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THREE-PARTICLE PRODUCTION NEAR THRESHOLD WITH RESONANCE INTERACTION OF TWO PARTICLES

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The Watson-Migdal formula can be used to analyze three-particle production near threshold in the case of a resonance interaction between two of the particles. If the kinetic energy of the produced particles is small, this formula expresses the three-particle production amplitude in terms of the threshold-energy production amplitude and scattering length of the resonant interacting particles. Corrections to the Watson-Migdal formula obtained by taking into account the effective radius of the resonant interacting particles, and also nonresonance interaction of the first two particles with the third, are also considered. The corrections are given in terms of the threshold-energy three-particle production amplitude, effective radius and scattering lengths of the produced particles. Correction terms for the reaction $N + N \to N + N + \pi$ are considered.

THE study of reactions in which several particles are produced has thus far been the only source of information regarding the scattering amplitudes of unstable particles. A comparison of experimental data with the Breit-Wigner [1] and Watson-Migdal[2] formulas, or utilization of the method of Chew and Low,[3] furnishes information about the character of the interactions of unstable particles. However, there is still another way of obtaining information about the scattering amplitudes of unstable particles at low energies. This involves a study of the production of several particles near threshold.

There have been several investigations [4-7] of the amplitude of three-particle production close to threshold, when all the produced particles interact without resonance at low energies. In this case the amplitude can be represented by a series in which the first term is a constant, the second term is of the order $\sqrt{E}$ (E is the c.m.s. kinetic energy of the produced particles), the third is of the order E etc. When the kinetic energy of the produced particles is very small the later terms of this expansion are small compared with the preceding. By studying these correction terms we can, in principle, obtain the scattering lengths of the produced particles.

There are several reactions involving a resonance interaction of two of the produced particles, e.g., $N + N \to N + N + \pi$, $N + N \to N + \Lambda + K$, $N + N \to N + \Sigma + K$ etc. These reactions near threshold can be described by means of the Watson-Migdal formula.[2] Corrections of this formula are needed to take into account the effective radius of resonant interacting particles and the interaction of these two particles with the third particle. These corrections can be calculated by a method similar to that employed in [4-7]. Corrections of the order $\sqrt{E}$, depending only on the relative momenta of the produced particles, have been calculated by Gribov,[8] in a quantum-mechanical examination of three-particle systems. These corrections were expressed as single and double definite integrals.

In the present work the dispersion technique is used to calculate corrections of the order $\sqrt{E}$ to the Migdal-Watson formula, which depend both on the relative momenta of the produced particles and on the total kinetic energy. In Sec. 1 the production of three neutral spinless particles with different masses is considered; corrections of the order $\sqrt{E}$ have the form of definite simple integrals. Sec. 2 considers reactions in which the masses of the resonant interacting particles considerably exceed the mass of the third particle. In this case the corrections can be calculated in a general form in terms of analytic functions.

The reaction $N + N \to N + N + \pi$ is the only process of the considered type in which all quantities in the correction terms are known. In the present work the cross sections of $N + N \to N + N + \pi$ processes near threshold are calculated.

1. PRODUCTION OF THREE NEUTRAL SPINLESS PARTICLES WITH DIFFERENT MASSES

In [5] a detailed study was made of questions concerning the series expansion of three-particle production amplitudes in powers of the momenta of the produced particles for the case in which the scattering lengths of these particles are not too large. Before considering the case that is of interest to us, we shall review that situation very briefly. An amplitude with total orbital angular momentum $L = 0$ was expanded in terms of the relative momenta k_{il} of different pairs of produced particles (the amplitude with $L = 0$ depends on only three such momenta). The terms of the amplitude having singularities whose distances from the threshold were of the order of the masses of the involved particles, were expanded near the threshold in powers of k_{il}^2. The first term of this series is a constant λ, which is represented conveniently by diagram 1a.

The next terms, which are linear in k_{il}^2, are of the order $\lambda k_{il}^2/m^2$ (m is of the order of the masses of the particles involved in the reactions), i.e., of the order $\lambda E/m$. These terms have been referred to simply as terms of the order E.

The amplitude also has singularities with respect to k_{il}^2 at the threshold. These singularities are associated with the scattering of one produced particle by another, as represented in Figs. 1,b and c. Diagram 1b is equal to $\lambda i k_{12} a_{12}$; diagram 1c is of the order $\lambda a_{12} a_{23} E \sqrt{\mu_{12}\mu_{23}}$ (a_{il} are the scattering lengths of the particles, and μ_{il} are their reduced masses). If the scattering lengths a_{il} are small (of the order m^{-1}) the terms having singularities at threshold which are of the type represented in Fig. 1b are of the order $\sqrt{E}$, while terms of the type Fig. 1c are of the order E. In this case the diagrams with a large number of scatterings in the final state make a less important contribution at threshold.

We shall be interested in the amplitude of three-particle production with $L = 0$ in the case when the scattering length of any two particles, let us say particles 1 and 2, is large, so that $a_{12}\sqrt{2\mu_{12}E}$ is not small. As previously, the amplitudes a_{13} and a_{23} are taken to be of the order m^{-1}. This means that in the diagrams we must take into account the multiple scattering of particles 1 and 2 by each other, which in this case does not lead to an additional small factor. Only the scattering of the first two particles by the third particle results in an additional small factor.

