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ASYMPTOTIC EQUALITY OF DIFFERENTIAL CROSS SECTIONS FOR PARTICLES AND ANTIPARTICLES

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The asymptotic behavior of the ratio of the differential cross section $d\sigma_+/dt$ to $d\sigma_-/dt$ is investigated for the scattering of particles and antiparticles by the same target at high energies and at a given momentum transfer t. Necessary and sufficient conditions are given for this ratio to tend to unity, to a finite limit different from unity, and to infinity or zero. The special role played by the distribution of the zeros of the scattering amplitude A (s, u, t) in the plane of the invariant variable s is clarified.

1. In the lecture delivered at Nor-Amberd [1] we have called attention to the mathematical apparatus adequate for the Pomeranchuk theorem on the asymptotic equality of total cross sections for particles and antiparticles. [2] Due to the existence of crossing symmetry this apparatus consists of the generalized Phragmén-Lindelöf maximum principle and of certain theorems on the behavior of a function at the boundary of a region (see the appendix). Starting from these general theorems we have made more precise [3] the Pomeranchuk theorem and have studied the rate at which the difference between particle and antiparticle total cross sections tended to zero.

The proofs of asymptotic equality of total cross sections for particles and antiparticles given in [1,3] are such that if it is assumed that the limits of the differential cross sections for particle and antiparticle scattering in the forward direction exist in the limit of infinite energy then it automatically follows that these limits are equal. We have not assumed that the limits of the differential cross sections exist since we thought that it would be interesting to prove the equality of the limits of the total cross sections assuming only that these latter limits exist. In [3] we prove the equality of cross sections not only when the cross section tends to a constant but also under other conditions.

Recently Van Hove on the one hand, [4] and Logunov, Todorov, Khrustalev, and Nguen Van Hieu on the other, [5] proved under certain assumptions the asymptotic equality of differential cross sections for nonzero values of momentum transfer. Van Hove, like Pomeranchuk, did so by means of dispersion relations, whereas Logunov, Todorov, Khrustalev and Nguen Van Hieu used the method described in [1,3]. The last four authors also obtained by this method relations between cross sections for certain specific reactions.

In this work we study in general the question under what conditions are the amplitudes for particle scattering in the s and u channels asymptotically symmetric or asymmetric. As a consequence of crossing symmetry one obtains in the first case asymptotic equality of differential cross sections for particles and antiparticles, whereas in the second case these cross sections differ.

The investigation is carried out under very general assumptions about the scattering amplitude. For the orientation of the reader we give the formulation of the necessary mathematical theorems in the Appendix.

2. We denote by A (s, u, t) the amplitude for the scattering of a particle by some target. According to crossing symmetry passage from particle to antiparticle is equivalent to passing from the s to the u channel, i.e., the amplitude for the scattering of the antiparticle by the same target is given by A (u, s, t).

The invariants s, u, and t are connected by the relation

$$u + s + t = 2(\mu^2 + M^2), \tag{1}$$

where μ and M are the masses of the particle and the target. It follows from (1) that for fixed t the amplitudes A (s, u, t) and A (u, s, t) are functions of the one variable s, and passage from the s channel to the u channel is equivalent to passage from the point s to the point $2(\mu^2 = M^2) - t - s$, symmetric to s with respect to the fixed point $c = (\mu^2 + M^2) - t/2$.

The amplitude A (s, u, t) is analytic in the s plane with two cuts along the real axis. The right cut starts at the point $s_{0,r} = (m_1 + m_2)^2$, and the left cut starts at the point $s_{0,l} = -(m_3 + m_4)^2 + 2(\mu^2 + M^2) - t$. The threshold masses $(m_1 + m_2)$ and $(m_3 + m_4)$ depend on the type of the reaction. The momentum transfer t is negative in the physical region of the s channel so that as t increases in absolute magnitude the two cuts overlap. As long as the two cuts do not overlap the amplitude is real in the gap and therefore $A(u^*, s^*, t^*) = A^*(u, s, t)$.

