p_0$. Graphs which do not contain such an intersection at all, make after substraction by Eq. (2.6) a contribution $\sim p^2$ to $\varphi(p)$, which we shall neglect. The set of graphs which must be considered can be drawn in the following way:



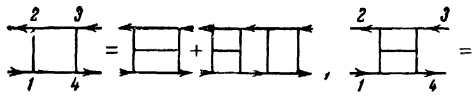
(2.9)

where V_1 is the sum of such graphs with four terminals, such that the bare vertex, into which enters the momentum p in the graph (2.9), cannot be separated from the graph by a cut on three lines with $\omega = 0$, $p \ll p_0$. Moreover, the graphs in V_1 which cannot be cut on two lines with $\omega = 0$, $p \ll p_0$. The square indicates the complete vertex part $\square$ with frequencies of the external lines $\omega_n = 0$.

The principal contribution to V_1 is made by integration over the region of high momenta and summation for $\omega_n \neq 0$. Therefore, in the region of small momenta, V_1 is a constant quantity with accuracy up to terms of the order p^2, which can be neglected. From the definitions of (2.6) and (2.7') we find the equation for φ (the parameters μ, T are omitted for brevity):

$$\varphi(p) = -\frac{T^2 V_1}{2(2\pi)^6}\int \frac{d^3q_1 d^3q_2}{(\eta + \varphi(q_1))(\eta + \varphi(q_2))} \times \left[\frac{\square(p, q_1, q_2, p + q_2 - q_1)}{\eta + \varphi(p + q_2 - q_1)} - \frac{\square(0, q_1, q_2, q_2 - q_1)}{\eta + \varphi(q_2 - q_1)}\right]. \qquad (2.10)$$

One must supplement the equation (2.10) with the equation for $\square$. It is convenient to introduce auxiliary quantities—the irreducible vertex parts $\mathcal{T}_{12;34}$, $\mathcal{T}_{13;24}$, $\mathcal{T}_{14;23}$. In particular, the vertex part which is "irreducible along the vertical," $\mathcal{T}_{12;34}$ is defined as the sum of graphs with four ends, with the exclusion of those which can be cut only along two lines with $\omega = 0$, $p \ll p_0$ in two parts, into one of which enter the momenta p_1 and p_2, and into the other, p_3 and p_4. In similar fashion the vertices $\mathcal{T}_{13;24}$, $\mathcal{T}_{14;23}$ are irreducible along the horizontal and diagonal. The complete vertex part $\square$ can be expressed in terms of each of the quantities $\mathcal{T}$ by means of an equation of the type



(2.11)

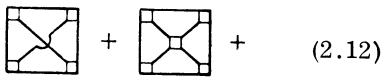
Equation (2.11) can be regarded as the analytic definition of $\mathcal{T}_{12;34}$.

We introduce the doubly nonintersecting vertex parts $\tilde{\mathcal{T}}_{12;34}$, $\tilde{\mathcal{T}}_{13;24}$, $\tilde{\mathcal{T}}_{14;23}$, which represent the set of graphs with four terminals which are not intersected along two lines with arbitrary frequencies and momenta. These vertex parts are distinguished from the nonintersecting vertices $\mathcal{T}_{12;34}$, $\mathcal{T}_{13;24}$, $\mathcal{T}_{14;23}$ by quantities which are constant in the region of small momenta.

We also introduce the absolutely irreducible vertex part which cannot be cut along two lines with arbitrary frequencies nor in any direction. The absolutely irreducible vertex part $\mathcal{T}^*$ can be represented in the form of a sum of graphs in each of which there enter only complete G func-

[1)] Such an approach to the problem was suggested by L. D. Landau and worked out by him and his co-workers in the Institute for Physics Problems of the USSR Academy of Sciences.

tions and complete vertex parts □. An exception is the first graph—the "bare" vertex V_0:



(2.12)

We are interested in the case in which to all terminals of the vertices there correspond $\omega = 0$ and small momenta $p \ll p_0$. In this case one can assume all the frequencies of the internal lines in the graphs (2.12) are equal to zero. The contribution from non-zero frequencies leads to a renormalization of the bare vertex V_0.

The vertex parts □, $\tilde{\mathcal{T}}$, $\mathcal{T}^*$ are connected by the logical identity

$$\square = \tfrac{1}{2}(\tilde{\mathcal{T}}_{12;34} + \tilde{\mathcal{T}}_{13;24} + \tilde{\mathcal{T}}_{14;23} - \mathcal{T}^*).$$

From this identity, we get the equation

$$\square = \tfrac{1}{2}(\mathcal{T}_{12;34} + \mathcal{T}_{13;24} + \mathcal{T}_{14;23} - \mathcal{T}^*) + \text{const}. \quad (2.13)$$

The set of equations (2.10)—(2.13) is complete.

3. THE FUNCTIONS φ AND □ ON THE TRANSITION LINE

We begin with the solution of Eqs. (2.10)—(2.13) on the phase transition line ($\eta = 0$). We seek $\varphi(p)$, $\square(p_i)$ and $\mathcal{T}$ in the form of homogeneous functions of their arguments of degree α, β, γ, respectively. At first glance, such an assumption appears to be arbitrary. However, one can hardly suppose any other physically reasonable dependence of the quantities φ and □ on the momenta. Actually, the value of φ vanishes for $p = 0$ more slowly than p^2, and has the characteristic dimension p_0. Therefore, in lowest order in p/p_0 it should have the form Ap^α. It is true that one could assume a priori that A and α are oscillating quantities of the type $A_0 + A_1 \cos(p_0/p)$. But fast oscillations for small p are not reproduced in integral equations.

For estimate of α, β, γ, we carry out a calculation of the powers in Eqs. (2.10)—(2.13). From the Dyson equation (2.10) we find

$$4\alpha - \beta = 6. \quad (3.1)$$

We now consider two possible cases separately:

1. $\alpha \geq 3/2$. In this case, $\beta \geq 0$ follows from (3.1). We have to deal with Eq. (2.11). We first consider the integral term—the second component on the right hand side. A formal calculation of the powers for this term leads to the expression $\beta + \gamma - 2\alpha + 3 = \gamma + \beta/2$ [use is made of the relation (3.1)]. But if this quantity is positive, then the integral formally diverges in the region of large momenta. In fact, this means that upon integration, the real part of the momenta $\sim p_0$ and the integral does not depend on the momenta of the external terminals.

Of the two vertices □ and $\mathcal{T}$, the larger is that whose degree of homogeneity is less. Thus we find

$$\min(\beta, \gamma) = \gamma + \beta/2. \quad (3.2)$$

It is not difficult to verify that the only solution of Eq. (3.2) which satisfies the condition $\beta \geq 0$ is $\beta = \gamma = 0$, $\alpha = 3/2$.

Here we consider one of the possibilities, in which the integral term of the equation is not small. In other cases, a consideration of the complete set of equations is necessary.

2. $\alpha < 3/2$. In this case, using Eqs. (4.14), (4.15), and (4.19) (see below), one can show that $\partial\Sigma(p, \omega_n)/\partial_\mu$ is large. As will be shown in Sec. 4, this leads to a contradiction.

Thus only the following values are possible: $\alpha = 3/2$, $\beta = \gamma = 0$.

One can attempt to construct a scheme in which φ and □ contain different powers of $L_1 = \ln(p_0/p)$ (and also the powers of $L_2 = \ln L_1$ etc.). Here it is seen that the series for an absolutely irreducible vertex can be broken off at the second term. As a result, either this equation or the equation for $\mathcal{T}$ cannot be satisfied. This is connected with the impossibility of compensating L_n for any finite n. In our variant, as will be shown below, such compensation takes place at the expense of all the terms of the series for $\mathcal{T}^*$ [see Eq. 3.9)].