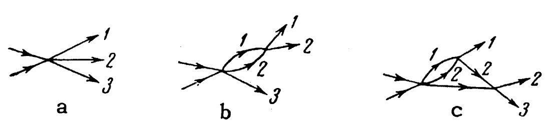

FIG. 1.

Figures 2—4 show the diagrams that are important for the amplitude when terms of the order $\sqrt{E}$ are taken into account. As already mentioned, the first diagram in Fig. 2 is a constant λ, and the second diagram represents $\lambda i k_{12} a_{12}$.[5,6] Each successive loop leads to an extra factor $i k_{12} a_{12}$, which is not small; therefore the sum of the diagrams in Fig. 2 is

$$\lambda/(1 - ik_{12}a_{12}). \tag{1}$$

This is the Watson-Migdal formula, for which corrections of the order $\sqrt{E}$ arise when the interaction of the first two particles with the third is taken into account. Diagrams of this type are shown in Figs. 3 and 4.

Diagrams 3, a and b, were studied in detail in[3,4]. The calculation of the diagrams of Fig. 3c and 4 are entirely similar with regard to procedure but are considerably more complicated. We shall consider these calculations briefly.

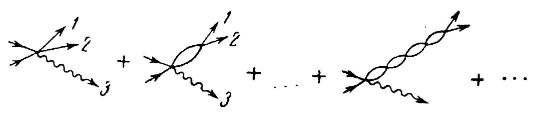

FIG. 2.

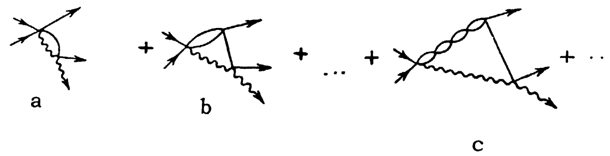

FIG. 3.

Diagrams 3 and 4 depend on only two variables—on one of the relative momenta k_{il}^2 of the produced particles and on the total kinetic energy E. They can therefore be calculated using the dispersion relations

$$A(k_{il}E) = A(E) + \frac{1}{\pi} k_{il}^2 \int_{C_1} dk_{il}'^2 \frac{A_1(k_{il}', E)}{k_{il}'^2 (k_{il}'^2 - k_{il}^2 - i\varepsilon)}, \tag{2a}$$

$$A(E) = \frac{1}{\pi} E \int_{C_2} dE' \frac{A_1(E')}{E'(E' - E - i\varepsilon)}. \tag{2b}$$

$A_1(k_{il}E)$ and $A_1(E)$ are the absorption parts in

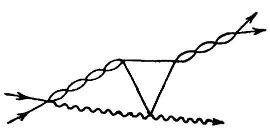

FIG. 4.

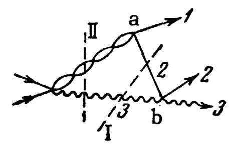

FIG. 5.

channels where the total energies are $k_{il}^2/2\mu_{il}$ and E, respectively. For example, for the diagram of Fig. 5 the absorption part $A_1(k_{il}E)$ arises from the division of type I, and $A_1(E)$ from the division of type II ($k_{23}=0$). If strongly interacting particles do not form bound states, the integrations in (2a) and (2b) go from 0 to ∞. In the opposite case some of the diagrams have anomalous singularities and the integrations in (2a) and (2b) follows a more complicated contour. $A(k_{il}E)$ vanishes at $E=0$ in the physical region in accordance with the fact that the amplitude of three-particle production at zero energy is λ.

We now proceed to calculate the sum of the diagrams in Fig. 3 for the case in which the second particle interacts with the third. We denote this sum by $A_2^{\mathrm{I}}(k_{23}E)$; the absorption part is

$$A_{21}^{\mathrm{I}}(k_{23}E)=a_{23}k_{23}\int\limits_{-1}^{1}\frac{dz}{2}\,\frac{\lambda}{1-ik'_{12}a_{12}}. \tag{3}$$

The term $\lambda(1-ik'_{12}a_{12})^{-1}$ in the integrand arose from the summation of all diagrams to the left of a type-I division (shown in Fig. 5); z is the cosine of the angle between the relative momentum of particles 2 and 3 in the intermediate state (for which division I occurs in Fig. 5) and the momentum of the first particle in the final state, in the system where the center of mass of particles 2 and 3 is at rest; k'_{12} is related to z and k_{23} as follows:[6]

$$\begin{aligned}(m_1+m_2)(m_2+m_3)\,x'^2_{12}&=m_1m_3x^2_{23}+m_2(m_1+m_2+m_3)\\&\times(E-x^2_{23})+2z\,[x^2_{23}(E-x^2_{23})\\&\times(m_1+m_2+m_3)\,m_1m_2m_3]^{1/2};\\x^2_{il}&=k^2_{il}/2\mu_{il}.\end{aligned} \tag{4}$$

The variables χ^2_{il} appearing in (4) will be used very frequently hereafter.