If passage from the s channel to the u channel is equivalent to the replacing of the point s by the point symmetric to it with the center of symmetry at c, then passage to u^* is equivalent to passage from the point s to the point symmetric to it with respect to the vertical line Re s = c. At that point the amplitude equals $A^*(u, s, t)$ (see Fig. 1). It follows by analytic continuation that the relation

$$A(u^*; s^*; t^*) = A^*(u; s; t),$$

$$u = 2(\mu' + M') - t - s \qquad (2)$$

holds true also in the absence of a gap between the cuts.

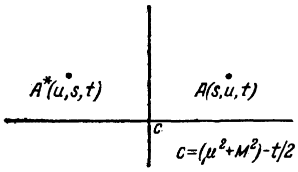

FIG. 1

It follows from (2) that the zeros of the amplitude A(s, u, t) are symmetric to the zeros of the amplitude A(u, s, t) with respect to the line Re s = c.

3. We shall discuss the amplitudes A(s, u, t) and A(u, s, t) only in the upper half-plane Im s > 0. As has been proved previously,[6] as a consequence of the causality principle the amplitudes A(s, u, t) and A(u, s, t) increase slower than an arbitrary exponential, i.e. for any $\epsilon > 0$ the amplitudes $|A(s, u, t)|$ and $|A(u, s, t)|$ are for sufficiently large $|s|$ smaller than $e^{\epsilon |s|}$. We introduce the additional requirement of convergence of the integrals[1)]

$$\int_{-\infty}^{\infty} \ln^+ |A(s, u, t)| \frac{ds}{1+s^2} < +\infty,$$

$$\int_{-\infty}^{\infty} \ln^+ |A(u, s, t)| \frac{ds}{1+s^2} < +\infty. \qquad (3)$$

(The convergence of either integral follows from the convergence of the other.)

[1)]By the symbol $\ln^+|a|$ we mean $\ln|a|$ when $|a| > 1$ and zero when $|a| < 1$.

If s_k and $\tilde{s}_k$ are zeros of the amplitudes A(s, u, t) and A(u, s, t) in the upper half plane, then it follows from (3) and the limitation on the growth of the amplitudes in the complex plane that (see[7], p. 158) the series

$$-\sum \mathrm{Im} \frac{1}{s_k} < \infty, \qquad -\sum \mathrm{Im} \frac{1}{\tilde{s}_k} < +\infty \qquad (4)$$

converge ($\tilde{s}_k$ is symmetric to s_k with respect to the line Re s = c). In turn the convergence of these series implies the convergence of the Blaschke products (see[7], p. 158)

$$\pi(s, u, t) = \prod_k \frac{1 - s/s_k}{1 - s/s_k^*}, \qquad \pi^*(u, s, t) = \prod_k \frac{1 - s/\tilde{s}_k}{1 - s/\tilde{s}_k^*}. \qquad (5)$$

Each of these products has magnitude equal to unity on the real axis.

We assume that A(s, u, t) and A(u, s, t) have no real zeros so that the functions

$$\hat{A}(s, u, t) = \frac{A(s, u, t)}{\pi(s, u, t)}, \qquad \hat{A}(u, s, t) = \frac{A(u, s, t)}{\pi^*(u, s, t)} \qquad (6)$$

have no zeros in the half-plane Im s ≥ 0 and on the real axis coincide in modulus with the functions A(s, u, t) and A(u, s, t). For brevity we shall refer to these functions as the reduced amplitudes.

For a given momentum transfer t the differential cross section dσ/dt is equal to $|A|^2/s^2 = |\hat{A}|^2/s^2$, from which the significance of the reduced amplitudes is clear.

