THE FREDHOLM METHOD IN THE RELATIVISTIC SCATTERING PROBLEM

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We investigate the solutions found by the Fredholm method for the linear integral equation describing the scattering of spinless particles with equal masses. The problem is treated over a restricted energy range but with arbitrary momentum transfer. The analytic properties of the scattering amplitude are studied as a function of the complex energy (or momentum) and angular momentum. The asymptotic form of the partial amplitude as $|l| \to \infty$ is found and it is shown that one can go over to the total amplitude by using the Watson-Sommerfeld transformation. The analyticity of the total amplitude as a function of t is demonstrated, and conditions for Regge asymptotic behavior when $t \to \infty$ or $u \to \infty$ are formulated.

1. INTRODUCTION

THIS paper describes an investigation of the analytic properties and asymptotic form of the amplitude for elastic scattering of two particles. In nonrelativistic theory, for a potential of the form

$$V(r) = \int_{\mu^2}^{\infty} d\nu U(\nu) \frac{e^{-\nu r}}{r} \tag{1.1}$$

it has been possible to obtain the Mandelstam representation[1] (under very general assumptions about $U(\nu)$), and to use the Watson-Sommerfeld transformation to get the asymptotic scattering amplitude for large values of the cosine of the scattering angle.[2] In quantum field theory information about the asymptotic scattering amplitude is usually obtained from the unitarity relations and from the Mandelstam representation, whose validity has as yet not been demonstrated. The Mandelstam representation is used here to prove the analyticity of the partial scattering amplitude in some part of the l-plane and to establish the connection between the singularities in the l-plane and the asymptotic behavior of the scattering amplitude at high energies.[3] To get this result it is, however, not necessary to use the Mandelstam representation. It may happen that there is no Mandelstam representation but, nevertheless, the partial amplitude is analytic in some portion of the l-plane, and the asymptotic behavior of the scattering amplitude in $\cos\theta$ (where θ is the scattering angle in the system of the center of inertia) is given by the singularities of the partial amplitude with respect to l. It is therefore desirable to develop a method for studying the analytic properties of the scattering amplitude and its asymptotic behavior as a function of $\cos\theta$ directly, without using the assumption that there is a Mandelstam representation.

For this purpose it is very convenient to describe the scattering amplitude and the bound states of the two particles using a Schroedinger-type equation with a generalized complex potential.[4] For the case of scattering of spinless particles of equal mass, this equation has the form[4]

$$\sqrt{p^2+m^2}\,(k^2-p^2)\,\psi_{\mathbf{k}}(\mathbf{p}) = \int d^3q V(\mathbf{p}, \mathbf{k}; k^2)\,\psi_{\mathbf{k}}(\mathbf{q}), \tag{1.2}$$

where $\mathbf{k}$ is the momentum of the particles in the cms, which is related to the square of the total energy $s = 4E^2$ by the formula $s = 4(k^2 + m^2)$. It is convenient to introduce the invariant scattering amplitude $T(\mathbf{p}, \mathbf{k})$ by the relation

$$\psi_{\mathbf{k}}(\mathbf{p}) = \delta^{(3)}(\mathbf{p}-\mathbf{k}) + \frac{T(\mathbf{p}, \mathbf{k})}{(k^2-p^2+i0)\sqrt{p^2+m^2}}. \tag{1.3}$$

Substituting (1.3) in (1.2), we get the equation for the scattering amplitude:

$$T(\mathbf{p}, \mathbf{k}) = V(\mathbf{p}, \mathbf{k}; k^2) + \int \frac{d^3q}{\sqrt{q^2+m^2}} \frac{V(\mathbf{p}, \mathbf{q}; k^2)\,T(\mathbf{q}, \mathbf{k})}{k^2-q^2+i0}. \tag{1.4}$$

In general $V(\mathbf{p}, \mathbf{q}; k^2)$ is a very complicated function of p^2, q^2, $\mathbf{p}\cdot\mathbf{q}$ and k^2. It was shown earlier,[5] however, that one can construct potentials of a much simpler type which lead to the same values of the scattering amplitude $T(\mathbf{p}, \mathbf{k})$ on the mass shell, i.e., for $p^2 = k^2 = E^2 - m^2$. Using the results of[5], we assume that the even and odd projections of the amplitude $T^{\pm}$ satisfy the equations

$$T^{\pm}(\mathbf{p}, \mathbf{k}) = V^{\pm}((\mathbf{p}-\mathbf{k})^2, k^2) + \int \frac{d^3q}{\sqrt{q^2+m^2}} \frac{V^{\pm}((\mathbf{p}-\mathbf{q})^2, k^2)\,T^{\pm}(\mathbf{q}, \mathbf{k})}{k^2-q^2}, \tag{1.5}$$

where the potentials $V^{\pm}$ are representable in the form

$$V^{\pm}((\mathbf{p}-\mathbf{q})^2, k^2)=\int_{\mu^2}^{\infty} d\nu \frac{U^{\pm}(\nu, k^2)}{\nu+(\mathbf{p}-\mathbf{q})^2}, \qquad (1.6)$$

and that there is [1] a $\rho\ (-1 \le \rho \le 0)$, such that

$$\int_{\mu^2}^{\infty} d\nu U^{\pm}(\nu, k^2)\nu^{\rho} < \infty. \qquad (1.7)$$

We shall assume in addition: a) The functions $U^{\pm}(\nu, k^2)$ defined for $\mu^2 \le \nu < \infty$ in some domain K of the complex k plane, are analytic in this domain except possibly for branch points on the real axis. We note that in addition to branch points, which correspond to thresholds for various inelastic processes, the spectral functions $U^{\pm}(\nu, k^2)$ may have some additional branch points arising from the presence of $\sqrt{q^2+m^2}$ in Eq. (1.5).[2] A more detailed discussion of such branch points will be given below.

We shall also assume b) that the spectral functions are real in some interval $I \subset K$ of the real k^2 axis, containing the point $k^2 = 0$. This assumption insures that the potentials are real in the region corresponding to elastic scattering and to bound states.