$A_{21}^{\mathrm{I}}(x_{23}E)$ is easily calculated:

$$\begin{aligned}A_{21}^{\mathrm{I}}(x_{23}E)&=i\,\frac{\lambda a_{23}(m_1+m_2)}{a_{12}}\left[\frac{m_3}{m_1m_2(m_1+m_2+m_3)}\right]^{1/2}x_{23}(E-x^2_{23})^{-1/2}\\&+\frac{\lambda a_{23}(m_1+m_2)^2}{a^2_{12}\,2\sqrt{2}\,m_1m_2}\left[\frac{m_2+m_3}{m_1(m_1+m_2+m_3)}\right]^{1/2}(E-x^2_{23})^{-1/2}\ln\frac{y_+}{y_-};\\y_\pm&=1-i\sqrt{2}\,a_{12}\left[\frac{m_1m_2}{(m_1+m_2)^2(m_2+m_3)}\right]^{1/2}\{\pm\sqrt{m_1m_3}\,x_{23}\\&+[m_2(m_1+m_2+m_3)(E-x^2_{23})]^{1/2}\}.\end{aligned} \tag{5}$$

When $a_{12}>0$, $A_2^{\mathrm{I}}(x_{23}E)$ is obtained from the integral (2a), where the contour C_1 is the segment of the real axis from 0 to ∞ (Fig. 6a). This

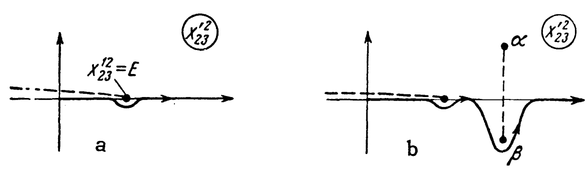

FIG. 6.

figure shows the root-type singularities of the absorption part. To the right of the point $x'_{23}=E$ on the real axis the value of $(E-x'^2_{23})^{1/2}$ is $i\,|E-x'^2_{22}|^{1/2}$ and the absorption part is real.

If $a_{12}<0$ the contour C_1 is shown in Fig. 6b. In this case the contour passes below the singularity of the logarithm of the absorption part.

To calculate the absorption part of the diagrams in Fig. 3 we make all possible type-II divisions (Fig. 5); the relative momentum of emitted particles 2 and 3 is set equal to zero. The result (for $a_{12}>0$) is

$$\begin{aligned}A_{21}^{\mathrm{I}}(E)&=\frac{\lambda a_{12}a_{23}(m_1+m_2)^2}{2\pi m_1m_2}\int\limits_0^{2\mu_{12}E}dk'^2_{12}\int\limits_{-1}^{1}dz\,\frac{k'_3k'_{12}}{1+k'^2_{12}a^2_{12}}\\&\times\left\{-\frac{2m_1m_3}{m_1+m_2+m_3}E-\frac{2(m_2+m_3)(m_1+m_2)}{m_1+m_2+m_3}(E-x'^2_{12})\right.\\&\left.-4z\left[\frac{m_1m_3(m_1+m_2)(m_2+m_3)}{(m_1+m_2+m_3)^2}E(E-x'^2_{12})\right]^{1/2}\right\}^{-1}.\end{aligned} \tag{6}$$

Here k'_{12} is the relative momentum of particles 1 and 2 in the intermediate state, and k'_3 is the momentum of the third particle in the c.m. system of all three particles. The expression inside the square brackets is the propagation function of the second particle (in Fig. 5 the line corresponding to this propagation function extends between vertices a and b), z is the cosine of the angle between the momentum of this second particle and the momentum of the first particle in the final state.

Further uncomplicated calculations lead to

$$\begin{aligned}A_{21}^{\mathrm{I}}(E)&=+\frac{\lambda a_{23}(m_1+m_2)^2}{a^2_{12}2\sqrt{2}\,m_1m_2}\left[\frac{m_2+m_3}{m_1(m_1+m_2+m_3)}\right]^{1/2}\\&\times\left(2\sqrt{2}a_{12}\frac{m_1\sqrt{m_2m_3}}{(m_1+m_2)\sqrt{m_2+m_3}}+E^{-1/2}\ln\frac{\varphi_-}{\varphi_+}\right);\\\varphi_\pm&=[(m_1+m_2)(m_2+m_3)(1+2\mu_{12}a^2_{12}E)]^{1/2}\\&\pm a_{12}\sqrt{2m_1m_3\mu_{12}E}.\end{aligned} \tag{7}$$

For $E>0$ the absorption part $A_{21}^{\mathrm{I}}(E)$ is real, as expected.

For $a_{12}>0$ the value of $A^{\mathrm{I}}(E)$ is determined by the integral (2b) with the absorption part (7), the integration going along the real axis from 0 to ∞. If $a_{12}<0$, there is also a bound state which

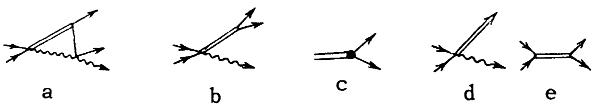

FIG. 7.

must be taken into account in summing over the intermediate states. An additional term is given by diagram 7a. This diagram contains the vertices 7, b and c, which are, respectively,

$$\lambda' = -\lambda/2a_{12}\sqrt{-\pi\mu_{12}a_{12}},$$

$$G = 4\,(m_1 + m_2)\,\sqrt{\pi}/\sqrt{-\mu_{12}a_{12}}. \qquad (8)$$

These relations are easily derived if we note that the vertices (8) appear in diagrams 7, d and e. The residues at the poles of these diagrams, and, accordingly, of the functions $\lambda(1 - ik_{12}a_{12})^{-1}$ and $a_{12}(1 - ik_{12}a_{12})^{-1}$, must coincide because the latter are exact expressions (neglecting the correction terms) for the production and scattering amplitudes of resonant interacting particles and contain these pole diagrams. It was not necessary to consider diagrams with bound states separately in (3), since the integrand is the total three-particle production amplitude in zeroth approximation, independently of the sign of a_{12}.