4. Let us introduce the function

$$g(s, t) = \ln \frac{\hat{A}(s, u, t)}{\hat{A}(u, s, t)}. \qquad (7)$$

It follows from the crossing symmetry (2) and the symmetry of the zeros of the amplitude that under the replacement $s \to u^*$ the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ goes over into the ratio $A^*(u, s, t)/A^*(s, u, t)$, and that on the real axis

$$\mathrm{Re}\, g(c - s, t) = -\mathrm{Re}\, g(s, t),$$

$$\mathrm{Im}\, g(c - s, t) = \mathrm{Im}\, g(s, t). \qquad (8)$$

(a) Let us suppose that the function g(s, t) is bounded on the real axis. It will be shown below that from this follows the boundedness of the function g(s, t) in the entire upper half plane under some very general and liberal requirements.

We denote by H_1 and H_2 the manifolds of limiting values of the function g(s, t) as $s \to -\infty$ and as $s \to +\infty$. It follows from (8) that H_1 and H_2 are symmetric to each other with respect to the imaginary axis. It therefore follows that the manifolds H_1 and H_2 intersect at some point $2i\alpha$ on the imaginary axis, because if they did not intersect then each of the bounded manifolds H_1 and H_2 would lie in its own half plane and the function g(s, t) would not be bounded in the upper half-plane (see Appendix, theorem V).

Since the point $2i\alpha$ belongs to the manifolds H_1 and H_2 there exists a sequence of energies $\{s_n\}$, $s_n \to +\infty$, for which Re $g(s_n, t) \to 0$, i.e.

$$\lim |A(s_n, u_n, t)/A(u_n, s_n, t)| = 1. \quad (9)$$

For this sequence of energies the ratio of the differential cross sections for particle and antiparticle tends to unity.

(b) Let us assume in addition that the limit of the ratio of magnitudes of the amplitudes exists:

$$\lim_{s\to+\infty} |A(s, u, t)/A(u, s, t)| = \gamma. \quad (10)$$

Since γ is independent of the choice of the sequence $s \to \infty$, and for $\{s_n\}$ the limit of the ratio equals unity, it follows that $\gamma = 1$.

In this manner it follows from (a) and (b) that as $s \to \infty$ the ratio $|A(s, u, t)|/|A(u, s, t)|$ tends to unity, i.e. that the ratio of the differential cross sections for particles and antiparticles tends to unity.

We emphasize that no assumptions were made about the manner in which the differential cross sections vary as $s \to \infty$.

5. It follows from the general theory [8] that if a function f(s) is regular in the upper half plane, has no zeros, and satisfies the condition

$$\lim_{R\to\infty} \frac{1}{R}\int_0^\pi \ln^+|f(Re^{i\varphi})|\sin\varphi d\varphi = 0, \quad (11)$$

and if also the following integral along the real axis converges

$$\int_{-\infty}^{\infty} \ln^+|f(s)| \frac{ds}{1+s^2} < +\infty, \quad (12)$$

then the ln f(s) may be expressed in the form of the integral

$$\ln f(s) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \ln|f(s')| \frac{ss'+1}{s'-1}\frac{ds}{1+s^2} + i \arg f(i). \quad (13)$$

By hypothesis the amplitude A(s, u, t) [and hence also A(u, s, t)] satisfies conditions (11) and (12). It therefore follows that the reduced amplitudes also satisfy these conditions, as well as the ratio of the reduced amplitudes $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ and the ratio of the amplitudes themselves A(s, u, t)/A(u, s, t) (see Appendix II). Consequently

$$g(s, t) = \ln\frac{\hat{A}(s, \mu, t)}{\hat{A}(u, s, t)} = \frac{1}{\pi i}\int_{-\infty}^{\infty} \ln\left|\frac{A(s, u, t)}{A(u, s, t)}\right| \frac{ss'+1}{s'-s}\frac{ds}{1+s^2} + i\arg\frac{\hat{A}(s, u, t)}{\hat{A}(u, s, t)}\bigg|_{s=i}. \quad (14)$$

The representation (14) makes it possible to study the behavior of g(s, t) in the entire half plane as a function of the behavior of |A(s, u, t)/A(u, s, t)| on the real axis.