2. SOLUTION OF THE EQUATIONS FOR THE PARTIAL AMPLITUDES. MEROMORPHY IN THE l-PLANE

Expanding the projections of the amplitudes and potentials in partial waves [3]

$$T^{\pm}(\mathbf{p}, \mathbf{p}')=\frac{1}{2pp'}\sum_{l=0}^{\infty}(2l+1) f_l^{\pm}(p, p') P_l\left(\frac{\mathbf{pp}'}{pp'}\right), \qquad (2.1)$$

$$V^{\pm}((\mathbf{p}-\mathbf{p}'), k^2) = \frac{1}{2pp'}\sum_{l=0}^{\infty}(2l+1) F_l^{\pm}(p, p'; k^2) P_l\left(\frac{\mathbf{pp}'}{pp'}\right), \qquad (2.2)$$

and using the relation

$$\int d\Omega_q P_l\left(\frac{\mathbf{pq}}{pq}\right) P_{l'}\left(\frac{\mathbf{qp}'}{qp'}\right) = \delta_{ll'}\frac{4\pi}{2l+1} P_l\left(\frac{\mathbf{pp}'}{pp'}\right), \qquad (2.3)$$

one can easily get the equations for the partial amplitudes:

$$f_l^{\pm}(p, p') = F_l^{\pm}(p, p'; k^2) + \int_0^{\infty} dq \frac{2\pi F_l^{\pm}(p, q; k^2) f_l^{\pm}(q, p')}{\sqrt{q^2+m^2}(k^2-q^2)}. \qquad (2.4)$$

From the well-known expansion

$$\frac{1}{\nu+(\mathbf{p}-\mathbf{q})^2} = \frac{1}{2pq}\left(\frac{p^2+q^2+\nu}{2pq} - \frac{\mathbf{pq}}{pq}\right)^{-1} = \frac{1}{2pq}\sum_{l=0}^{\infty}(2l+1) Q_l\left(\frac{p^2+q^2+\nu}{2qp}\right) P_l\left(\frac{\mathbf{pq}}{pq}\right) \qquad (2.5)$$

and formulas (1.6) and (2.2) we get for $F_l^{\pm}$ the representation

$$F_l^{\pm}(p, q; k^2) = \int_{\mu^2}^{\infty} d\nu U^{\pm}(\nu, k^2)\, Q_l\left(\frac{p^2+q^2+\nu}{2pq}\right). \qquad (2.6)$$

To solve (2.4) it is convenient to introduce the function (we shall omit the ± superscript)

$$R_l(p, p') = \varphi(p) f_l(p, p') \varphi(p'),$$

$$\varphi(p) = \sqrt{2\pi}\,[\sqrt{p^2+m^2}(k^2-p^2)]^{-1/2}.$$

The function R_l satisfies the equation

$$R_l(p, p') = K_l(p, p'; k^2) + \int_0^{\infty} dq\, K_l(p, q; k^2) R_l(q, p'),$$

$$K_l(p, p'; k^2) = \varphi(p) F_l(p, p'; k^2) \varphi(p'). \qquad (2.7)$$

Thus the function R_l is the resolvent of the kernel K_l and can be represented as a ratio of two Fredholm series:[7]

$$R_l(p, p') = D_l(p, p'; k^2) D_l^{-1}(k^2). \qquad (2.8)$$

Here

$$D_l(k^2) = \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\int_0^{\infty}\prod_{i=1}^{n} dq_i\, K_n\begin{pmatrix} q_1\ldots q_n \\ q_1\ldots q_n\end{pmatrix}, \qquad (2.9)$$

$$D_l(p, p'; k^2) = \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\int_0^{\infty}\prod_{i=1}^{n} dq_i\, K_{n+1}\begin{pmatrix} pq_1\ldots q_n \\ p'q_1\ldots q_n\end{pmatrix}, \qquad (2.10)$$

$$K_0 = 1, \qquad K_n\begin{pmatrix} q_1\ldots q_n \\ q_1\ldots q_n\end{pmatrix} = \det_{(i,j)} \| K_l(q_i, q_j; k^2)\,|. \qquad (2.11)$$

The series (2.9) and (2.10) converge if[8]

$$\int_0^{\infty} dq_1 dq_2\, |K_l(q_1, q_2; k^2)|^2 < \infty. \qquad (2.12)$$

Using the condition (1.7) and the asymptotic formula for the Legendre functions

$$Q_l(z) \approx z^{-l-1}\sqrt{\pi}\,\Gamma(l+1)/2^{l+1}\Gamma(l+{}^3/_2)$$

as $|z| \to \infty$,

it is easy to show that the integral (2.12) is bounded for $\mathrm{Re}\, l > -1-\rho$, if k^2 lies in the region K′ ob-

[1]Conditions (1.7) and (1.8) mean that the potential in r-space $V(r, k^2)$, which is related to $U(\nu, k^2)$ by formula (1.1) (to within a numerical factor), can be represented as a sum of generalized Yukawa potentials, and that for $r \to 0$ it increases more slowly than r^{-2}.

[2]Polivanov, Zav'yalov and Khoruzhii [6] have shown that for the potential (1.1) additional singularities appear in (1.5) in the second approximation.

[3]We note that the momenta p and p′ are not on the mass shell, i.e., are not related to E. The transition to the mass shell can be made by setting p = p′ = k.

tained from K by making the cut $k^2 \geq 0$. Since Q_l is analytic in l for $l \neq -1, -2, \ldots$ (it has simple poles at these points), because of the convergence of the series (2.9) and (2.10) the functions $D_l(k^2)$ and $D_l(p, p'; k^2)$ are analytic in the halfplane $\operatorname{Re} l > -1-\rho$ if k^2 is in K'. In this half-plane R_l is a meromorphic function of l and can have poles only at the zeros of the function $D_l(k^2)$.

3. STRUCTURE OF THE PARTIAL AMPLITUDE. SPECTRAL REPRESENTATION OF $D_l(k^2)$

We shall now show that the Fredholm method automatically leads to the N/D structure (cf., for example,[9]) for the partial amplitude f_l, and we shall find the expression for f_l in terms of $D_l(k^2)$.