If the diagrams in Fig. 3 were calculated using Feynman's technique, it would also be unnecessary to consider diagram 7a separately. In calculating diagrams by the dispersion technique all states must be taken into account in the unitarity condition, including the bound state of particles 1 and 2. The division of diagram 7a, as shown in the figure, gives an addition to the absorption part (7), which is

$$\Delta A_1^{\rm I}(E) = -\frac{\lambda a_{23}(m_1+m_2)^2}{\sqrt{2a_{12}^2 m_1 m_2}}\left[\frac{m_2+m_3}{(m_1+m_2+m_3)\,m_1}\right]^{1/2} E^{-1/2}\ln\frac{\varphi_-}{\varphi_+}. \qquad (9)$$

The total absorption part for $a_{12} < 0$ is the sum of (7) and (9). $A^{\rm I}(E)$ is obtained by the integration (2b) of these absorption parts; as previously, the integration contour for (7) is the real axis from 0 to ∞. The integration of the addition (9) goes along the contour shown in Fig. 8c beginning at the point $E' = -(2\mu_{12}a_{12}^2)^{-1}$, where there is a root-type singularity, the branch cut from which is shown in the figure. To the right of the cut on the real axis $(1 + 2\mu_{12}a_{12}^2E)^{1/2}$ is positive. α and β are singular points of the logarithm in (9). The contribution to $A_2^{\rm I}(E)$ from $\Delta A_{21}^{\rm I}(E)$ can easily be calculated from the residues.

We now proceed to calculate the diagrams de-

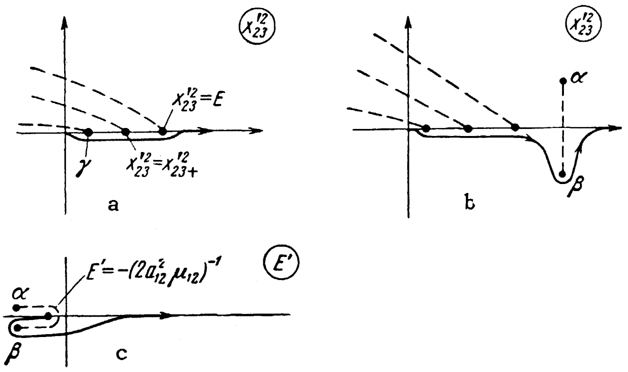

FIG. 8.

pending on k_{12} (of the type represented in Fig. 4). The sum of all these diagrams is denoted by $A^{\rm II}(k_{12}E)$. The summation of the loops with interaction of particles 1 and 2 in the final state gives

$$A^{\rm II}(k_{12}E) = [B_1^{\rm II}(k_{12}E) + B_2^{\rm II}(k_{12}E)]\,(1 - ik_{12}a_{12})^{-1}, \qquad (10)$$

where $B_2^{\rm II}(k_{12}E)$ is the sum of the diagrams represented in Fig. 9. $B_1^{\rm II}(k_{12}E)$ is the sum of diagrams in which particles 1 and 2 have been exchanged. $B_2^{\rm II}(k_{12}E)$ can be represented in the form of the dispersion integrals (2). $B_{21}^{\rm II}(k_{12}E)$ is determined by the division I in Fig. 9, and $B_{21}^{\rm II}(E)$ by the divisions II and III. The calculation of $B_2^{\rm II}(k_{12}E)$ is somewhat more complicated than that of $A_2^{\rm I}(k_{23}E)$, but is completely analogous. We present only the final result:

$$B_2^{\rm II}(k_{12}E) = B_2^{\rm II}(E) + ik_{12}a_{12}A_2^{\rm I}(E)$$
$$+\frac{a_{12}(m_2+m_3)\sqrt{m_1+m_2}}{2\sqrt{2}\,\pi\sqrt{m_3(m_1+m_2+m_3)}}$$
$$\times\int\limits_{C_1} dx_{23}'^2 A_{21}^{\rm I}(x_{23}'E)\Big[\,i(E-x_{12}^2)^{-1/2}\ln\frac{x_-}{x_+} + i(E-x_{12}^2)^{-1/2}$$
$$\times\ln\frac{z_-(x_{12})}{z_+(x_{12})} - iE^{-1/2}\ln\frac{z_-(0)}{z_+(0)}$$
$$-\frac{4i\sqrt{m_1m_2m_3(m_1+m_2+m_3)}\,x_{12}}{(m_1+m_2)(m_2+m_3)\,x_{23}'^2}\Big],$$
$$x_\pm = (m_1+m_2)(m_2+m_3)\,x_{23}'^2$$
$$-\left[\sqrt{m_1m_3}\,x_{12} \pm \sqrt{m_2(m_1+m_2+m_3)(E-x_{12}^2)}\right]^2,$$
$$z_\pm(x_{12}) = m_2(m_1+m_2+m_3)\,x_{23}'^2 + \Big[\sqrt{m_1m_3(x_{23}'^2-E)}$$
$$\pm i\sqrt{(m_1+m_2)(m_2+m_3)(E-x_{12}^2)}\Big]^2. \qquad (11)$$

The expression for $A_{21}^{\rm I}(x_{23}E)$ has already been given in (5). The first two logarithms in the integrand have singularities at

$$x'^2_{23\pm} = [\sqrt{m_1 m_3}\, x_{12} \pm \sqrt{m_2 (m_1 + m_2 + m_3)(E - x_{12}^2)}]^2$$
$$\times (m_1 + m_2)^{-1} (m_2 + m_3)^{-1}.$$

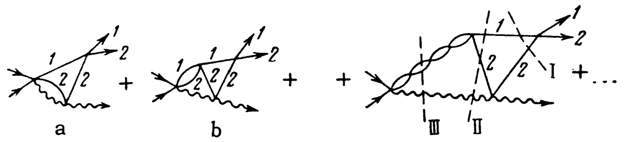

FIG. 9.