We shall seek a solution for φ in the form

$$\varphi(\mathbf{p}) = Ap^{3/2}, \quad (3.3)$$

and assume □ to be a homogeneous function of zero order. Equation (2.9) takes the following form under these assumptions

$$2(2\pi)^6 \frac{A^4}{V_1 T^2} = -\int \frac{d^3q_1 d^3q_2}{q_1^{3/2} q_2^{3/2}} \left(\frac{\square(\mathbf{n}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{n} - \mathbf{q}_1 + \mathbf{q}_2)}{|\mathbf{n} - \mathbf{q}_1 + \mathbf{q}_2|^{3/2}} - \frac{\square(0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_2 - \mathbf{q}_1)}{|\mathbf{q}_2 - \mathbf{q}_1|^{3/2}} \right), \quad (3.4)$$

where $\mathbf{n}$ is an arbitrary unit vector. The integral on the right hand side of (3.4) converges only in the case in which $\square(p_1, p_2, p_3, p_4)$ becomes a constant, when two arguments are significantly larger than the other two. In the opposite case, an uncompensated divergence arises in integration, for example, over q_1. This property of $\square(p_i)$ agrees with the equations for the vertex part. In general, there arise two different limits $\square(p, p, 0, 0) = \square(p, 0, 0, p)$ and $\square(p, 0, -p, 0)$

$= \square(0, p, 0, -p)$. In what follows, it will be shown that

$$\square(p, p, 0, 0) = 0. \tag{3.5}$$

The value of the quantity $\square(p, 0, -p, 0)$ is not known to us.

We now proceed to Eq. (2.11). It is written explicitly in the following fashion:

$$\square(p_1, p_2, p_3, p_4) = \mathcal{T}_{12;34}(p_1, p_2, p_3, p_4)$$
$$-T\int d^3q\, \square(p_1, p_2, q, q + p_1 - p_2)$$
$$\times \mathcal{T}_{12;34}(q, q - p_1 - p_2, p_3, p_4)$$
$$\times G(q)\, G(q + p_1 - p_2) + \text{const}, \tag{3.6}$$

where the constant represents the contribution from the region of momenta $\sim p_0$ and frequencies different from zero. We shall regard it as an equation for the determination of $\mathcal{T}_{12;34}$ for given $\square(p_i)$. Equation (2.11) is linear and inhomogeneous.

The analysis given in the Appendix shows the solution of Eq. (3.6) is determined uniquely if we take into account the conditions imposed on $\mathcal{T}_{12;34}$ by Eqs. (2.12), (2.13). The solution of $\mathcal{T}_{12;34}$ is seen to be a homogeneous function of zero order satisfying the condition

$$\mathcal{T}_{12;34}(p, p, 0, 0) = \text{const}. \tag{3.7}$$

The same results are also valid for the function $\mathcal{T}_{14;23}$. These confirmations also apply to the quantity $\mathcal{T}_{13;24}$ if the equality $\square(p, 0, -p, 0) = 0$ is satisfied. If $\square(p, 0, -p, 0) \neq 0$, then $\mathcal{T}_{13;24}$ is seen to be a small quantity of the order of $1/\ln[p_0/|p_1 + p_3|]$. We now assume $\mathcal{T}_{12;34}$, $\mathcal{T}_{13;24}$ and $\mathcal{T}_{14;23}$ are known functionals of $\square$, and substitute in (2.12) the absolutely irreducible vertex part $\mathcal{T}^*$ for the series (2.13). A closed equation is obtained for $\square$.

Let us consider the isolated graph entering into the series (2.13) for the absolutely irreducible vertex $\mathcal{T}^*$. A simple calculation of the powers shows that each graph diverges logarithmically for large momenta. We emphasize that the divergence is connected with the region in which all the momenta of the internal lines are large. This takes place for the reason that in the graphs for $\mathcal{T}^*$ there are no parts which have an internal structure and which are joined with the remainder by only four lines. Any part of the graph which is joined to the remainder by a larger number of lines converges in the region of large momenta.

Thus the equation for $\square$ takes the form

$$\Phi_0(\square)\ln(p_0/p) + \Phi_1(\square) + V_0 + V' = 0. \tag{3.8}$$

Here p is the maximum in the values of the arguments of $\square$. The quantity Φ_0 represents the sum of coefficients for $\ln(p_0/p)$ from each of the diagrams. Φ_1 is represented in the form of a sum of the quantities $2\square - \mathcal{T}_{12;34} - \mathcal{T}_{13;24} - \mathcal{T}_{14;23}$ and contributions from different diagrams for $\mathcal{T}^*$ from the residues of the diverging parts. The quantity V' represents a constant which arises in integration over the region $p \sim p_0$ and summation over $\omega_n \neq 0$. The functional $\Phi_0(\square)$ does not depend on external momenta since integration in each of the graphs (2.13) takes place over a region of momenta of the internal lines $q_i \gg p$. The principal contribution to $\Phi_1(\square)$ is made by the region $q_i \sim p$ (because of the homogeneity of $\square$). Therefore $\Phi_1(\square)$ depends on the external momenta p_i. In order to make (3.8) agree with the initial assumptions on $\square$, it is necessary to set

$$\Phi_0(\square) = 0. \tag{3.9}$$

Equation (3.9) does not determine $\square$ completely, but imposes only a single numerical condition. Equation (3.6) now takes the following form:

$$\Phi_1(\square) = V; \qquad V = V_0 + V'. \tag{3.10}$$

From Eq. (3.10), the vertex part $\square$ is determined as a function of the momenta and the parameter V: $\square = \square(p_i, V)$. Substituting the value of $\square$ found from (3.10) in (3.9), we get an equation[2] for V:

$$\Phi_0(\square(p_i, V)) = 0. \tag{3.11}$$

We have not succeeded in finding the root of Eq. (3.11) or even of showing its existence since the structure of the equation is extremely complicated. If it were shown that Eq. (3.11) does not have a root, then all of the schemes created here would be invalid. In particular, the power approximations for φ in $\square$ would be invalid. We do not see any other physically realizable scheme in which the fundamental role is played by small momenta. Therefore we shall assume that Eqs. (3.10) and (3.11) have a real solution.

Let us analyze some properties of the assumed solutions. In the transition in the graphs for $\mathcal{T}^*$ from n-th to (n + 1)st order, there is added one vertex, two G functions, one integration over momentum space and the factor T. Therefore the series for $\Phi_0(\square)$ and $\Phi_1(\square)$ are power series in the quantity

[2] Equation (3.11) can be regarded as the condition for joining the solution for $p \ll p_0$ with the solution in the region $p \sim p_0$. See also Appendix 3.

$$\Gamma(\mathbf{p}_i) = T\square(\mathbf{p}_i)/(2\pi)^3A^2. \quad (3.12)$$

We introduce the "dimensionless charge" g by the relation

$$g = TV/A^2. \quad (3.13)$$

After this, the parameter V is eliminated from Eqs. (3.9)—(3.11), and they become completely universal. This means that the dimensionless charge represents a universal constant, while $\Gamma(p_i)$ is a universal function.

By substituting (3.12) in Eq. (3.4), we get the following equation for A

$$A^2 = QV_1T, \quad (3.14)$$

where the numerical constant Q is determined by the integral:

$$Q = \frac{1}{2(2\pi)^6}\int \frac{d^3\mathbf{q}_1 d^3\mathbf{q}_2}{q_1^{3/2}q_2^{3/2}}\left[\frac{\Gamma(0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_2-\mathbf{q}_1)}{|\mathbf{q}_2-\mathbf{q}_1|^{3/2}} - \frac{\Gamma(\mathbf{n}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{n}+\mathbf{q}_2-\mathbf{q}_1)}{|\mathbf{n}+\mathbf{q}_2-\mathbf{q}_1|^{3/2}}\right]. \quad (3.15)$$

Finally, substituting (3.14) in (3.12) we find

$$\square = QV_1\Gamma(\mathbf{p}_i)(2\pi)^3. \quad (3.16)$$

Equations (3.14) and (3.16) make it possible to estimate the values of A^2 and $\square$. We now attempt, albeit very roughly, to find the signs of these quantities. In a rough approximation, we replace the quantity Γ by a constant, determined by the equation

$$\Phi_0(\Gamma) = 0. \quad (3.9')$$

Consideration of the graphs shows that $\Phi_0(\Gamma)$ is a power series in Γ with alternating coefficients. Therefore, if the solution of (3.9′) exists, it is positive, which corresponds to an effective repulsion of particles. From (3.15), we find, in the approximation under consideration,

$$Q = \tfrac{4}{15}\Gamma. \quad (3.17)$$

Consequently, Eq. (3.14) can be solved only for $V_1 > 0$.