From formulas (2.8)—(2.11) it follows that

$$f_l(p, p') = \varphi^{-1}(p) R_l(p, p') \varphi^{-1}(p') = N_l(p, p'; k^2) D_l^{-1}(k^2); \tag{3.1}$$

$$N_l(p, p'; k^2) = \varphi^{-1}(p) D_l(p, p'; k^2) \varphi^{-1}(p') = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int_0^{\infty} \prod_{i=1}^{n} dq_i \frac{2\pi}{\sqrt{q_i^2 + m^2(k^2 - q_i^2)}} \times F_{n+1}\begin{pmatrix} p & q_1 \ldots q_n \\ p' & q_1 \ldots q_n \end{pmatrix}, \tag{3.2}$$

$$D_l(k^2) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int_0^{\infty} \prod_{i=1}^{n} dq_i \frac{2\pi}{\sqrt{q_i^2 + m^2(k^2 - q_i^2)}} \times F_{n+1}\begin{pmatrix} q_1 \ldots q_n \\ q_1 \ldots q_n \end{pmatrix}, \tag{3.3}$$

$$F_0 = 1; \qquad F_n\begin{pmatrix} q_1 \ldots q_n \\ q_1 \ldots q_n \end{pmatrix} = \det_{(i,j)} \| F_l(q_i, q_j; k^2) \|. \tag{3.4}$$

The obvious identity

$$\prod_i (q_i^2 - q^2)^{-1} = \sum_i (q_i^2 - q^2)^{-1} \prod_{j \neq i} (q_j^2 - q_i^2)^{-1}$$

enables one easily to get the relation between (3.2) and (3.3):

$$D_l(k^2) = 1 - \int_0^{\infty} dq \frac{2\pi}{\sqrt{q^2 + m^2(k^2 - q^2)}} \tilde{N}_l(q, q; q^2, k^2), \tag{3.5}$$

$$\tilde{N}_l(q, q; q^2, k^2) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int_0^{\infty} \sum_{i=1}^{n} dq_i \frac{2\pi}{\sqrt{q_i^2 + m^2(q^2 - q_i^2)}} F_{n+1}\begin{pmatrix} qq_1 \ldots q_n \\ qq_1 \ldots q_n \end{pmatrix}. \tag{3.6}$$

We note that (3.5) is obtained purely algebraically, and that the convergence of the Fredholm series is sufficient for its validity. This formula is analogous to the corresponding relation in the usual N/D method.

Let us consider $D_l(k^2)$ in the k^2-plane. On the segment I, where the potential has no branch points, we get from (3.5) for $k^2 \geq 0$ (where we denote the intersection of the segment I and the half-line $k^2 \geq 0$ by $(0, k_i^2)$) the relation

$$D_l(k^2 + i0) - D_l(k^2 - i0) = 2\pi^2 i \frac{1}{k\sqrt{k^2 + m^2}} N_l(k^2), \quad 0 \leqslant k^2 < k_i^2, \tag{3.7}$$

where

$$N_l(k^2) = N_l(k, k; k^2) = \tilde{N}_l(k, k; k^2, k^2) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int_0^{\infty} \prod_{i=1}^{\infty} dq_i \frac{2\pi}{\sqrt{q_i^2 + m^2(k^2 - q_i^2)}} F_{n+1}\begin{pmatrix} kq_1 \ldots q_n \\ kq_1 \ldots q_n \end{pmatrix}. \tag{3.8}$$

The function $N_l(k^2)$ has no singularity in the interval $0 \leq k^2 < k_i^2$, since for $k = q_i$ the determinant $F_{n+1}\begin{pmatrix} kq_1 \ldots q_n \\ kq_1 \ldots q_n \end{pmatrix}$ in formula (3.6) vanishes.

On the energy surface f_l has the form $f_l(k^2) = f_l(k, k) = N_l(k^2) D_l^{-1}(k^2)$. For $0 \leq k^2 < k_i^2$ it is given by the ratio of the limiting values of $D_l(k^2)$:

$$f_l(k^2 + i0) = \frac{N_l(k^2)}{D_l(k^2 + i0)} = i\frac{k\sqrt{k^2 + m^2}}{2\pi^2}\left[\frac{D_l(k^2 - i0)}{D_l(k^2 + i0)} - 1\right]. \tag{3.9}$$

From this we get the unitarity relation in the interval $0 \leq k^2 < k_i^2$:[4)]

$$\frac{f_l(k^2) - f_{l^*}^*(k^2)}{2i} = -\pi^2 \frac{1}{k\sqrt{k^2 + m^2}} f_l(k^2) f_{l^*}^*(k^2). \tag{3.10}$$

ANALYTIC PROPERTIES OF N_l IN THE k-PLANE.

In the preceding section we have already pointed out that the vanishing of the denominator in (3.8) does not give rise to branch points of $N_l(k^2)$. Thus $N_l(k^2)$ has no singularities if $0 \leq k^2 < k_i^2$. Considering N_l as a function of k (denoting it by $N_l(k)$ in this case), we find that it has no singularities if k is in the interval $-k_i < k < k_i$. From formulas (3.8), (3.4), and (3.6) it is now clear that in the analytic continuation along any curve, singularities can appear in $N_l(k)$ in only two cases: 1) if we hit a singularity of $U(\nu, k)$ or 2) if a singularity appears in one of the functions $Q_l[(k^2 + q_{i_{1,2}}^2 + \nu)/2kq_{i_{1,2}}]$ which cannot be eliminated by deforming the contour ("coalescing" singularities or a "boundary" singularity).

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4)Outside this interval one must add to the right side of (3.10) the expression

$$2\pi^2 \int_0^{\infty} dr \cdot r^2 |\psi_{kl}(r)|^2 \operatorname{Im} V(r, k^2),$$

where $\psi_{kl}(r)$ is the wave function in the r-representation.

Let us consider the second case. $Q_l(z)$ has singular points at $z = \infty$, $z = +1$ and $z = -1$. Using (2.16) we easily get rid of the first singularity. Dividing $Q_l[(k^2 + q^2 + \nu)/2kq]$ by $(kq)^{l+1}$, we also eliminate the singularity at $k = 0$. We therefore consider only the singularities at $z = \pm 1$. The equation for the singularity in the q-plane

$$(k^2 + q^2_{i_{1,2}} + \nu)/2kq_{i_{1,2}} = \pm 1 \tag{4.1}$$

has the solution $q_{i_{1,2}} = \pm k \pm i\sqrt{\nu}$. Here the sign of $i\sqrt{\nu}$ is not related to the choice of sign in (4.1).