Each integral separately has both singularities. In the first integral x'^2_{23-} is a singularity of the numerator of the logarithm, while x'^2_{23+} is a singularity of the denominator. In the second integral the denominator of the logarithm has both singularities; the root-type singularity $x'^2_{23} = E$ is located above the integration contour, and $\sqrt{x'^2_{23} - E}$ is positive on the real axis to the right of this point. It is easily seen that the sum of the first two logarithms has a singularity only at $x'^2_{23} = x'^2_{23+}$. The singularity at $x'^2_{23} = x'^2_{23-}$ is cancelled in the sum of logarithms. The integration contour C_1 for $a_{12} > 0$ is shown in Fig. 8a; the integration goes along the real axis from 0 to ∞. The same figure shows the cuts starting at the singularities.

On the real axis the first logarithm is real to the right of $x'^2_{23} = x'^2_{23+}$. The second logarithm is real in the interval between the points $x'^2_{23} = x'^2_{23+}$ and $x'^2_{23} = E$. On the integration contour to the left of $x'^2_{23} = x'^2_{23+}$ each logarithm acquires the imaginary addition $+ i\pi$. The third logarithm has two coincident singularities at

$$x'^2_{23} = m_2 (m_1 + m_2 + m_3)(m_1 + m_2)^{-1} (m_2 + m_3)^{-1} E.$$

These are singularities of the denominator. The third logarithm is real to the right of this point, which is denoted by γ. To the left of γ the third logarithm has the imaginary addition $2\pi i$.

For $a_{12} < 0$ the singularities of $A^{I}_{21}(x'_{23}E)$ deform the contour exactly as for the calculation of $A^{I}_{2}(x_{23}, E)$. Figure 8b shows the integration contour and the singularities for $a_{12} < 0$.

The absorption part $B^{II}_{21}(E)$ for $a_{12} > 0$ is

$$B^{II}_{21}(E) = \frac{\lambda a_{23}\, i\, (m_2 + m_3)(m_1 + m_2)^2}{a_{12} 4 m_1 m_2 (m_1 + m_2 + m_3)}$$
$$\times \left\{ + i \frac{\sqrt{(m_1 + m_2)(m_1 + m_2 + m_3)}}{\sqrt{2}\, a_{12} m_1 \sqrt{m_3}} E^{-1/2} \ln \frac{\varkappa^1_+}{\varkappa^1_-} \right.$$
$$\left. - \ln \frac{\varkappa^2_+}{\varkappa^2_-} - \left[\frac{1 + 2a_{12}^2 \mu_{12} E}{2a_{12}^2 \mu_{12} E} \right]^{1/2} \ln \frac{\varkappa^3_+}{\varkappa^3_-} + 2 \ln \frac{m_+}{m_-} \right\};$$

$$\varkappa^1_\pm = m_2 (m_1 + m_2 + m_3)$$
$$+ m_1 m_3 (\sqrt{2\mu_{12} E a_{12}^2} \pm \sqrt{1 + 2\mu_{12} a_{12}^2 E})^2,$$

$$\varkappa^2_\pm = m_2 (m_1 + m_2 + m_3)$$
$$- m_1 m_3 \pm 2i \sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)}$$
$$\times \sqrt{1 + 2\mu_{12} a_{12}^2 E},$$

$$\varkappa^3_\pm = (m_1 + m_2)(m_2 + m_3)$$
$$\pm 2i\, [m_1 m_2 m_3 (m_1 + m_2 + m_3)\, 2\mu_{12} a_{12}^2 E]^{1/2},$$

$$m_\pm = \sqrt{m_2 (m_1 + m_2 + m_3)} \pm i \sqrt{m_1 m_3}. \qquad (12)$$

$B^{II}_{21}(E)$ is real for $E > 0$. $B^{II}_{2}(E)$ is determined by a dispersion integral such as (2b). The integration with respect to E' goes along the real axis from 0 to ∞.

In the case $a_{12} < 0$ the absorption part contains an additional term arising from consideration of the bound state:

$$\Delta B^{II}_{21}(E) = -2B^{II}_{21}(E). \qquad (13)$$

Here $B^{II}_{21}(E)$ is determined from (12). The integration of the dispersion integral with the absorption part $\Delta B^{II}_{21}(E)$ goes from $E' = -(2\mu_{12} a_{12}^2)^{-1}$ to ∞. The absorption part of the second dispersion integral is obtained from (12), as previously, and is integrated from 0 to ∞.

We have thus far considered only the diagrams in which the second particle interacts with the third. The corresponding expressions for the diagrams in which the first particle interacts with the third are obtained from (5), (7), etc. with the indices 1 and 2 exchanged. The amplitudes $A^{I}_{i}(kE)$ and $A^{II}(k_{12}E)$ are seen to be expressed by definite simple integrals. If a_{12} is known, these integrals can be calculated comparatively easily.