4. VICINITY OF THE PHASE TRANSITION CURVE

Let us consider the behavior of the functions φ and $\square$ in the vicinity of the phase transition curve. We shall assume $\eta \neq 0$ for a fixed value of T. It is natural to expect that for sufficiently small η the functions $\varphi(p)$ and $\square$ do not change too strongly in the region where $\varphi(p) \geq \eta$. Therefore, for estimate of $\varphi(\mathbf{p})$, we substitute the quantity $\varphi(p) = Ap^{3/2}$ and $\square$ from the formula (3.14) in the integral on the right hand side of (2.10). We get

$$\varphi_\eta(\mathbf{p}) = Ap^{3/2}f_0(\eta/Ap^{3/2}), \quad (4.1)$$

where

$$f_0(x) = F_0(x)/F_0(0),$$

$$F_0(x) = \int \frac{d^3\mathbf{q}_1 d^3\mathbf{q}_2}{(x+q_1^{3/2})(x+q_2^{3/2})}\left\{\frac{\square(\mathbf{n}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{n}+\mathbf{q}_2-\mathbf{q}_1)}{x+|\mathbf{n}+\mathbf{q}_2-\mathbf{q}_1|^{3/2}} - \frac{\square(0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_2-\mathbf{q}_1)}{x+|\mathbf{q}_2-\mathbf{q}_1|^{3/2}}\right\}. \quad (4.2)$$

It is therefore natural to seek a solution of Eqs. (2.10)—(2.13) in the form

$$\varphi(\mathbf{p}) = Ap^{3/2}f(\eta/Ap^{3/2}), \quad (4.3)$$

$$\square = QV_1\Gamma_1(\mathbf{p}_i, (\eta/A)^{2/3}), \quad (4.4)$$

where f(x) is some standard function (which does not contain the parameters of the problem); $\Gamma_1(p_i, x)$ is a standard homogeneous function of zero order. The functions f(x) and $\Gamma(p_i, x)$ obey the normalization conditions:

$$f(0) = 1, \quad (4.5)$$

$$\Gamma_1(\mathbf{p}_i, 0) = \Gamma(\mathbf{p}_i); \quad (4.6)$$

$\Gamma(p_i)$ is determined in the preceeding section.

Equation (2.10) takes the form

$$f(x) = F(x)/F(0), \quad (4.7)$$

$$F(x) = \int \frac{d^3\mathbf{q}_1 d^3\mathbf{q}_2}{[x+q_1^{3/2}f(xq_1^{-3/2})][x+q_2^{3/2}f(xq_2^{-3/2})]} \times\left[\frac{\Gamma_1(\mathbf{n}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{n}-\mathbf{q}_2-\mathbf{q}_1, x^{2/3})}{x+|\mathbf{n}+\mathbf{q}_2-\mathbf{q}_1|^{3/2}f(x|\mathbf{n}+\mathbf{q}_2-\mathbf{q}_1|^{-3/2})} - \frac{\Gamma_1(0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_2-\mathbf{q}_1, x^{2/3})}{x+|\mathbf{q}_2-\mathbf{q}_1|^{3/2}f(x/|\mathbf{n}+\mathbf{q}_2-\mathbf{q}_1|)}\right]. \quad (4.8)$$

Equations (2.11) and (2.12) transform into standard equations for the function $\Gamma_1(p_i, 1)$. The dimensionless charge $g = VT/A^2$ as before reamins a universal constant (it would appear, the same is also true of the transition curve, since it is determined by the values of $\Gamma(p_i, 1)$ in the region $p_i \gg 1$). We note that the factors V and A^2 individually do not depend on μ and T and their derivatives on the transition curve are logarithmically large. The equations under consideration are universal with an accuracy to small terms of the order of η/T or $(p/p_0)^{1/2}$.

As $x \to \infty$, the function f(x) behaves as $x^{-1/3}$. Thus for $p \ll p_\eta = (\eta/A)^{2/3}$, we have $\varphi(p) = \text{const} \cdot p^2A^{-4/3}\eta^{-1/3}$. This result is to be expected

since the singularity in $G(p)$ should not arise for $\eta > 0$. The vertex part of $\Gamma_1(p_i, 1)$ for $p_i \ll 1$ becomes a constant quantity.

In order to make clear the behavior of thermodynamic quantities close to the transition curve, it is necessary to investigate $f(x)$ and $\Gamma_1(p_i, x)$ for small x. In the first place, it is evident that the quantity $f'(0)$ cannot be infinite. In the opposite case, one could neglect η in comparison with $q^{3/2} f(\eta/Ap^{3/2})$ and in the denominators of the G functions. But after this, the set of equations for G and $\square$ would have been the same as for $\eta = 0$, and we would reach the conclusion that $f(x)$ does not depend on x, in contrast with what was assumed. Therefore, $f(x)$ has the form

$$f(x) = 1 + f_1 x + \ldots \quad (f_1 = \text{const}). \tag{4.9}$$

We consider the structure of any term of the series which represents $\mathcal{J}^*$. For small $\eta \ll Ap_i^{3/2}$, we expand $G(p, \eta)$ in a series in $\eta/Ap^{3/2}$:

$$G(p) = -\frac{1}{Ap^{3/2}} + \frac{\eta}{A^2 p^3} + \ldots . \tag{4.10}$$

The contribution to the increment of $\mathcal{J}^*$ from the internal line of the graph with momentum q (for small x) has the structure

$$\eta \int \frac{d^3q}{A^2q^3} \left(\prod_i d^3q_i / \prod_j Aq_j^{3/2} \right) \prod_s \Gamma_0(q_s), \tag{4.11}$$

where q_i are the remaining independent momenta of the external lines, q_j are all the momenta of the internal lines, q_s are the momenta of lines entering into a single vertex. The integral over q diverges logarithmically for small q. Clearly one must cut off the integration over q from below in (4.11) for $q \sim (\eta/A)^{2/5}$. The integral of (4.11) converges on the high momentum side. The principal contribution to the integral is made by the region of momenta $q \sim p_i$.

Thus the estimate of (4.11) for small η gives

$$\Gamma_1(p_i; x^{2/3}) = \Gamma(p_i) + \gamma_1 \frac{x}{p^{3/2}} \ln \frac{p^{3/2}}{x}. \tag{4.12}$$

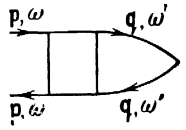
Here $\Gamma(p_i)$ is the value of $\Gamma_1(p_i, x^{2/3})$ for $x = 0$ (see the previous section), p is the maximum of the arguments of the function Γ_1, and γ is a constant.