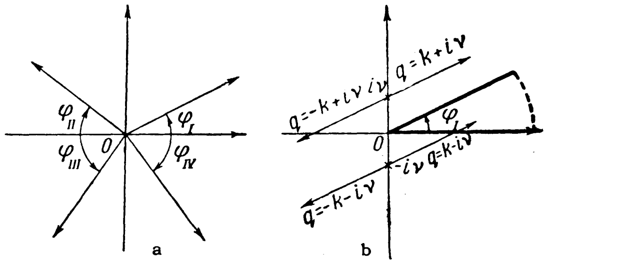

FIG. 1. Deformation of the integration contour in the expression for N_l; a — rays along which the analytic continuation is carried out in the k-plane; b — trajectories of singularities and turning of the contour in the q-plane, for k in the first quadrant.

Suppose that k moves from the origin along a ray in the first quadrant of the k-plane (cf. Fig. 1). If we turn the contours for the $q_{i_{1,2}}$ integrations through the angle φ, the line of singularities will not intersect these contours. We turn the contour for the other q's through the angle φ. The arguments of the corresponding Legendre functions will be $(q_i^2 + q_j^2 + \nu e^{-2i\varphi})/2q_iq_j$, so that for $0 \leq \varphi \leq \pi/2$ the Legendre functions will be regular. The function $1/\sqrt{q^2 + m^2}$ has singular points only along the imaginary axis ($q = \pm im$), i.e., for $\varphi \neq \pm \pi/2$ we do not meet its singularities either. It is possible to turn the contour because the integral along the arcs joining the old and new contours is zero. Carrying out the same procedure with k moving along rays in the other quadrants, we accomplish the analytic continuation of $N_l(k)$ over the whole complex k-plane, except for the cuts along the imaginary axis and cuts from the potential.[5)]

The branch points on the imaginary axis arise because in the motion of k along the imaginary axis the line of singularities hits the boundary of the integration contour ("boundary" singularity). There are no "coalescing" singularities in the present case. The closest branch points on the imaginary axis at $\pm i\mu/2$ appear because of the terms containing $Q_l(1 + \nu/2k^2)$. The next branch points at $\pm i\mu$ occur because of the terms containing the product

$$Q_l\left(\frac{k^2 + q^2 + \nu_1}{2kq}\right) Q_l\left(\frac{k^2 + q^2 + \nu_2}{2kq}\right), \tag{4.2}$$

etc. We remark that the presence of the factor $1/\sqrt{q^2 + m^2}$ in these integrals gives additional branch points along the imaginary axis. For example an expression containing (4.2), and equal to the second Born approximation for the amplitude f_l, has additional branch points at $\pm i(m + \mu)$ and others.[6)]

This is seen most easily by using the procedure described for the case of $\varphi = \pi/2$ (cf. Fig. 2).

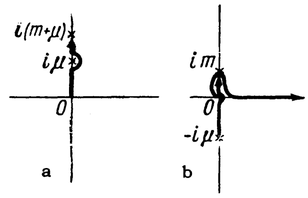

FIG. 2. Appearance of additional singularities $i(m + \mu)$: a — the line along which the analytic continuation in the k-plane is carried out; b — deformation of the integration contour in the q-plane.

Circling the point $k = 0$ and moving on along the imaginary axis, we meet the singularity im of the integrand. These additional singularities have no physical meaning and should not be contained in the amplitude f_l. They are apparently compensated by corresponding singularities in the potential [cf. Sec. 1, assumption (a)]. The compensation can occur automatically in constructing the potential of the form (1.7) from the given amplitude on the mass shell.[5]

We note that in the k^2-plane the function $M_l(k^2) = N_l(k^2)/k^{2(l+1)}$ has no branch points for $k^2 = 0$. In fact the singularity at $k^2 = 0$, which is contained in the functions Q_l, is clearly eliminated by the factor $k^{2(l+1)}$; the appearance of branch points of the root type in going from the k-plane to the k^2-plane is prohibited by the invariance of the function $N_l(k^2)/k^{2(l+1)}$ to the replacement of k by $-k$. This allows one to get a useful representation for D_l:

$$D_l(k^2) = \frac{2\pi^2 i (k^2)^l}{1 + e^{2\pi i l}} \sqrt{\frac{k}{k^2 + m^2}} M_l(k^2) + \Phi_l(k^2), \tag{4.3}$$

where $\Phi_l(k^2)$ has no branch point at $k^2 = 0$, but has a left cut $(-\infty, -\mu^2/4)$, so that the jump of $\Phi_l(k^2)$ on this cut compensates the jump in

5)If the potential is defined over a finite part of the plane, we can carry out the analytic coninuation into this region.

6)These singularities are treated in [6].

$M_l(k^2)$, while this cut does not appear in $D_l(k^2)$. To derive the representation (4.3) we define the function $\sqrt{k^2/(k^2+m^2)}$ in the k^2-plane so that it takes on positive values on the upper side of the cuts $(-\infty, -m^2)$ and $(0, +\infty)$ and negative values on their lower sides, and we choose that branch of the function $(k^2)^l$ for which $\arg k^2 = 0$ for $k^2 > 0$. Then the change in the function $D_l(k^2)$ in circling the point $k^2 = 0$ in the positive direction is the same as the jump in $D_l(k^2)$ computed from formula (3.7).

Formula (4.3) allows the definition of the function D_l in the k-plane. The function $D_l(k)$ thus obtained is a generalization of the well-known Jost function [10] in terms of which the S matrix is expressed by the formula $S_l(k) = D_l(ke^{i\pi})/D_l(k)$. From the representation (4.3) it follows that for nonintegral l the point $k = 0$ is a branch point of the function $D_l(k)$.

5. ASYMPTOTIC SCATTERING AMPLITUDE FOR $l \to \infty$.

To determine the asymptotic form of the scattering amplitude f_l for $|l| \to \infty$, it is convenient first to transform the expression (3.10) for N_l so that the dependence on l becomes simpler. To do this we use the relation [11]

$$Q_l(z_1)\, Q_l(z_2) = \int\limits_{z^+}^{\infty} dz \frac{Q_l(z)}{\sqrt{K(z_1 z_2 z)}}, \tag{5.1}$$

where

$$z^+ = z_1 z_2 + \sqrt{z_1^2 - 1}\sqrt{z_2^2 - 1},$$

$$K(z_1 z_2 z) = z_1^2 + z_2^2 + z^2 - 2 z_1 z_2 z - 1.$$

Applying formula (5.1) successively, one can transform the product of any number of Legendre functions into an integral of a single Legendre function, but the expressions obtained are quite complicated and will not be reproduced here.