The foregoing calculations are fairly complicated. It is therefore desirable to consider limiting cases for the purpose of comparing them with expressions obtained independently. A limit can be reached by making a_{12} small, after which a comparison can be made with the results in [6]. To the first order in a_{12}, $A^{I}_{21}(x_{23}E)$ and $A^{I}_{21}(E)$ should become the absorption parts of diagrams 3, a and b, while $B^{II}_{21}(xE)$ and $B^{II}_{21}(E)$ become the expressions for diagram 9a [Eqs. (10) and (14) of [6]]. It can be shown that the results agree.

We have thus far considered corrections of the order $\sqrt{E}$ arising from the interaction of the first two particles with the third. Corrections arising from consideration of the effective radius of nucleon interaction are also of the order $\sqrt{E}$, and are obtained because

$$a_{12}(k_{12}^2) = a_{12} - \tfrac{1}{2} k_{12}^2 a_{12} r_{12}, \qquad (14)$$

where r_{12} is the effective interaction radius of particles 1 and 2, which is of the order m^{-1}. Thus the correction terms in the expansion of $a_{12}(k_{12}^2)$ are of the order $\sqrt{E}$.

If in the Watson-Migdal formula describing the production process in zeroth approximation we make the substitution $a_{12} \to a_{12} - \frac{1}{2}k_{12}^2 a_{12} r_{12}$ and expand the resulting expression in powers of $-\frac{1}{2}k_{12}^2 a_{12} r_{12}$ we obtain the required correction terms. It must be remembered that λ can be represented by

$$\lambda = a_{12}\alpha + m^{-1}\beta = \lambda_0 + \Delta\lambda, \tag{15}$$

where α and β are generally of the same order. Since $a_{12} \gg m^{-1}$, we have $\lambda_0 \gg \Delta\lambda$. The fact that a_{12} is a factor in several terms comprising λ can be seen from the reaction $1 + 2 \to 1 + 2 + 3$. Specifically, the diagrams of Fig. 10 are included in the definition of λ. These diagrams have the structure $a_{12}\alpha$. Taking the effective interaction radius into account, the Watson-Migdal formula becomes

$$\frac{\lambda}{1 - ik_{12}a_{12}} \to \frac{\lambda}{1 - ik_{12}a_{12}} - \frac{\frac{1}{2}k_{12}^2 a_{12} r_{12} \lambda_0}{(1 - ik_{12}a_{12})^2}. \tag{16}$$

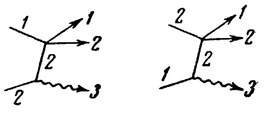

FIG. 10.

In the second term λ_0 can be replaced by λ, since the resulting error affects only higher-order terms (of the order E).

The entire three-particle production amplitude at threshold can be written, accurately up to terms of the order $\sqrt{E}$, as

$$A(k_{12}k_{13}k_{23}) = \frac{\lambda}{1 - ik_{12}a_{12}} - \frac{\frac{1}{2}k_{12}^2 a_{12} r_{12}\lambda}{(1 - ik_{12}a_{12})^2}$$

$$+ \lambda \frac{a_{23}B_2(k_{12}E) + a_{13}B_1(k_{12}E)}{1 - ik_{12}a_{12}}$$

$$+ \lambda a_{13}A_1(k_{13}E) + \lambda a_{23}A_2(k_{23}E),$$

$$A_i^I(k_{i3}E) = \lambda a_{i3}A(k_{i3}E), \qquad B_i^{II}(k_{12}E) = \lambda a_{i3}B_i(k_{12}E). \tag{17}$$

The constant λ in (17) is a common normalizing factor. All other quantities are determined by the amplitudes of two-particle interactions at zero energy and effective radius r_{12}. In all terms of (17) with the exception of the first, λ can be replaced by λ_0, because the resulting error would be of a higher order than the terms considered in (17).

If $a_{12} < 0$, particles 1 and 2 can form a bound state (1, 2). The production amplitude of this bound state and the third particle is obtained from (17) by setting $A_i(k_{i3}E) = 0$ and multiplying the right-hand side by the quantity

$$-(1 - ik_{12}a_{12})/2a_{12}\sqrt{-\pi\mu_{12}a_{12}}.$$

Gribov [8] calculated the parts of the corrections $A_i(k_{i3}E)$ and $B_i(k_{12}E)$ depending only on k_{il}, i.e., $A_i^I(E)$ and $B_i^{II}(E)$ were not calculated. $A_i(k_{i3}E)$ had the form of a definite simple integral, while $B^{II}(k_{12}E)$ had the form of a definite double integral. Because of the different method of calculation the definite integrals in [8] are completely different from those in the present work and it would be extremely difficult to make a direct comparison between them. The expressions for $A_i(k_{i3}E)$ and $B_i(k_{12}E)$ in [8] can be compared only indirectly; for example, they can be expanded in powers of a_{12} for comparison with the terms calculated in [6,7]. In this way it can be shown that the expression for $B_i(k_{12}E)$ in [8] contains an error.

2. CALCULATION OF CORRECTIONS FOR THE REACTION $N + N \to N + N + \pi$ WITH $m_1 = m_2 \gg m_3$

For the purpose of applying our formulas to the reaction $N + N \to N + N + \pi$, we shall assume $m_1 = m_2 = M \gg m_3 = \mu$. An analysis, similar to what follows, can also be performed without assuming that the masses of the first two particles are equal.