The coefficients in the expansions of $f(x)$ and $\Gamma_1(p, x^{2/3})$ in powers of x could be obtained by solving the equation for $f(x)$ and Γ_1 by the method of successive approximations. Of course the actual solution of these equations is scarcely possible at the present time, since the equations themselves are very complicated and the solution of $\Gamma(p)$ is not known at $x = 0$. Some evidence on the solutions can be obtained by means of the Ward identity (see [3]) which we write in the following fashion:

$$\frac{\partial \Sigma(\mathbf{p}, \omega_n)}{\partial \mu} = T \sum_{\omega_k} \int d^3q \square(\mathbf{p}\omega, \mathbf{p}\omega, \mathbf{q}\omega'; \mathbf{q}\omega') G^2(\mathbf{q}, \omega'). \tag{4.13}$$

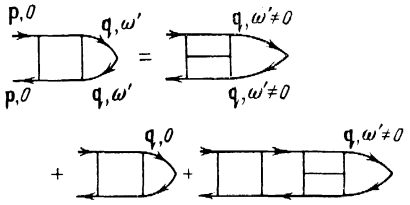
The right side of Eq. (4.13) can be drawn graphically:

$$\frac{\partial \Sigma(p,\omega)}{\partial \mu} =$$



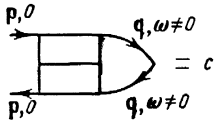
(4.14)

We first consider the case $\omega = 0$, $p \ll p_0$. We represent the set of graphs (4.14) in the form



(4.15)

Here a set of diagrams is chosen which are not cut by the two lines with $\omega = 0$, $p \ll p_0$. It is not difficult to see that the quantity



(4.16)

does not depend on p. From (4.14)—(4.16), we get

$$\frac{\partial \Sigma(\mathbf{p}, 0)}{\partial \mu} = c + (1 - c) P(x), \tag{4.17}$$

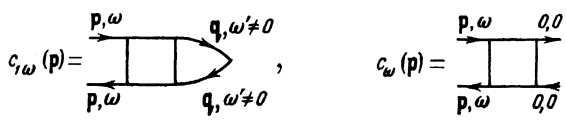
$$P(x) = T \int d^3q \square(\mathbf{p}, \mathbf{p}, \mathbf{q}, \mathbf{q}) G^2(\mathbf{q}) = \int \Gamma(\mathbf{n}, \mathbf{n}, \mathbf{q}, \mathbf{q}, x^{1/3}) \frac{d^3q}{[x + q^{3/2} f(x/q^{3/2})]^2}. \tag{4.18}$$

In the integral on the right hand side of (4.18), the quantities $\square$ and G are taken for $\omega = 0$.

In similar fashion we get, for $\omega \neq 0$ or $p \sim p_0$:

$$\frac{\partial \Sigma(\mathbf{p}, \omega_n)}{\partial \mu} = c_{1\omega}(p) + c_\omega(p)(1 - c) \int G^2 (1 - P)\, d^3q, \tag{4.19}$$

where $c_{1\omega}$, c_ω are determined by the graphical equations:



(4.20)

We shall show that $P(x)$ does not diverge as $x \to \infty$. If the contrary is assumed, then, by virtue of (4.17)—(4.19), all the derivatives $\partial\Sigma(p, \omega)/\partial\mu$ on the transition curve become infinite. In the equations for G and □, the value of μ enters only through the Green's function G:

$$G = [\mu - \Sigma_\mu(\mathbf{p}\ \omega_n) + i\omega_n - \varepsilon_p^{\}}]^{-1}.$$

In accord with the assumption, one can neglect $\mu - \mu_0$ in comparison with $\Sigma_\mu(p) - \Sigma_{\mu_0}(p)$. After such a neglect, the explicit dependence on μ vanishes from the equations of the theory and they have a unique (by continuity) solution $\Sigma_\mu(p) = \Sigma_{\mu_0}(p)$, in contradiction with the assumption.

Thus, as $x \to \infty$, the value of $P(x)$ tends to a universal constant value $P(\infty)$. It follows from (4.17), (4.19), that $\partial\Sigma(p, 0)/\partial\mu\,|_{\mu=\mu_0, p \ll p_0}$ is also a bounded quantity, whereas $\partial\Sigma(p, \omega_n)/\partial\mu$ diverges logarithmically.

Using the Ward identity (4.17)—(4.18), we can get the equation for

$$f'(x) = -\frac{P(x) - P(\infty)}{1 - P(\infty)}. \qquad (4.21)$$

Equation (4.21) can be used in place of (4.7) and (4.8), with the initial condition $f(0) = 1$. In deriving (4.21), we have made use of the fact that

$$\partial\eta/\partial\mu = -(1 - \partial\Sigma(0)/\partial\mu)$$

$$= -(1 - c)(1 - P(\infty)). \qquad (4.22)$$

The quantity $\partial\Sigma(0)/\partial\mu$ is not universal. Therefore, the vanishing of $1 - \partial\Sigma(0)/\partial\mu$ can only take place accidentally, at isolated points. The value of $1 - \partial\Sigma(0)/\partial\mu$ cannot be negative. In the opposite case, η could be negative for $\mu - \mu_0 < 0$ and the Green's function would have a pole for $\varphi(p) = -\eta$. This means that the mean occupation number of the particles could be negative for p close to the pole, which is physically impossible. Thus we finally obtain[3)]

$$\eta = -a(\mu - \mu_0) \qquad (0 < a < 1),$$

$$a = (1 - P(\infty))(1 - c). \qquad (4.23)$$

If μ and T change simultaneously, then η can be written in the form of the linear combination

$$\eta = -a(\mu - \mu_0) + b(T - T_0) \quad (b > 0) \qquad (4.24)$$

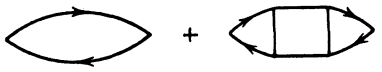
with coefficients a and b depending on the point (μ_0, T_0) on the transition curve. Such a possibility arises from the fact that the phase transition curve can be written in the form $\eta = 0$. On the other hand, the derivative $(\partial\mu/\partial T)_\lambda$ on the transition curve is generally finite and different from zero.

5. THE THERMODYNAMICS OF THE PHASE TRANSITION

We shall find the value of $\partial N/\partial\mu$:

$$\frac{\partial N}{\partial\mu} = T\sum_{\omega_n}\int d^3\mathbf{p}G^2(\mathbf{p}, \omega_n)\left(1 - \frac{\partial\Sigma(\mathbf{p}, \omega_n)}{\partial\mu}\right). \qquad (5.1)$$

Equation (5.1) is represented graphically in the form

$$\frac{\partial N}{\partial\mu} =$$  $$. \qquad (5.2)$$

Using the same considerations as in Sec. 4 [see Eqs. (4.13)—(4.19)], we find

$$\frac{\partial N}{\partial\mu} = c_1 + (1 - c)^2(1 - P(0))T$$

$$\times\int\frac{d^3\mathbf{p}}{[\eta + Ap^{3/2}f(\eta/Ap^{3/2})]^2}. \qquad (5.3)$$

The values of c_1 and c are finite on the transition curve.

The integral on the right hand side of (5.3) diverges logarithmically for large p. Therefore $\partial N/\partial\mu$ can be written in the form

$$\frac{\partial N}{\partial\mu} = \frac{R}{V}\ln\frac{T_0}{\eta} + R_1, \qquad (5.4)$$

$$R = \frac{(1 - c)^2(1 - P(0))}{2\pi^2}g, \qquad (5.5)$$

where R and R_1 are finite on the transition curve. Integrating (5.4), we get

$$N - N_0 = -\frac{R}{aV}\eta\ln\frac{T_0}{\eta}, \qquad (5.6)$$

where N_0 is the regular part of N.