Let us consider the expression $N_l(k^2) = f_l(k^2)\, D_l(k^2)$, where $D_l(k^2)$ for $k^2 < 0$ has branch points only when the potential has singular points in this region. Integrating Eq. (2.7), we can get the Born expansion for f_l:

$$f_l(k^2) = \int\limits_{\mu^2}^{\infty} d\nu U(\nu, k^2)\, Q_l\left(1 + \frac{\nu}{2k^2}\right)$$
$$+ \int\limits_{\mu^2}^{\infty} d\nu_1 \int\limits_{\mu^2}^{\infty} d\nu_2\, U(\nu_1, k^2)\, U(\nu_2, k^2)$$
$$\times \int\limits_0^{\infty} dq \frac{2\pi}{\sqrt{q^2 + m^2(k^2 - q^2)}}\, Q_l\left(\frac{k^2 + q^2 + \nu_1}{2kq}\right)$$
$$\times Q_l\left(\frac{q^2 + k^2 + \nu_2}{2kq}\right) + \ldots . \tag{5.2}$$

To simplify the writing of the formulas, we consider the case when $U(\nu, k^2) = U(k^2)\,\delta(\nu - \mu)$. We transform the second integral in (5.2) using the relation (5.1). After straightforward transformations we find

$$f_l(k^2) = U(k^2)\, Q_l\left(1 + \frac{\mu^2}{2k^2}\right)$$
$$+ \pi U^2(k^2) \int\limits_{4\mu^2}^{\infty} \frac{d\nu}{\sqrt{4\nu}}\, Q_l\left(1 + \frac{\nu}{2k^2}\right)$$
$$\times \int\limits_{q_-^2}^{q_+^2} \frac{dq^2}{k^2 - q^2} \frac{1}{[(q^2 + m^2)(q^2 - q_-^2)(q^2 - q_+^2)]^{1/2}}, \tag{5.3}$$

where $q_\pm^2(\nu)$ is determined from the equation $\nu = (k^2 + q_\pm^2 + \mu^2)\, q_\pm^2 - 4k^2$. The integral over q^2 in (5.3) can be expressed in terms of elliptic functions. We shall, however, not give a detailed investigation of the Born series, but shall formulate the general result.

One can show that the function $N_l(k^2)$ is representable in the form

$$N_l(k^2) = \int\limits_{\mu^2}^{\infty} d\nu\, Q_l\left(1 + \frac{\nu}{2k^2}\right) f(\nu, k^2)\, D_l(k^2), \tag{5.4}$$

$$f(\nu, k^2) = U(\nu, k^2)$$
$$+ \theta(\nu - 4\mu^2)\, f_1(\nu, k^2) + \theta(\nu - 9\mu^2)\, f_2(\nu, k^2). \tag{5.5}$$

We note that the convergence of the series in (5.4) is guaranteed by the factor $D_l(k^2)$, which goes to zero at those points where the amplitude $f_l(k^2)$ has a pole and where its Born expansion (5.2) diverges.

From the representation (5.4) we see that the asymptotic behavior of N_l is determined by the behavior of Q_l and D_l. The asymptotic form of Q_l is (cf., for example, [12])

$$Q_l(\operatorname{ch}\alpha) \underset{|l|\to\infty}{\approx} \sqrt{\frac{\pi}{2(l + 1/2)\operatorname{sh}\alpha}}\, e^{-\alpha(l + 1/2)},$$

$$|\arg(\operatorname{ch}\alpha - 1)| < \pi, \qquad |\arg(l + 1/2)| \leqslant \pi - \delta. \tag{5.6}$$*

From formulas (3.3), (3.4), (2.6) and (5.5), using the assumption (1.7) [7)] we find the following asymptotic form for D_l (valid for $\operatorname{Re} l > -1/2$):

$$D_l(k^2) \underset{|l|\to\infty}{\approx} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{C^n}{(l + 1/2)^{3n/2}} \int\limits_0^{\infty} \prod_{i=1}^{n}$$
$$\times \frac{q_i^2\, dq_i}{\sqrt{q_i^2 + m^2(k^2 - q_i^2)}}\, K\begin{pmatrix} q_1 \ldots q_n \\ q_1 \ldots q_n \end{pmatrix}, \tag{5.7}$$

$$K\begin{pmatrix} q_1 \ldots q_n \\ q_1 \ldots q_n \end{pmatrix} = \det_{(i,k)} \| \sqrt{\operatorname{sh}\alpha_{ik}} \exp\{-(l + 1/2)\,\alpha_{ik}\} \|, \tag{5.8}$$

*sh = sinh, ch = cosh.

7)To simplify we set $\rho = 0$ in (1.7). The generalization to the case of $-1 \leq \rho < 0$ presents no difficulties.

$$\operatorname{ch} \alpha_{ik} = (q_i^2 + q_k^2 + \mu^2)/2q_iq_k, \tag{5.9}$$

where C is a constant. To derive (5.7) we substitute the asymptotic formula (5.6) in the formula for D_l, change from the variables ν to variables α, and make one integration by parts. The convergence of the q integrals in (5.7) is easily verified.

From the asymptotic formula (5.7) it follows that $D_l\ (k^2) \rightarrow 1$ for $|l| \rightarrow \infty$ if $\operatorname{Re} l > -1/2$.