6. Suppose that the ratio of the amplitudes on the real axis is bounded in magnitude from below and from above, i.e.,

$$M^{-1}(t) < |A(s, u, t)/A(u, s, t)| < M(t); \quad (15)$$

from (15) follows the analogous inequality for the magnitude of the ratio of the reduced amplitudes for arbitrary complex s since

$$\mathrm{Re}\,\frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{ss'+1}{s'-s}\frac{ds'}{1+s'^2} = 1. \quad (16)$$

In other words, in the upper half-plane

$$-\ln M(t) < \mathrm{Re}\, g(s, t) < \ln M(t). \quad (17)$$

It is seen from (14) that under the restriction (15) on the real axis, g(s, t) grows in the complex plane no faster than ln s. If it is known beforehand that the function g(s, t) is bounded on the real axis then it follows from the generalized maximum principle (see Appendix I) that the function g(s, t) is bounded in the entire upper half plane.

Since

$$g(s, t) = \ln\left|\frac{\hat{A}(s, u, t)}{\hat{A}(u, s, t)}\right| + i\arg\frac{\hat{A}(s, u, t)}{\hat{A}(u, t, s)}$$

and since the boundedness of Re g(s, t) in the entire half plane follows from the boundedness of the ratio A(s, u, t)/A(u, s, t), everything is determined by the growth of the phase of the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$, it being sufficient to study the growth of the phase on the real axis.

On the real axis

$$\mathrm{Im}\, g(s, t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \ln\left|\frac{A(s, u, t)}{A(u, s, t)}\right| \frac{ss'+1}{1+s'^2} + \mathrm{Im}\, g(i, t); \quad (18)$$

the integral being understood in the principal value sense.

Let us study the influence of assumption (b) of Sec. 4 on the existence of the limit (10).

It follows from crossing symmetry that as $s \to -\infty$ the magnitude of the ratio of the amplitudes tends to γ^{-1}. After introduction of these limits into (18) we find that as $s \to +\infty$ the quantity Im g(s, t) or, what is the same, the phase of the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ grows like $2\pi^{-1} \ln \gamma \ln s$. This estimate proves the following assertion:

If as $s \to +\infty$ the limit of the ratio $|A(s, u, t)|/|A(u, s, t)|$ exists, then that limit is equal to unity if and only if the phase of the ratio of the reduced amplitudes $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ grows slower than ln s.

The case $\gamma = 1$ and the phase infinitely increasing occurs.

If the amplitude A (s, u, t) has only a limited number of zeros (this number may increase with t)[2] then it is not necessary to introduce the reduced amplitudes $\hat{A}(s, u, t)$, $\hat{A}(u, s, t)$ and the phase referred to in the preceding assertion is simply the phase of the ratio $A(s, u, t)/A(u, s, t)$. If the phase of this ratio grows slower than ln s and, consequently, $\gamma = 1$, then $|A(s, u, t)/A(u, s, t)|$ tends uniformly to unity not only along the real axis but in the entire upper half plane as $|s| \to \infty$.

The simplest case occurs when in the limit as $s \to +\infty$ the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ itself, and not only its magnitude, tends to a limit. In that case the phase of the ratio is bounded and $\gamma = 1$, therefore

$$\lim_{s\to+\infty} \frac{\hat{A}(s, u, t)}{\hat{A}(u, s, t)} = e^{2i\alpha}.$$

In view of crossing symmetry, the limit of this ratio as $s \to -\infty$ is also equal to $e^{2i\alpha}$. In that case, according to the Lindelöf theorem, as $|s| \to \infty$ in the upper half plane the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ tends uniformly to $e^{2i\alpha}$. If the amplitude A (s, u, t) has a finite number of zeros then the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ may be replaced everywhere by the ratio $A(s, u, t)/A(u, s, t)$.