The thermodynamic potential Ω is expressed in term of N by the equation

$$\Omega = \Omega_0 - \frac{R}{2a^2V}\eta^2\ln\frac{T_0}{\eta} \qquad (5.7)$$

(the integration is carried out for fixed T). From (5.6) and (5.7) we get the result that

$$\Omega = \Omega_0 - \frac{R}{2a^2V}\eta^2\ln\frac{T_0}{\eta}. \qquad (5.8)$$

We express C_V in the variable μ and T (see [4], p. 88):

$$C_v = T\frac{\Omega_{\mu T} - \Omega_{\mu\mu}\Omega_{TT}}{\Omega_{\mu\mu}}. \qquad (5.9)$$

It is not difficult to establish the fact that, because of the linear dependence (4.23) of η on $\mu - \mu_0$ and $T - T_0$, the logarithmic term does not appear in C_p. For calculation of C_p, we use the well-known formula

$$C_p = C_v - T\left(\frac{\partial p}{\partial T}\right)_{vN}^2\Big/\left(\frac{\partial p}{\partial v}\right)_{TN} \qquad (5.10)$$

[3)] See the note at the end of the article.

and express the derivative entering into (5.10) in the variables

$$\left(\frac{\partial p}{\partial T}\right)_{vN} = s - \frac{N\Omega_{\mu T}}{\Omega_{\mu\mu}}, \quad \left(\frac{\partial p}{\partial v}\right)_{TN} = \frac{N^2}{\Omega_{\mu\mu}}. \tag{5.11}$$

Here s is the entropy per unit volume of the liquid, N is the density. Substituting (5.11) in (5.10) and using (5.8), (4.23), we find

$$C_p = \frac{T(s+Nb/a)^2}{N^2V} R \ln \frac{T_0}{\eta}. \tag{5.12}$$

The ratio b/a is identical in value with the derivative $(d\mu/dT)_\lambda$ along the phase transition curve. Actually, we get for (4.23)

$$(d\mu/dT)_\lambda = b/a. \tag{5.13}$$

Furthermore, we express $(d\mu/dT)_\lambda$ in terms of the entropy s and the measured physical quantities p and T. We have

$$\left(\frac{d\mu}{dT}\right)_\lambda = -\frac{s}{N} + \frac{1}{N}\left(\frac{dp}{dT}\right)_\lambda. \tag{5.14}$$

Substituting (5.13) and (5.14) in (5.12), we get

$$C_p = \frac{T_0}{N^2}\left(\frac{dp}{dT}\right)_\lambda^2 \frac{R}{V} \ln \frac{T_0}{T - T_0}. \tag{5.15}$$

We note that the following equation is satisfied in the phase transition curve.

$$(\partial p/\partial T)_{vN} = (\partial p/\partial T)_\lambda.$$

The fact that the coefficient in front of the logarithm increases as the interaction in (5.15) decreases should cause no surprise, since the heat capacity in an ideal gas increases more rapidly close to T_0. Thus

$$C_p^{id} \sim V\overline{T_0/(T - T_0)}. \tag{5.16}$$

Comparing (5.15) with (5.16) we find the region of logarithmic phase transition:

$$(T - T_0)/T_0 \sim (V_0/T_0)^2. \tag{5.17}$$

The same estimate can be made in another way. One can define $T - T_0$ in order of magnitude as the kinetic energy of the particle when it becomes equal to the potential:

$$T - T_0 \sim p_0^2/2m \sim Ap_0^{3/2}. \tag{5.18}$$

From the latter equality, we find $p_0 = 2mA^2$ and, substituting it in the first, we reach the same estimate.

The thermodynamic relations also lead to logarithmic singularities in the compressibility and the coefficient of linear expansion.

6. REGION OF EXISTENCE OF CONDENSATION

Below the λ curve, a condensation appears in momentum space: a macroscopically large fraction of the particles of the liquid are in the state with p = 0. For a description of a Bose liquid in the presence of a condensation, we make use of the mathematical apparatus developed by Belyaev [2] (see also [3]). Here we shall take it into account that the density of the condensation N_0 is small close to the λ curve.

Below the phase transition curve the Green's function G(p, ω) has the form [2]

$$G(p,\omega) = \frac{\mu - \Sigma_1(p,\omega) - p^2/2m - \omega}{(\mu - \Sigma_1(p,\omega) - p^2/2m + \omega)(\mu - \Sigma_1(p,\omega) - p^2/2m - \omega) - \Sigma_2(p,\omega)^2}. \tag{6.1}$$

Here Σ_1, Σ_2 are the self-energy parts, Σ_1 corresponds to a graph with one input and one output, and Σ_2 to one out two inputs (outputs) of condensation particles.

In all the quantities of interest to us we select the sum of graphs which do not contain condensation lines, and note the contribution of such graphs by the superscript (0). The remaining quantities we denote by the index (1). All quantities with index (0) are analytic continuations of the corresponding quantities above the λ curve. For example, $\Sigma_1^{(0)}(p, \omega)$ is the analytic continuation of the function $\Sigma(p, \omega)$ above the λ curve. Obviously, $\Sigma_2^{(0)} = 0$.

Let us consider the quantities Σ and □ for ω = 0 (the argument ω = 0 will be omitted below). The relation derived by Belyaev (see [3]) connects $\Sigma_1(0)$ and $\Sigma_2(0)$ with μ:

$$\mu = \Sigma_1(0) - \Sigma_2(0). \tag{6.2}$$

From (6.2), we immediately get

$$\Sigma_1^{(1)}(0) - \Sigma_2(0) = -\eta = \mu - \Sigma_1^{(0)}(0). \tag{6.3}$$

The equations of Pines and Hugenholtz [5] (see also [3], p. 293) make it possible to get the exact relation between $\Sigma_1^{(0)}$ and Σ_2:

$$\partial(\Sigma_1^{(1)} - \Sigma_2)/\partial N_0 = \Sigma_2/N_0. \tag{6.4}$$

From (6.3) and (6.4) we get

$$\Sigma_2 = -(\partial \ln N_0/d\eta)^{-1}. \tag{6.5}$$

It is natural to assume that N_0 is proportional to η for small η (this will be supported by a calculation). It then follows from (6.5) (for small η) that

$$\Sigma_2(0) = -\eta, \tag{6.6}$$

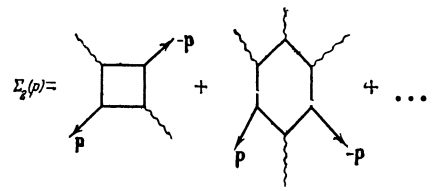
and from (6.6) and (6.3) we get

$$\Sigma_1^{(1)}(0) = 2\Sigma_2(0) = -2\eta. \tag{6.7}$$

Above the transition curve $\eta > 0$, below, $\eta < 0$.

Therefore $\Sigma_1^{(0)}(0)$ and $\Sigma_2(0)$ are positive.

We now determine N_0 as a function of η. For this purpose, we represent $\Sigma_2(p)$ in the form of a power series in N_0:



(6.8)

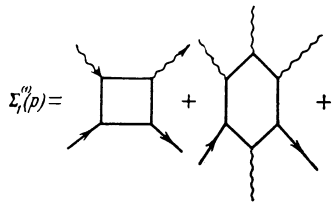
All the multipoles entering into (6.8) do not contain condensation lines. In particular, for $p = 0$, a 2n-pole behaves as $C_n V^{n-1}/\eta^{n-2}$, where C_n is a universal constant. Therefore, we get the following equation for N_0 from (6.7) and (6.8):

$$y\left(\frac{N_0 V}{\eta}\right) = 0, \qquad y(x) = 1 + \sum_{n=2}^{\infty} C_n x^n. \tag{6.9}$$

Thus N_0 is associated with the root x_0 ($x_0 \neq 0$) of the standard function $y(x)$ in the following fashion:

$$N_0 V/\eta = x_0. \tag{6.10}$$

Knowing N_0, we can find $\Sigma_2(p)$ as a function of η and p by using Eq. (6.8). It is not difficult to see that $\Sigma_2(p)$ is a standard function of $\eta/Ap^{3/2}$. The function $\Sigma_1^{(1)}(p)$ also takes the same form and can be drawn in the series:



(6.11)

It is important to note that in the region $Ap^{3/2} \gg \eta$ the quantity $\Sigma_1^{(1)}$ becomes a small quantity of the order of $\eta^2/Ap^{3/2}$. This follows from the fact that in this region all the 2n-poles with $n \geq 3$ fall off as $p^{-3(n-2)/2}$. In Sec. 4 it was shown that the vertex part of $\square$ (p, p, 0, 0) also falls off (as $\eta/Ap^{3/2}$).