Now we investigate the asymptotic behavior of f_l for different values of k^2. Its behavior is determined by the behavior of N_l. To determine the asymptotic form of N_l when $|l| \rightarrow \infty$, we substitute the asymptotic form (5.6) in (5.4), make one integration by parts, first changing from the integration variable ν to the variable α. The result is

$$N_l\,(k^2) \underset{|l|\to\infty}{\approx} C\,(k^2)\, e^{-\alpha\lambda}\, \lambda^{-3/2}; \qquad \operatorname{ch} \alpha = 1 + \mu^2/2k^2. \tag{5.10}$$

We set

$$l + 1/2 = \lambda = \lambda_1 + i\lambda_2 = |\lambda|\, e^{i\psi}, \quad -\pi < \psi < \pi; \tag{5.11}$$

$$\alpha = \alpha_1 + i\alpha_2, \quad \alpha_1 > 0, \quad -\pi < \alpha_2 < \pi. \tag{5.12}$$

Using the identity

$$\begin{aligned}\operatorname{ch} \alpha &= \operatorname{ch} \alpha_1 \cos \alpha_2 + i \operatorname{sh} \alpha_1 \sin \alpha_2 \\ &= 1 + {}^1/_2\mu^2\, |k|^{-2}\, [\cos \arg k^2 - i \sin \arg k^2]\end{aligned} \tag{5.13}$$

it is easy to show that

$$\alpha_2 = 0, \quad +\infty > \alpha_1 > 0 \quad \text{for} \quad 0 < k^2 < +\infty; \tag{5.14a}$$

$$-\pi < \alpha_2 < 0, \quad \alpha_1 > 0 \quad \text{for } \operatorname{Im} k^2 > 0; \tag{5.14b}$$

$$0 < \alpha_2 < \pi, \quad \alpha_1 > 0 \quad \text{for } \operatorname{Im} k^2 < 0; \tag{5.14c}$$

$$\alpha_2 = -\pi, \quad \alpha_1 > 0 \quad \text{for} \quad -\mu^2/4 < k^2 < 0 \text{ and } k^2 \text{ on the upper side of the cut } (-\infty, 0); \tag{5.14d}$$

$$-\pi < \alpha_2 < 0, \quad \alpha_1 = 0 \quad \text{for} \quad -\infty < k^2 < -\mu^2/4 \text{ and } k^2 \text{ on the lower side of the cut } (-\infty, 0). \tag{5.14e}$$

Substituting (5.11) and (5.12) in (5.10), we find the asymptotic form of $N_l(k^2)$ (and consequently, of $f_l(k^2)$) for $|l| \rightarrow \infty$ in the different regions of variation of k^2. We have (for $|l| \rightarrow \infty$)

$$f_l\,(k^2) \sim l^{-3/2} \exp\{-|l|\, \alpha_1 \cos\psi\} \quad \text{for} \quad k^2 > 0, \tag{5.15a}$$

$$f_l\,(k^2) \sim l^{-3/2} \exp\{|l|\,(-\alpha_1 \cos\psi + \alpha_2 \sin\psi)\} \quad \text{for } \operatorname{Im} k^2 \gtrless 0, \tag{5.15b, c}$$

$$f_l\,(k^2) \sim l^{-3/2} \exp\{|l|\,(-\alpha_1 \cos\psi + \pi \sin\psi)\} \quad \text{for} \quad -\mu^2/4 < k^2 < 0, \tag{5.15d}$$

$$f_l\,(k^2) \sim l^{-3/2}\{\exp|l|\,\alpha_2 \sin\psi\} \quad \text{for} \quad -\infty < k^2 < -\mu^2/4. \tag{5.15e}$$

We note that the regions of exponential fall-off and growth of the amplitude f_l in the l-plane are separated from one another by the line $\tan\psi = \alpha_1/\alpha_2$. On the real axis f_l drops off like $l^{-3/2}\, e^{-l\alpha_1}$ so long as k^2 does not lie on the cut $(-\infty, -\mu^2/4)$. In the latter case f_l oscillates on the real axis, while it decreases in the upper half-plane if k^2 is on the upper side of the cut and increases if k^2 is on the lower side of the cut. For any k^2 the amplitude $f_l(k^2)$ cannot increase on the imaginary axis of l faster than $l^{-3/2} \exp\{\pi|\operatorname{Im} l|\}$.

6. THE WATSON-SOMMERFELD TRANSFORMATION. ANALYTICITY AND ASYMPTOTIC FORM OF THE SCATTERING AMPLITUDE IN TERMS OF THE MOMENTUM TRANSFER

For working with the Watson-Sommerfeld transformation, we need some asymptotic formulas for the function $P_l(z)/\sin \pi l$ as $|l| \rightarrow \infty$. Setting

$$z = \operatorname{ch} \zeta, \quad \zeta = \zeta_1 + i\zeta_2, \quad \zeta_1 > 0, \quad -\pi < \zeta_2 < \pi, \tag{6.1}$$

we use the well-known asymptotic formula (cf., for example, [12])

$$\begin{aligned}P_l\,(\operatorname{ch}\zeta)/\sin \pi l \underset{|l|\to\infty}{\sim}\; & l^{-1/2}\, [\exp|\lambda|\,(\zeta_1 \cos\psi - \zeta_2 \sin\psi - \pi\,|\sin\psi|)\} \\ &+ e^{\pm i\pi/2}\{\exp|\lambda|\,(-\zeta_1 \cos\psi + \zeta_2 \sin\psi - \pi\,|\sin\psi|)\}], \\ & -\pi/2 \leqslant \psi \leqslant \pi/2.\end{aligned} \tag{6.2}$$

From this general formula it is easy to show that

a) if $-1 < z < 1$, then $\zeta_1 = 0$ and

$$P_l\,(z)/\sin \pi l \underset{|l|\to\infty}{\sim} l^{-1/2} \exp\{-|l|\,|\sin\psi|\,(\pi - |\zeta_2|)\}; \tag{6.3a}$$

b) if $1 < z < \infty$, then $\zeta_2 = 0$ and along the imaginary axis λ ($\psi = \pm\pi/2$)

$$P_l\,(z)/\sin \pi l \underset{\operatorname{Im} l\to\infty}{\sim} l^{-1/2}\, e^{-|l|\pi}; \tag{6.3b}$$

c) if $-\infty < z < 1$, then $\zeta_2 = \pi$ and along the imaginary λ axis

$$P_l\,(z)/\sin \pi l \text{ oscillates as } \operatorname{Im} l \rightarrow \infty; \tag{6.3c}$$

d) if z is not on the cut $(-\infty, -1)$, then along the imaginary λ axis

$$P_l\,(z)/\sin \pi l \underset{\operatorname{Im} l\to\infty}{\sim} l^{-1/2} \exp\{-|\lambda|\,(\pi - |\zeta_2|)\}. \tag{6.3d}$$