7. So far we have investigated conditions under which it follows that if, as $s \to +\infty$, the ratio $|A(s, u, t)/A(u, s, t)|$ tends to a finite limit then that limit is equal to unity. We could say that we have studied the question under what conditions do the differential cross sections for particles and antiparticles follow the same regime with equal or unequal coefficients. It is obvious that the question of differential cross sections of particles and antiparticles following different regimes entails even greater asymmetry in the s and u channels and so is of even greater interest. Indeed representation (14) makes it possible to answer this question.

Let us suppose that as $s \to +\infty$ the ratio $|A(s, u, t)|/|A(u, s, t)|$ tends to infinity or to zero. This means different regimes. For the sake of definiteness let this limit be $+\infty$. It then follows from (18) that as $s \to \infty$ the phase of the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ grows faster than ln s. In the end we obtain the following result:

If as $s \to \infty$ the ratio $|A(s,u,t)/A(u,s,t)|$ tends to a definite limit γ, then $\gamma = 1$ if the phase of the ratio $\hat{A}(s,u,t)/\hat{A}(u,s,t)$ increases or decreases no faster than ln s; γ has a finite value, different from unity, if the phase of the ratio $\hat{A}(s,u,t)/\hat{A}(u,s,t)$ increases or decreases like ln s, and γ equals infinity or zero if the phase of the ratio $\hat{A}(s,u,t)/\hat{A}(u,s,t)$ increases or decreases faster than ln s.

These results remain unchanged if the amplitude has a finite number of real zeros.

8. The phase of the ratio of the reduced amplitudes is related to the phase of the ratio of the amplitudes as follows:

$$\arg \frac{\hat{A}(s, u, t)}{\hat{A}(u, s, t)} = \arg \frac{A(s, u, t)}{A(u, s, t)} - \arg \frac{\pi(s, u, t)}{\pi^*(u, s, t)}, \qquad (19)$$

and therefore in many cases there follow from the above results restrictions on the asymptotic distribution of the zeros of the amplitude A (s,u,t). For greater clarity we shift the origin of the coordinate system to the point c: $s' = s - c$; then

$$\frac{\pi(s, u, t)}{\pi^*(u, s, t)} = \nu \prod \frac{1 - s'/s_k'^*}{1 - s'/s_k'^*} \Big/ \prod \frac{1 + s'/s_k'^*}{1 + s'/s_k'^*},$$
$$\nu = \frac{\pi(s, u, t)}{\pi^*(u, s, t)}\Big|_{s=c}, \qquad s_k' = s_k - c. \qquad (20)$$

If one introduces the notation

$$s_k' = \rho_k e^{i(\pi/2 - \varepsilon_k)}, \qquad \alpha_k = \arg \frac{1 - s'/s_k'}{1 + s'/s_k'^*},$$

then

$$\sin \alpha_k = \sin 2\varepsilon_k / [1 + (\rho_k/s')^4 + 2(\rho_k/s')^2 \cos 2\varepsilon_k]^{1/2},$$

$$\arg \frac{\pi(s, u, t)}{\pi^*(u, s, t)} = 2 \arg \prod \frac{1 - s'/s_k'}{1 + s'/s_k'^*} = 2 \sum_k \alpha_k.$$

Let us consider a few examples.

1. If as $s \to +\infty$ the ratio of the magnitudes $|A(s,u,t)|/|A(u,s,t)|$ tends to a finite limit γ and the phase of the ratio $A(s,u,t)/A(u,s,t)$ varies slower than ln s, then $\gamma = 1$ if and only if the phase of $\Pi(1 - s'/s_k')/(1 + s'/s_k'^*)$ varies slower than ln s.