From the given property of $\Sigma_1^{(1)}(p)$, it follows that not only G, but also $\partial G/\partial\mu$ is discontinuous on the transition line. We recall that everything pointed out above refers only to the case $\omega \neq 0$, the value of $\Sigma_1^{(1)}$ is represented by the series (6.11) just as for $\omega = 0$. By neglecting all the terms of the series (6.11) except the first, we find

$$\Sigma_1^{(1)}(p, \omega) = -c_\omega (1 - P(0)) N_0. \tag{6.12}$$

The remaining terms in the series have a higher order of smallness in η/T. In similar fashion, we get the result that $\Sigma_2(p, \omega) \sim VN_0 (\omega \neq 0)$.

We again begin the calculation of the thermodynamic functions with the quantity $\partial N/\partial\mu$

$$\frac{\partial N}{\partial \mu} = -T \sum_{\omega} \int \frac{\partial G}{\partial \mu} d^3\mathbf{p} + \frac{\partial N_0}{\partial \mu}. \tag{6.13}$$

The continuity of $\partial G/\partial\mu$ on the transition line makes it possible to conclude immediately that $\partial N/\partial\mu$ is proportional to $\ln(T/|\eta|)$ below the transition curve as before, with the same coefficient.

In Fig. 1 are shown the experimental values of the heat capacity close to the transition point, taken from the review of Buckingham and Fairbank.[1]

In addition, for the logarithmic singularity, the heat capacity C_p experiences a finite jump at the phase transition point. It is not difficult to establish the fact that C_V does not have such a jump. Simple calculations with the use of Sec. 5 give

$$\Delta C_p = T \frac{(\partial p/\partial T)_\lambda^2}{VN^2} \left\{ \frac{\partial N_0}{\partial \mu} [1 - P(\infty)](1 - c) + T \int d^3\mathbf{p} \left[\left(\frac{\partial G}{\partial \mu}\right)_{\eta<0} - \left(\frac{\partial G}{\partial \mu}\right)_{\eta>0} \right] \right\}. \tag{6.14}$$

The first term on the right hand side of (6.14) is connected with the appearance of the condensation; the second evidently corresponds to the generation of a new branch of elementary excitations (phonons). Both components are positive and have the same order of $T(\partial p/\partial T)_\lambda^2/VN^2$. Thus the heat capacity curve as a function of $|T - T_0|$, for $T < T_0$, should be higher than for $T > T_0$. This conclusion is also substantiated by experiment (see Fig. 1).

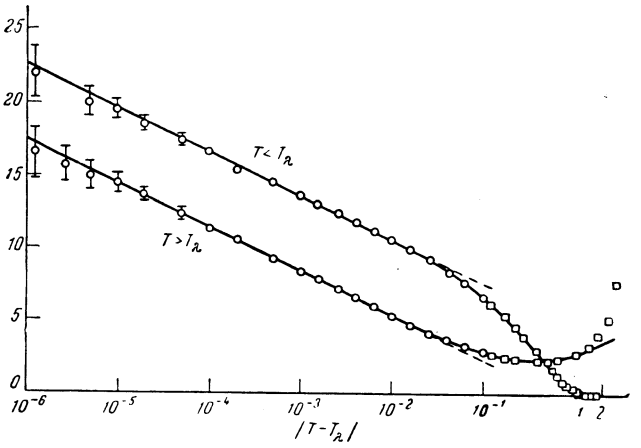


FIG. 1

7. THE PROBLEM OF SECOND ORDER PHASE TRANSITION

The theory proposed in the present work possesses greater generality than that put on it on the basis of the assumption. Actually, we see that in the most important region $\eta \ll T_0$, $p \ll p_0$, the mean occupation number n_p is seen to be large. This means that one can neglect the noncommutability of the operators a_p and a_p^+ and regard them as numbers. In other words, our theory applies to the classical wave field $\psi(r)$, whose Hamiltonian has the form

$$H[\psi] = \frac{1}{2}\int (\nabla\psi)^2 d^3\mathbf{r} - \mu\int \psi^2 d^3\mathbf{r} + \frac{V_0}{2}\int \psi^4(r)\, d^3\mathbf{r}. \quad (7.1)$$

The theory of the classical field $\psi(r)$ with the Hamiltonian (7.1) can be regarded as the generalization of the phase transition theory of Landau.[4] In the Landau theory, the quantity ψ plays the role of an ordering parameter and does not depend on the coordinates. The generalization given in the Landau theory is connected with the necessity of taking into account order fluctuations close to the phase transition curve. In our theory, singularities of thermodynamic quantities, if they arise, should have a logarithmic character. From this point of view, the logarithmic singularity of the heat capacity in the two-dimensional Ising model, found by Onsager,[6] does not seem to us accidental. Apparently a comparison of the phenomenological order parameter $\psi(r)$, for which the Hamiltonian takes the form (7.1) and parameters of the given model are by no means so simple. Therefore, we cannot comment at all on the temperature dependence of the average order parameters of the given model below the transition point (for example, on the total moment in the Ising model).

Then the non-arbitrary phase transition (or, in any case, the non-arbitrary model) can be reduced to the generalized Landau model. As an example, one can use the Hamiltonian model of Bardeen, Cooper, and Schrieffer (BCS) [7] which, in an exact calculation,[8] gives a finite jump in the heat capacity. This result is connected with the specific characteristics of the BCS model, in which the interaction is considered only in the measure in which it leads to pair formation. If one considers a phase transition in a system of Fermi particles with a more realistic interaction, then the phase transition is seen to be the same as in the problem considered by us. (A separate paper will be devoted to phase transitions in a superconductor.)

Our theory can also be generalized by inclusion in the Hamiltonian (2.1) of terms describing non-pair interaction, but preserving the number of particles in the system. It is not difficult to see that such terms lead only to renormalization of the interaction constant.

The theory developed in this paper is very complicated mathematically. Therefore, it seems appropriate to us to carry out simple physical considerations here in the spirit of the turbulence theory of Kolmogorov, which makes it possible to obtain one of the fundamental relations.

The energy of interaction of particles with small momenta $p \ll p_0$ on the phase transition curve is much larger than its kinetic energy. Let there be fluctuations of large scale $1/p$ in the system. Because of the interaction, the occupation number n_p changes materially in some region of momentum space. The dimension of this region can depend only on p and V. From dimensional considerations, it is evident that this dimension is equal to p. Therefore the interaction energy of particles in such fluctuations has the order $Vn_p^2p^3$. In the state of thermodynamic equilibrium, the energy entering into the degree of freedom is of the order of magnitude T. From these considerations, we find $n_p \sim (V/Tp^3)^{1/2}$. The mean energy of "quasiparticles" can be determined from the approximate equality $\varepsilon_p n_p \sim T$. These considerations lead to the conclusion that the phenomenon of the λ transition can be described by the ideal gas model of quasiparticles with dispersion $\varepsilon_p = Ap^{3/2}$. Apparently, such considerations do not permit us to determine the power of the logarithm and, consequently, the thermodynamic behavior of the system.

Recently measurements [9] have shown that, close to the critical point, the heat capacity also has a logarithmic singularity. This result, from our point of view, testifies to the fact that, close to the critical point, the principal role is played by fluctuations of small amplitude but of large dimensions (at this point, not of the order but of the density). Evidently, even critical phenomena can in some measure be described within the framework of the generalized Landau model.