For the case $m_1 = m_2 \gg m_3$ the absorption parts and the integrand in (11) can be expanded in powers of μ/M, and $A_i^I(k_{i3}E)$ and $A^{II}(k_{12}E)$ can be calculated explicitly. We note that, since $m_1 = m_2$, we have $A_1(kE) = A_2(kE) \equiv A(xE)$ and $B_1(kE) = B_2(kE) \equiv B(xE)$.

Let us consider the absorption part $A(k_{23}E)$, which for the case $M \gg \mu$ (introducing the notation $a_{12} = a$) equals

$$\frac{i\sqrt{2\mu}\,x_{23}}{a\sqrt{M}\sqrt{E - x_{23}^2}} + \frac{1}{a^2\sqrt{M}\sqrt{E - x_{23}^2}}$$

$$\times \ln\frac{1 - ia\sqrt{M}\sqrt{E - x_{23}^2} - ia\sqrt{\mu/2}\,x_{23}}{1 - ia\sqrt{M}\sqrt{E - x_{23}^2} + ia\sqrt{\mu/2}\,x_{23}}. \tag{18}$$

When the logarithm is expanded in powers of μ/M it is easily seen that on the complex plane of x_{23}^2 there exists a region in which $|1 - ia\sqrt{M}\sqrt{E - x_{23}^2}| \sim |a\sqrt{\mu/2}\,x_{23}$; the expansion therefore ceases to be valid. This difficulty can be obviated by shifting the contour of integration in the dispersion integral to the lower half-plane.

This procedure can be followed for both $a > 0$ and $a < 0$.

The calculations give the following expression for $A(kE)$ when $a > 0$:

$$A(E) = \frac{\sqrt{2\mu}}{\pi a\sqrt{M}}\Big[-\ln\left(\tfrac{1}{4}Ma^2E\right) + i\pi + (1 + Ma^2E)^{-1/2} \times \left(\ln\frac{\sqrt{1+Ma^2E}-1}{\sqrt{1+Ma^2E}+1} - i\pi\right)\Big], \tag{19a}$$

$$A(xE) = \frac{a^2Mx^2\sqrt{2\mu(E+a^{-2}M^{-1})}}{(1+a^2ME)(1+a^2ME-a^2Mx^2)} \times \left(\frac{2}{\pi}\operatorname{arc\,sh}(a\sqrt{ME})^{-1} + i\right) + \frac{x\sqrt{2\mu}}{1+a^2ME-a^2Mx^2} \times \left(-\frac{2}{\pi}a\sqrt{M(E-x^2)}\arccos\frac{x}{\sqrt{E}} + i\right) + A(E). \tag{19b*}$$

When $a < 0$ the expression for $A(E)$ contains the additional term

$$\Delta A(E) = -2i\frac{\sqrt{2\mu}}{a\sqrt{M}}\left(1 - \frac{1}{\sqrt{1+Ma^2E}}\right). \tag{20}$$

A similar calculation of $B(kE)$ for $a > 0$ gives

$$B(xE) = B(E) + ia\sqrt{M}\,xA(E) + i\sqrt{2\mu(E-x^2)} + \sqrt{2\mu(E+a^{-2}M^{-1})}\,[(1+ia\sqrt{M}x)^{-1} + ia\sqrt{M}\,x(1+a^2ME)^{-1} - 1] \times\left(\frac{2}{\pi}\operatorname{arc\,sh}(a\sqrt{ME})^{-1} + i\right) + \sqrt{2\mu(E-x^2)}(1+ia\sqrt{M}\,x)^{-1} \times\left(\frac{2}{\pi}a\sqrt{M}x\arcsin\frac{x}{\sqrt{E}} - i\right); \tag{21a}$$

$$B(E) = \frac{\sqrt{2\mu}}{\pi a\sqrt{M}}\Big[\ln\left(\tfrac{1}{4}Ma^2E\right) - i\pi + \sqrt{1+Ma^2E} \times\left(\ln\frac{\sqrt{1+Ma^2E}+1}{\sqrt{1+Ma^2E}-1} + i\pi\right)\Big]. \tag{21b}$$

For $a < 0$, $B(E)$ acquires the addition

$$\Delta B(E) = 2i\frac{\sqrt{2\mu}}{a\sqrt{M}}(1 - \sqrt{1+Ma^2E}). \tag{22}$$

It must also be remembered that $A(E)$ in (21a) changes in accordance with (20) when $a < 0$.

It is easily verified that the amplitude $A(kE)$ has no pole on the physical sheet at the point $x^2 = E + a^{-2}M^{-1}$. An expansion in powers of μ/M was used to calculate $\Delta A(E)$ and $\Delta B(E)$. It is easily shown that, as a result, (20) and (22) are invalid when $E \sim -M^{-1}a^{-2}$.

It can be shown that the given formulas for $A(kE)$ and $B(kE)$ when expanded in powers of a become the corresponding formulas of Gribov[5] for the terms that are linear and quadratic in the momenta. The expressions for the linear and quadratic terms in [5] must, of course, also be expanded in powers of μ/M.