This last requirement constitutes a quantitative measure of the symmetry of the distribution of the zeros s_k' with respect to the imaginary axis. If, for example, all the roots s_k' except for a finite number lie to the right of the imaginary axis then

[2] In that case it is sufficient to consider the function A(s,u,t)/A(u,s,t) in a region which is obtained from the half plane by deformation of the finite part of the real axis and which does not contain zeros of the amplitude.

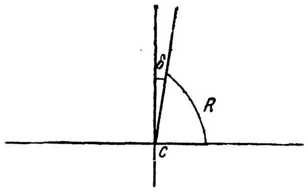

FIG. 2

the number of roots of the amplitude A(s, u, t) that lie inside a circle of radius R and in the angle $\arg s' < \pi/2 - \delta$ grows for any fixed δ slower than ln R (see Fig. 2).

2. If as $s \to +\infty$ the ratio $|A(s,u,t)|/|A(u,s,t)|$ tends to a finite limit and the phase of the ratio $A(s,u,t)/A(u,s,t)$ grows slower than ln s, then the asymmetry condition, i.e. $\gamma \neq 1$, consists in the growing of $\arg \Pi(1 - s'/s_k')/(1 + s'/s_k'^*)$ like ln s.

3. If the phase of the ratio $A(s,u,t)/A(u,s,t)$ increases no faster than ln s, then the magnitude of this ratio tends to infinity if and only if the phase of $\Pi(1 - s'/s_k')/(1 + s'/s_k'^*)$ increases faster than ln s.

As an illustration we consider Gribov's example of an amplitude with asymmetric asymptotic behavior. Let

$$A(s, u, t) = 2\exp(-\sqrt{-s}) + \exp(-\sqrt{-u}), \quad u = -s, \quad c = 0. \tag{21}$$

The roots are defined so as to have real parts positive in the cut plane.

It is clear that

$$\lim_{s \to +\infty} \frac{A(s, u)}{A(u, s)} = 2, \qquad \lim_{s \to -\infty} \frac{A(s, u)}{A(u, s)} = \frac{1}{2}.$$

According to the theory developed above, as $s \to +\infty$ the phase of $\hat{A}(s,u)/\hat{A}(u,s)$ increases as $2\pi^{-1} \ln 2 \ln s$, and the growth of the quantity

$$2 \arg \Pi \frac{1 - s/s_k}{1 + s/s_k^*}.$$

should be the same.

Let $s_k = \rho_k e^{i(\pi/2 - \varepsilon_k)}$. It is easy to see that

$$\rho_k \approx \frac{(2k+1)^2}{2}\pi, \qquad \varepsilon_k \approx \frac{2 \ln 2}{(2k+1)\pi},$$

$$\arg \Pi \frac{1 - s/s_k}{1 + s/s_k^*} \sim 2 \sum \frac{s^2}{|s|^2 + s_k^2} \varepsilon_k.$$

The last sum grows like ln s.

I am grateful to V. N. Gribov for useful discussion.

APPENDIX I

I. The generalized Phragmén-Lindelöf maximum principle. If the function f(z) is regular in the upper half-plane and bounded on the real axis $|f(z)| < M$, then either $|f(z)| < M$ in the entire upper half-plane or f(z) grows faster than a certain exponential, i.e., there exists an $\alpha > 0$ such that $\max |f(Re^{i\varphi})|$ on a semicircle in the upper half-plane of sufficiently large radius R is larger than $e^{\alpha R}$.

The theorem remains true for a region obtained from the half-plane by deformation of a finite part of the boundary.

II. The Lindelöf-Iverson theorem. Let f(z) be a function meromorphic in a singly-connected region G, and let Γ_1 and Γ_2 be two branches of the boundary of G that go off to infinity relative to the region in the negative and positive directions, i.e., on moving along Γ_1 the region lies to the right, on moving along Γ_2 the region lies to the left.

If the function f(z) tends to different limits (one of them may be infinity) along Γ_1 and Γ_2, then the function f(z) assumes in the region G all values, with the possible exception of two, an infinite number of times.