The idea of the generality of the nature of all second order phase transitions was advanced by Landau. To him also belongs the mathematical formulation of the problem on the basis of the Hamiltonian (7.1). These ideas were developed by his co-workers A. A. Abrikosov, L. P. Gor'kov, I. E. Dzyaloshinskiĭ, L. P. Pitaevskiĭ, and I. M. Khalatnikov. They constructed a three-dimensional graph technique and showed the identity of problems of phase transitions in a classical fluctuating

system and a Bose liquid. Searchers for solutions of the type $\langle n_p \rangle \sim p^{-3/2}$ were also undertaken under this initiative of L. D. Landau, and his colleagues at the Institute for Physics Problems, and also by A. A. Vedenov. The authors regard their own work as an extension of the research of this group of physicists, with whom they are closely associated, and convey to them their sincere gratitude.

We consider it our pleasant duty to thank A. A. Vedenov for many discussion, contributing to the clarification of the physical side of the problem. The authors are grateful to A. I. Larkin, V. V. Sudakov, D. V. Shirkov, G. M. Éliashberg and other participants in the Second Odessa Symposium on theoretical physics for extended discussion, to E. G. Batyev, S. K. Savvinykh and G. I. Surdutovich for useful comments, permitting them to eliminate some errors.

We should like to note the role of Yu. B. Rumer, whose unchanging enthusiasm maintained interest in research in this field over the course of many years.

APPENDIX A

ANALYSIS OF THE EQUATION FOR VERTEX PARTS

The set of equations for vertex parts consists of equations of the Bethe-Salpeter type (2.11), the logical identity (2.13) connecting the irreducible vertexes with the reducible, and Eq. (2.12) for the absolutely irreducible vertex part. Let us first find the equations of the Bethe-Salpeter type. We shall consider the total vertex part as known and determine the irreducible vertex parts from equations of the type (2.11).

The following analytic formula can be written down for the vertex part $\mathcal{T}_{12;34}$ of Eq (2.11):

$$\mathcal{T}_{12;34}(p_1, p_2, p_3, p_4) = \square(p_1, p_2, p_3, p_4) + \frac{T}{(2\pi)^3}\int \square(p_1, p_2, q, q - p_1 + p_2) \times \mathcal{T}_{12;34}(q, q - p_1 + p_2, p_3, p_4) \times G(q) G(q - p_1 + p_2)\, d^3q. \quad (A.1)$$

The kernel of this equation is not of the Fredholm type, since the integral

$$\int \square^2(p, p, q, q) G^2(p) G^2(q)\, d^3p d^3q \quad (A.2)$$

diverges logarithmically. In this connection, the homogeneous equation corresponding to (A.1) has an irreducible spectrum of eigenvalues.[4] The entire divergence of the integral of (A.2) is associated with regions in which two of the momenta are approximately equal and are much larger than all the others; the non-uniqueness of solution of Eq. (A.1) is also connected with this region. Therefore, the problem of the uniqueness of the solution of Eq. (A.1) can be solved in a much narrower setting.

Let us consider Eq. (A.1) for the case $p_1 = p_2$, $p_3 = p_4 = p'$. In this case, $\square$ and $\mathcal{T}$ are functions only of the ratios of the moduli of their vector arguments and the angle θ between them. We represent them in the form of series in Legendre polynomials of cos θ with coefficients $\square_n(p/p')$. The equations for the different harmonics separate:

$$(\mathcal{T}_{12;34})_n(x) = \square_n(x) + \lambda \int_0^\infty \square_n\left(\frac{x}{x'}\right)(\mathcal{T}_{1;34})_n(x')\frac{dx'}{x'},$$

$$\lambda = \frac{T}{2\pi^2 A^2}. \quad (A.3)$$

From the properties of $\square(p_i)$ it follows that

$$\square_n(x) = \square_n(1/x), \qquad \square_n(0) = 0. \quad (A.4)$$

Equations (A.3) are easily solved with the help of the Mellin transform. Denoting the Mellin transform of $\varphi(s)$ by $\hat{\varphi}(s)$, we get

$$(\hat{\mathcal{T}}_{12;34})_n(s) = \hat{\square}_n(s)/[1 - \lambda\hat{\square}_n(s)]. \quad (A.5)$$

The inverse transform of (A.5) gives a function possessing the property

$$(\mathcal{T}_{12;34})_n(x) = (\mathcal{T}_{12;34})_n(1/x). \quad (A.6)$$

To the solution of (A.5) of Eq. (A.3) can be added an arbitrary superposition of solutions of the corresponding homogeneous equation. They all have the form $x^{s(\lambda)}$, where $s(\lambda)$ is a root of the equation

$$1 - \lambda\hat{\square}_n(s) = 0. \quad (A.7)$$

The function $\mathcal{T}_n(x)$ must satisfy the condition (A.6) (symmetry relative to the permutation of the arguments). Therefore, the solutions of the homogeneous equation must always lead to the combination $x^{s(\lambda)} + x^{-s(\lambda)}$. Substituting x in the ratio p/p', we find that each solution of the homogeneous equation becomes infinite as $p/p' \to 0$ and as $p/p' \to \infty$.

We now compare the results obtained with Eqs.

[4] V. E. Zakharov called our attention to this fact, for which we express our gratitude.

(2.11), (2.13). From Eqs. (2.12), which do not contain vertex parts, it is clear that the absolutely irreducible vertex part $\mathcal{T}^*$ does not contain divergences when two of the momenta vanish. It then follows from Eq. (2.13) that the coefficients in the solution of the homogeneous equations are equal to zero.

The situation is somewhat different for the vertex part $\mathcal{T}_{13;24}$. The Bethe-Salpeter equation for it is written in the form

$$\mathcal{T}_{13;24}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) - \frac{T}{2(2\pi)^3}\int \square(\mathbf{p}_1, \mathbf{q}, \mathbf{p}_3, \mathbf{p}_1 + \mathbf{p}_3 - \mathbf{q}) \times \mathcal{T}_{13;24}(\mathbf{q}_1, \mathbf{p}_2, \mathbf{p}_1 + \mathbf{p}_3 - \mathbf{q}, \mathbf{p}_4)\, G(\mathbf{q}) \times G(\mathbf{p}_1 + \mathbf{p}_3 - \mathbf{q})\, d^3\mathbf{q} = \square(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4). \quad (A.8)$$

In this case the kernel of the corresponding integral equation generally tends toward a constant value when the momentum of the integration increases. The equation will have a solution only for explicit furnishing of the integration limit p_0. Its solution is a small quantity of the order of $V/\ln[p_0/|\mathbf{p}_1 + \mathbf{p}_3|]$. We can establish this fact by direct substitution.

Thus, in both the cases considered, the solution of the Bethe-Salpeter equation is uniquely determined.

The equations considered here have a great resemblance to the equations of quantum field theory, the solutions of which were studied in a number of well-known researches.[10] These solutions are materially different from those found by us, which is explained by important differences in the equations themselves. A more detailed consideration of this question goes beyond the limits of the present communication.

APPENDIX B

STRUCTURE OF THE SERIES $\Phi_0(\square)$, $\Phi_0(\square)$.

We shall consider the structure of the terms of the series which correspond to the quantity $\mathcal{T}^*$ for the simplest example of an "open envelope" diagram (Fig. 2). The contribution of this diagram to the λ curve is equal to the integral

$$-\frac{T^3}{A^6(2\pi)^9} \times \int \frac{\square_1\square_2\square_3\square_4\, d^3\mathbf{q}_1 d^3\mathbf{q}_2 d^3\mathbf{q}_3}{q_1^{3/2}q_2^{3/2}q_3^{3/2}\,|\mathbf{p}_1 - \mathbf{q}_1 - \mathbf{q}_2|^{3/2}\,|\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{q}_4|^{3/2}\,|\mathbf{p}_4 - \mathbf{p}_1 - \mathbf{q}_3|^{3/2}}, \quad (B.1)$$

where $\mathbf{q}_4 = \mathbf{q}_1 - \mathbf{q}_2 + \mathbf{q}_3$; $\square_1$, $\square_2$, $\square_3$, $\square_4$ are the complete vertex parts, the arguments of which are given in Fig. 2. Since a logarithmic divergence arises for large values of all the momenta q_i, it is convenient to transform to new variables. As one of these, we introduce the quantity Q:

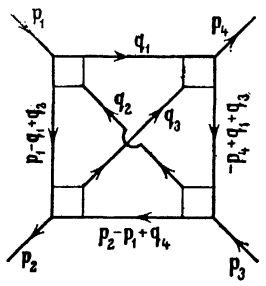


FIG. 2

$$Q^2 = \sum_i q_i^2. \quad (B.2)$$

We shall not write down the remaining variables (angles in nine-dimensional space) explicitly.