The foregoing formulas enable us to calculate the cross sections for $N + N \to N + N + \pi$. Different isospin and spin states are taken into account in the conventional manner. If the nucleons are unpolarized, then for processes $p + p \to p + p + \pi^0$ and $p + p \to p + n + \pi^+$ up to terms of the order $\sqrt{E}$ we have, respectively,

$$\frac{d\sigma^0}{d\Omega} = \frac{|\lambda_s|^2}{1+k_{12}^2a_s^2} + 2\,\mathrm{Re}\,[\lambda_s^*(1-ik_{12}a_s)(\lambda_s\beta_s + \lambda_t\gamma_t)], \tag{23a}$$

$$\frac{d\sigma^+}{d\Omega} = 3|\lambda_t|^2(1+k_{12}^2a_t^2)^{-1} + |\lambda_s|^2(1+k_{12}^2a_s^2)^{-1} + 2\,\mathrm{Re}\,\{\lambda_t^*(1-ik_{12}a_t)^{-1}(3\lambda_t\beta_t + 3\lambda_t\gamma_t) + \lambda_s^*(1-ik_{12}a_s)^{-1}(\lambda_s\beta_s + \lambda_s\gamma_s)\}. \tag{23b}$$

Here a_s and a_t are the singlet and triplet scattering lengths of the nucleons; β_t, γ_t, β_s, and γ_s are defined by

$$\beta_t = \tfrac{1}{2}k_{12}^2a_tr_t(1+ik_{12}a_t)^{-2} - a_t\left(\tfrac{2}{3}b_1 + \tfrac{4}{3}b_3\right)(1+ik_{12}a_t)^{-1}B_t(k_{12}) + \left(\tfrac{1}{3}b_1 + \tfrac{2}{3}b_3\right)[A_t(k_{13}) + A_t(k_{23})], \tag{24a}$$

$$\gamma_t = \left(-\tfrac{1}{3}b_1 + \tfrac{1}{3}b_3\right)[A_t(k_{12}) - A_t(k_{23})], \tag{24b}$$

$$\beta_s = \tfrac{1}{2}k_{12}^2a_sr_s(1+ik_{12}a_s)^{-2} - a_s\left(\tfrac{4}{3}b_1 + \tfrac{2}{3}b_3\right)(1+ik_{12}a_s)^{-1}B_s(k_{12}) + \left(\tfrac{2}{3}b_1 + \tfrac{1}{3}b_2\right)[A_s(k_{13}) + A_s(k_{23})], \tag{24c}$$

$$\gamma_s = \left(-\tfrac{2}{3}b_1 + \tfrac{2}{3}b_3\right)[A_s(k_{12}) - A_s(k_{23})], \tag{24d}$$

where r_t and r_s are the effective radii of the nucleons in the triplet and singlet states, b_1 and b_2 are the scattering lengths of pions on nucleons with isospins ½ and 3/2. These quantities have the values, with the pion mass taken as unity:

$$a_s = -17.0, \quad a_t = 3.8, \quad r_s = 1.8, \quad r_t = 1.1, \quad b_1 = 0.17, \quad b_3 = -0.09. \tag{25}$$

$A_s(k)$ and $B_s(k)$ [or $A_t(k)$ and $B_t(k)$] are obtained from $A(kE)$ and $B(kE)$ by means of the substitution $a = -a_s$ (or $a = -a_t$).

The real and imaginary parts of the complex constants λ_s and λ_t are not independent, because they are related by an expression including the scattering phase angles of the initial particles at the threshold energy.[9] Specifically,

*arc sh = $\sinh^{-1}$.

$$\lambda_s = \rho_s \exp\left(i\,{}^3\delta_0^P\right), \qquad \lambda_t = \rho_t \exp\left(i\,{}^3\delta_1^P\right), \tag{26}$$

where ρ_s and ρ_t are real constants not known in advance; ${}^3\delta_0^P$ and ${}^3\delta_1^P$ are the scattering phase angles of P protons with total angular momentum 0 and 1, having the values [10]

$${}^3\delta_0^P \approx -10°, \qquad {}^3\delta_1^P \approx -28°.$$

When the total kinetic energy of the produced particles is very small (of the order 1—3 MeV in the c.m.s.), all correction terms have a single order of magnitude and are considerably larger than the succeeding correction terms, which are of the order E. At 5—10 MeV in the c.m.s. only the first term in (24a) and the first two terms in (24c) remain relatively large, making an important contribution to the cross sections. The other quantities are small, being of the order E/μ (μ is the pion mass) and it would be meaningless to consider them in the present case.

It would be difficult to indicate in advance the upper limit of the total kinetic energy of produced particles at which the foregoing formulas begin to lose their validity. A good experimental criterion is found in the fact that terms of the order E/μ contribute to the angular distribution of pions (relative to the direction of the initial particles). Therefore the large anisotropy of pions must indicate that the production process proceeds through amplitudes with $L > 0$ and that the equations (24) are no longer valid.

It can be expected that (24) correctly represents the cross sections for $N + N \to N + N + \pi$ processes up to 10 MeV (c.m.s.). However, we at present have no experimental data for such reactions at the given energies.

We can also consider similar reactions such as $N + N \to N + \Lambda + K$, $N + N \to N + \Sigma + K$ etc. The experimental investigation of these reactions should in principle enable us to determine the scattering amplitudes of the produced unstable particles and the effective radii of resonant interacting pairs of particles. For this reason a comparison between experimental data obtained for $N + N \to N + N + \pi$ and the equations (24) acquires additional interest.

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