We now go over from the series (2.1) to the Watson-Sommerfeld integral along a contour C_1

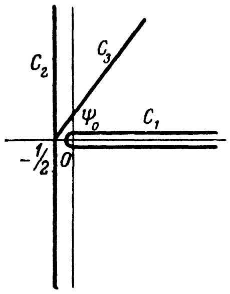

FIG. 3. Integration contours in the l-plane.

encircling the positive semiaxis in the l-plane (cf. Fig. 3):

$$T(k^2, \cos\theta) = \frac{1}{4ik^2}\int_{C_1} dl\,(2l+1)\,f_l\,(k^2)\,\frac{P_l(-\cos\theta)}{\sin\pi l}. \quad (6.4)$$

For $-1 < \cos\theta < 1$ and $k^2 > 0$, the convergence of this integral follows from (5.15a) and (6.3a).

Using the meromorphy of the amplitude in the l-plane, which was proved in Sec. 2, one can further deform the contour C_1 to a contour C_2 along the line Re $l = -1/2$ (cf. Fig. 3). We have

$$T(k^2, \cos\theta) = \frac{1}{4ik^2}\int_{-1/2+\varepsilon-i\infty}^{-1/2+\varepsilon+i\infty} dl\,(2l+1)\,f_l\,(k^2)\,\frac{P_l(-\cos\theta)}{\sin\pi l}$$
$$+\sum_i b_i\,(k^2)\,\frac{P_{\alpha_i(k^2)}(-\cos\theta)}{\sin\pi\alpha_i\,(k^2)}, \quad (6.5)$$

where α_i are the positive poles of f_l inside the contour C_2. From formulas (5.15a) and (6.3a) when $-1 < \cos\theta < 1$, $k^2 > 0$, it follows that the integral over the semicircle at infinity is equal to zero.

The representation (6.4) enables us to complete the analytic continuation of the amplitude to complex values of $\cos\theta$. Since $P_l(-\cos\theta)$ is an analytic function in the complex $\cos\theta$-plane with the cut $(1, +\infty)$, and the integral in (6.5) converges when $k^2 > 0$ (by virtue of (5.15a) and (6.3d)), formula (6.2) shows that when $k^2 > 0$ the amplitude $T(k^2, \cos\theta)$ is analytic in the complex $\cos\theta$-plane with the cut $1 \le \cos\theta < \infty$.

One can show that the cut in the $\cos\theta$-plane begins not at 1 but at $1 + \mu^2/2k^2$. In order to see this we note that the asymptotic formula (5.15a) quarantees the convergence of the series (2.1) inside the ellipse with semiaxes $1 + \mu^2/2k^2$ and $[(1 + \mu^2/2k^2)^2 - 1]^{1/2}$, where the major semiaxis $1 + \mu^2/2k^2$ is along the real axis in the $\cos\theta$-plane.[13] Thus when $k^2 > 0$ the series (2.1) defines a function which is analytic inside this ellipse. Since along the real axis the amplitude f_l drops off like $l^{-3/2}e^{-l\alpha_1}$ (cf. the end of the preceding Section), for arbitrary k^2 not on the cut $(-\infty, -\mu^2/4)$ there is an ellipse in the $\cos\theta$-plane containing the segment $-1 < \cos\theta < 1$ such that the series (2.1) converges if $\cos\theta$ is inside this ellipse.

Let us consider the representation (6.5) for $1 < -\cos\theta < \infty$. According to (6.3b) and (6.15), the integral in (6.5) converges for arbitrary k^2, so that this representation is convenient for determining the asymptotic behavior of the amplitude $T(k^2, \cos\theta)$ for $\cos\theta \to -\infty$. To study the asymptotic behavior as $\cos\theta \to +\infty$, it is not convenient to use (6.5) since when $-\infty < \cos\theta < -1$ the integral in (6.5) diverges for all k^2 (except possibly for $k^2 > 0$), and we therefore consider the modified Watson-Sommerfeld representation[2]

$$T(k^2, \cos\theta) = \frac{1}{4ik^2}\int_{C_1} dl\,(2l+1)\,\frac{P_l(\cos\theta)}{\sin\pi l}\,e^{-i\pi l}f_l\,(k^2). \quad (6.6)$$

The integral (6.6) converges for the same reasons as (6.4). But now one cannot deform the integration contour C_1 into the contour C_2 when $k^2 > 0$, $-1 < \cos\theta < 1$, since this is prevented by the exponential growth of $e^{-i\pi l}$ in the upper l-halfplane. According to (6.3a) and (5.14a) one can deform the contour C_1 into the contour C_3 consisting of the two rays $\psi = -\pi/2$ and $\psi = \psi_0$, where $\tan\psi_0 = \alpha_1/|\zeta_2|$ (cf. Fig. 3).

Now let $-\mu^2/4 < k^2 < 0$ (Im $k^2 \to 0$) and $-1 < \cos\theta < 1$. Then according to (5.15a) and (6.3a) the integration contour C_3 can be deformed into C_2, giving the representation

$$T(k^2, \cos\theta) = \frac{1}{4ik^2}\int_{-1/2+\varepsilon-i\infty}^{-1/2+\varepsilon+i\infty} dl\,(2l+1)\,f_l\,(k^2)\,e^{-i\pi l}\frac{P_l(\cos\theta)}{\sin\pi l}$$
$$+\sum_i b_i\,(k^2)\,e^{-i\pi\alpha_i}\,\frac{P_{\alpha_i(k^2)}(\cos\theta)}{\sin\pi\alpha_i\,(k^2)}. \quad (6.7)$$

From (5.14), (5.15) and (6.3b) it follows that when $1 < \cos\theta < \infty$ the integral in (6.7) converges if Im $k^2 \ge 0$.[8)]