The integral (B.1) in the new variables has the form

$$-\frac{T^3}{A^6(2\pi)^9}\int \frac{dQ}{Q} \times \frac{\square_1\square_2\square_3\square_4\, d\Omega}{\nu_1^{3/2}\nu_2^{3/2}\nu_3^{3/2}\,|\mathbf{p}_1/Q - \boldsymbol{\nu}_1 - \boldsymbol{\nu}_2|^{3/2}\,|(\mathbf{p}_1 + \mathbf{p}_2)/Q - \boldsymbol{\nu}_4|^{3/2}\,|\mathbf{p}_4/Q - \boldsymbol{\nu}_1 - \boldsymbol{\nu}_3|^{3/2}}. \quad (B.3)$$

Here $d\Omega$ is the element of solid angle in nine-dimensional space. The rest of the notation is obvious. Integration over Q is cut off above at a value $\sim p_0$. Below, it is cut off in practice at $Q \sim p$, where p is the maximum of the momenta p_i. Thus the logarithmic term in the integral (B.3) has the form

$$-\frac{T^3}{(2\pi)^9 A^6}\int \frac{\square_1\square_2\square_3\square_4\, d\Omega}{\nu_1^{3/2}\nu_2^{3/2}\nu_3^{3/2}\nu_4^{3/2}\,|\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2|^{3/2}\,|\boldsymbol{\nu}_1 + \boldsymbol{\nu}_3|^{3/2}} \ln\frac{p_0}{p}, \quad (B.4)$$

where the external momenta can be set equal to zero.

For calculation of the next term of the expansion, we make use of the following method. Let the integral I have the form

$$I = \int_0^{x_0} f(x)\frac{dx}{x} \qquad (x_0 \gg 1), \quad (B.5)$$

where $f(\infty) = f_0 \neq 0$, and $f(x) \sim x^{1+\alpha}$ $(\alpha > 0)$ for $x \to 0$. Then integration by parts yields

$$I = f_0 \ln x_0 - \int_0^{\infty} f'(x)\ln x\, dx. \quad (B.6)$$

Making use of this relation, we obtain the next term of the expansion of the integral (B.1) in the form

$$\frac{T^3}{A^6(2\pi)^9}\int_0^\infty d\Omega\, dQ \ln Q \frac{\partial}{\partial Q}\left[\frac{\square_1\square_2\square_3\square_4}{\nu_1^{3/2}\nu_2^{3/2}\nu_3^{3/2}|p_1/Q-\nu_1-\nu_2|^{3/2}|(p_1+p_2)/Q-\nu_4|^{3/2}|p_4/Q-\nu_1-\nu_3|^{3/2}}\right]. \quad (B.7)$$

In view of the homogeneity of the □ quantities, the integral (B.7) converges for Q ~ p. From the form of the integral (B.7), it is evident that if, say, p_1, p_3 are much greater than p_2, p_4, then we can neglect the latter. Because of the homogeneity of the integrand, the integral (B.7) will not depend on p_1 in this case.

Generalization of the results that have been obtained to the general term of the series for $\mathcal{J}^*$ follows directly. We thus obtain (3.6).

We proceed to an estimate of the later terms of the series $\Phi_0(\square)$. The contribution from each diagram can be illustrated by an integral of the form

$$\frac{T^{n-1}}{A^{2(n-1)}(2\pi)^{3n}}\int d\Omega_n \prod_{j=1}^{2(n-1)} \nu_j^{-3/2} \prod_{k=1}^{n} \square_k. \quad (B.8)$$

Integration over $d\Omega_n$ gives, on the surface,

$$\sum_{i=1}^{n-1} \nu_i^2 = 1 \quad (B.9)$$

(summation only over the independent momenta). In the important region of integration, all the ν_i have the identical order $1/\sqrt{n}$. The region of integration itself has the order $n^{-3n/2}$. Therefore, with factorial accuracy, the integration is of the order $(T\square/A^2)^n$.

We can only estimate very crudely the number of diagrams of n-th order by the upper limit (n − 1)!! Although this estimate is evidently quite rough, the factorial increase in the number of graphs raises no special doubt. Therefore, the series for $\Phi_0(\square)$ is evidently asymptotic.

APPENDIX C

ONE DIMENSIONAL ANALOG OF THE EQUATION FOR □

The one-dimensional analog of Eqs. (2.11)–(2.13) can be written in the following fashion:

$$F(\square) = \int_x^{x_0} \Phi(\square, x')\frac{dx'}{x'}. \quad (C.1)$$

Here □(x) is the desired function, F and Φ are certain given functions. It is assumed that $\Phi_0(\square) = \Phi(\square, 0)$ is a finite quantity. We shall somewhat simplify the situation by assuming that Φ does not depend explicitly on x. Then Eq. (C.1) can easily be reduced to a differential equation with separable variables. The solution of the latter has the form

$$\int_{\square_1}^{\square} \frac{F'(\square)}{\Phi(\square)} d\square = -\ln\frac{x}{x_0}, \quad (C.2)$$

where $\square_1$ is a root of the equation $F(\square) = 0$. We investigate the solution (B.2) for x close to zero. Let $\Phi(\square)$ have a simple root $\square_0$. We introduce the notation $F'(\square_0) = F_0'$, $\Phi'(\square_0) = \Phi_0'$. For x close to zero, we shall seek the solution of □ close to $\square_0$. A simple calculation gives

$$|\square - \square_0| = \exp\left\{-\frac{F_0'}{\Phi_0'}\ln\frac{x}{x_0}\right\}. \quad (C.3)$$

By analyzing Eq. (3.8) in the region of small momenta, we obtained Eq. (3.11), which determines the value of V. On the other hand, this constant itself makes an important contribution from the region of large momenta. At first glance, such a situation appears to be contradictory. In fact, there is no contradiction. The change in the interaction in the region of high momenta produces such a change in the region of small momenta that the constant V does not change.

We shall illustrate what has been pointed out by the simple example we have considered. Here the value of $\square(0) = \square_0$ is also determined from the condition of solvability of the equations for small x. On the other hand, it follows from Eq. (C.1) that

$$F(\square_0) = \int_0^{x_0} \Phi(\square)\frac{dx}{x}. \quad (C.4)$$

Equation (C.4) shows that there exists a contribution (for $\square_0$) of the region of "large" $x(x \sim x_0)$. We simulate the change of the interaction by the change in x_0. Then (C.3) shows that the value of □ also changes in the region of small momenta. For correction of the initial assumptions, it is necessary to satisfy two conditions: 1) the existence of the root $\Phi(\square)$; 2) the satisfaction of the inequality $F_0'/\Phi_0' < 0$. However, the second condition is a consequence of the simplifying assumption made by us on the independence of Φ on x. It is not difficult to establish the fact that if $\partial\Phi/\partial x_{\square=\square_0, x=0} \neq 0$, then the second condition drops out, and □ has the form $\square = \square_0 + ax$ for small x.

Note added in proof (February 22, 1964). Experiment shows that $\mu - \mu_0 > 0$ above the transition curve, and in Eq. (4.23), $a < 0$. It is shown that $|a|/b \ll 1$. The consequences of this fact are discussed in the work of E. Batyev and the authors [JETP 46, No. 6 (1964), Soviet Phys. JETP 19, No. 6 (1964)].

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