A special case of this theorem is the Lindelöf theorem.

III. The Lindelöf theorem. If f(z) is regular in the region G and tends to different limits along Γ_1 and Γ_2, then f(z) is unbounded in the region G.

If the region G coincides with the half plane then it follows from the generalized maximum principle that f(z) grows faster than a certain exponential.

IV. If in the conditions of theorems II and III the function f(z) tends to the same limits along Γ_1 and Γ_2 then the following alternative holds: either as $z \to \infty$ f(z) tends in the region G uniformly to the common limit, or the assertions of theorem II or III respectively are true.

V. General Theorem. Let f(z) be a function meromorphic in the region G. We denote by H_1 the set of limit values of the function f(z) as $z \to \infty$ along Γ_1, by H_2 the set of limit values as $z \to \infty$ along Γ_2. If a closed curve can be drawn through the point w_0 such as to separate the sets H_1 and H_2 from each other then the function f(z) assumes the value w_0 an infinite number of times.

From that rule at most two values w_0 can be excluded. If the sets H_1 and H_2 have points in common then a separating curve, obviously, does not exist. In particular if both sets H_1 and H_2 are unbounded then they have in common the infinitely distant point.

It follows from this theorem that if both sets H_1 and H_2 are bounded and have no points in common, then the function f(z) is unbounded in the region G.

The theorems I, II, III, IV, in somewhat different formulation, appear in many books, in particular in Nevanlinna's book.[9] Theorem V follows from considerations given in Nevanlinna's book (p. 68).

APPENDIX II

The Blaschke product $\pi(s, u, t)$ is less than unity in magnitude and satisfies the following equalities in the indicated limits (see [10]):

$$\lim \frac{1}{R}\int_0^\pi \ln|\pi(Re^{i\theta})|\sin\theta d\theta = 0,$$

$$\lim \int_{-\infty}^{\infty} \ln|\pi(s+i\varepsilon)|\, ds/(s^2+1) = 0, \qquad R\to\infty, \qquad \varepsilon\to 0.$$

It therefore follows that the amplitude $\hat{A}(s, u, t)$ satisfies the conditions (11) and (12) simultaneously with A(s, u, t). By expressing the harmonic function $\ln \hat{A}(s, u, t)$ in terms of the limiting values and making use of the conditions (11) and (12) we deduce the convergence of the integrals

$$\frac{1}{2\pi R}\int_0^\pi \ln^+\left|\frac{1}{\hat{A}(Re^{i\theta}, u, t)}\right|\sin\theta d\theta < +\infty,$$

$$\int_{-\infty}^{\infty} \ln^+\left|\frac{1}{\hat{A}(s, u, t)}\right|\frac{ds}{1+s^2} < +\infty.$$

This is also true of the amplitude $\hat{A}(u, s, t)$ and consequently the corresponding integrals for the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ also converge.

It follows from crossing symmetry (relation 8) that

$$\frac{1}{2\pi R}\int_0^\pi \ln\left|\frac{\hat{A}(Re^{i\theta}, u, t)}{\hat{A}(u, Re^{i\theta}, t)}\right|\sin\theta d\theta = \frac{1}{2\pi R}\int_0^\pi \ln^+\left|\frac{\hat{A}(Re^{i\theta}, u, t)}{\hat{A}(u, Re^{i\theta}, t)}\right|$$

$$\times \sin\theta d\theta - \frac{1}{2\pi R}\int_0^\pi \ln^+\left|\frac{\hat{A}(u, Re^{i\theta}, t)}{\hat{A}(Re^{i\theta}, u, t)}\right|\sin\theta d\theta = 0,$$

but according to the results of [8] the limit of one of the integrals on the right is equal to zero, hence both the integrals on the right tend to zero and the ratio $\hat{A}(s, u, t)/\hat{A}(u, s, t)$ satisfies condition (11).

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144