Standard arguments[2] permit one to find the asymptotic form of the amplitudes $T^\pm(k^2, \cos\theta)$ for $\cos\theta \to \pm\infty$ (we recall that our whole treatment above applied to the even and odd projections of the scattering amplitude T):

$$T^\pm(k^2, \cos\theta) \underset{\cos\theta\to-\infty}{\approx} b_m^\pm(k^2)\,\frac{2^{\alpha_m^\pm}\Gamma(\alpha_m^\pm + 1/2)}{\sqrt{2}\,\Gamma(\alpha_m^\pm + 1)}\,\frac{(-\cos\theta)^{\alpha_m^\pm(k^2)}}{\sin\pi\alpha_m^\pm(k^2)}, \quad (6.8)$$

$$T^\pm(k^2, \cos\theta) \underset{\cos\theta\to+\infty}{\approx} b_m^\pm(k^2)\,\frac{2^{\alpha_m^\pm}\Gamma(\alpha_m^\pm + 1/2)}{\sqrt{2}\,\Gamma(\alpha_m^\pm + 1)}\,e^{-i\pi\alpha_m^\pm}\,\frac{(\cos\theta)^{\alpha_m^\pm(k^2)}}{\sin\pi\alpha_m^\pm(k^2)}. \quad (6.9)$$

8)In the case of Im $k^2 < 0$ one can get analogous results. But to obtain the physical consequences it is sufficient to consider values of k^2 in the upper halfplane.

Here $\alpha_m^{\pm}(k^2)$ are the positions of the poles farthes to the right in the l-plane for the amplitudes $f_l^{\pm}$ respectively.

Remembering that the total amplitude T is related to the amplitudes $T^{\pm}$ by the formula (cf. for example[14])

$$T(k^2, \cos\theta) = \tfrac{1}{2}[T^+(k^2, \cos\theta) + T^-(k^2, \cos\theta) + T^+(k^2, -\cos\theta) - T^-(k^2, -\cos\theta)], \quad (6.10)$$

we find from (6.8) and (6.9) the asymptotic form of T:

$$T(k^2, \cos\theta) \underset{\cos\theta\to\mp\infty}{\approx} b_m^+(k^2)\frac{2^{\alpha_m^+}\Gamma(\alpha_m^+ + 1/2)}{\sqrt{2}\Gamma(\alpha_m^+ + 1)}\frac{1+e^{-i\pi\alpha_m^+}}{\sin\pi\alpha_m^+}(\mp\cos\theta)^{\alpha_m^+} + b_m^-(k^2)\frac{2^{\alpha_m^-}\Gamma(\alpha_m^- + 1/2)}{\sqrt{2}\Gamma(\alpha_m^- + 1)}\frac{1-e^{-i\pi\alpha_m^-}}{\sin\pi\alpha_m^-}(\mp\cos\theta)^{\alpha_m^-}. \quad (6.11)$$

Treating the amplitude T as a function of $t = -2k^2(1-\cos\theta)$ or of $u = -2k^2(1+\cos\theta)$, we can find its asymptotic form for $t\to+\infty$ or $u\to+\infty$, i.e., for large energies in the t- or u-channels of the reaction (cf. also[15]):

$$T(k^2, \cos\theta) \underset{t\to\infty}{\sim} \frac{1\pm e^{-i\pi\alpha}}{\sin\pi\alpha}t^{\alpha}, \quad (6.12)$$

$$T(k^2, \cos\theta) \underset{u\to\infty}{\sim} \frac{1\pm e^{-i\pi\alpha}}{\sin\pi\alpha}u^{\alpha}. \quad (6.13)$$

We emphasize that to derive (6.12) and (6.13) it is sufficient that 1) the potentials $V^{\pm}$, representable in the form (1.6), exist only in some domain K containing a segment of the real axis of the k^2-plane, and 2) that the spectral functions $U^{\pm}$ satisfy the condition (1.7). As shown earlier,[5] for small k^2 the representation (1.6) can be proved using perturbations in quantum field theory. The convergence of the integral (1.6) and the condition (1.7) are, however, complementary assumptions, and if they are not satisfied the asymptotic amplitude for $t\to\infty$ or $u\to\infty$ may have a form different from (6.12) and (6.13). For example, if the integral

$$\int_{\mu^2}^{\infty} d\nu\, U(\nu, k^2)\,\nu^{-1}$$

diverges, the amplitude can have branch points in the l-plane, which have an essential effect on the asymptotic behavior (compare[16,17]). We remark that the method developed in this paper is convenient for solving the problem of the effect of branch points in the l-plane on the asymptotic form of the scattering amplitude, since it permits a simple comparison with the results from solving the problem using a potential scattering model.[16]

7. CONCLUSION

In conclusion we mention some results which can be gotten from the present work. Although on the whole the method considered here is most effective for studying partial waves and the dependence of the total amplitude on the variables t and u, it can also be used to study its dependence on s (or k^2), if one has sufficient information about the properties of the generalized potential as a function of k^2.[9)] If the potential is an analytic function of k^2 with branch points (on the real axis) corresponding to the thresholds for inelastic processes, then using the results of Secs. 5 and 6 one can show that for $t < 0$ the scattering amplitude is an analytic function of $s = 4(k^2 + m^2)$ with cuts along the real axis. Here, however, we come upon the problem of additional singularities (cf. Secs. 1 and 4), so that the question of the validity of the Mandelstam representation within the framework of our approach requires further investigation.

We emphasize again that this complication is related to the presence in (1.4) of the relativistic factor $\sqrt{q^2 + m^2}$. If this factor were not present, the equation would become the usual equation of nonrelativistic scattering theory with a complex, energy-dependent potential. If this potential is representable in the form (1.6), satisfies condition (1.7), is analytic in the k^2-plane with a cut along the positive real axis starting from the inelastic threshold, and if the absorptive condition ($\mathrm{Im}\, U(\nu, k^2) \le 0$) is satisfied on the cut, then the arguments given here allow one easily to obtain the Mandelstam representation for the scattering amplitude (cf.[19]).

Finally we mention that the equation $D_l(k^2) = 0$ is apparently a convenient means for studying Regge trajectories. In particular the behavior of the trajectories as $k^2\to\infty$ and $k^2\to 0$ is obtained very simply in the first approximation.

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9)On the other hand, for studying the analyticity and asymptotic form of the amplitude in the variable s, one can simply interchange the roles of s and t in the fundamental equations, and obtain an equation with some new potential defined in a bounded region in t, but for arbitrary s. There is an obvious analogy of such an approach with the "pole approximation" of Chew and Frautschi.[18]

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