INTENSITY EFFECTS IN COMPTON SCATTERING

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At very high photon densities processes involving several incident photons become important. Formulas are obtained for the effective cross sections of these processes, which are valid at arbitrary intensities. The Klein-Nishina formula is obtained in the limiting case of small photon density.

Is the Compton effect correctly described by the well known Klein-Nishina formula at extremely large photon densities, such as can be produced with lasers, for example? And under such conditions, how probable are processes in which several photons are absorbed in a single act, with subsequent emission of a harder photon? These questions are especially interesting in connection with recent discussions of a method for producing hard γ rays by the reflection of a laser beam from relativistic electrons.[1-3]

We shall give here a calculation of these processes which is valid at arbitrary photon densities. The characteristic dimensionless and relativistically invariant parameter for a plane wave with photon density N and wavelength l is

$$\xi^2 = 2e^2\hbar l N/m^2c^3.$$

The Klein-Nishina formula turns out to be correct if the condition $\xi \ll 1$ is satisfied. The method of treatment used in what follows is to calculate the emission of a photon by an electron moving in the field of a plane electromagnetic wave. The interaction of the incident photons with the electron is described by using Wolkov's exact solution,[2] and the emission of the photon is treated in first-order perturbation theory.

We write the solution of the Dirac equation in the field of the plane wave in the form

$$\psi = \begin{pmatrix} m + \lambda + \sigma_z \boldsymbol{\sigma\pi} \\ \sigma_z (m - \lambda) + \boldsymbol{\sigma\pi} \end{pmatrix} v \exp \{i [\mathbf{pr} + S (z - t) - \lambda t]\}. \quad (1)$$

Here all vectors lie in the plane normal to the direction of propagation of the wave $\mathbf{A}(z - t)$; v is a two-component spinor; and

$$S(\tau) = \frac{1}{2\lambda} \int^{\tau} (\boldsymbol{\pi}^2 + m^2 - \lambda^2)\, d\tau, \qquad \boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}. \quad (2)$$

The energy ϵ and the momentum p_z of the electron are expressed in terms of λ in the following way:

$$\varepsilon = (\mathbf{p}^2 + m^2 + \lambda^2)/2\lambda, \qquad p_z = (\mathbf{p}^2 + m^2 - \lambda^2)/2\lambda. \quad (3)$$

In order to normalize ψ we shall suppose that A and ψ are periodic in a cube of volume $L^3 = 1$; we then have

$$\mathbf{p} = 2\pi \mathbf{n}/L, \qquad [\mathbf{p}^2 + m^2 (1 + \xi^2) - \lambda^2]/2\lambda = 2\pi n_\tau/L, \quad (4)$$

where $\mathbf{n}$, n_τ are integers, $\xi^2 = e^2 \overline{\mathbf{A}}^2/m^2$, and it is assumed that $\overline{\mathbf{A}} = 0$. It can be verified that the $\psi_{\mathbf{p}\lambda}$ form a complete orthogonal system of functions at any time, and the normalization constant that must be introduced in Eq. (1) is given by $(v^+ v = 1)$

$$c_{\mathbf{p}\lambda} = 2^{-1/2} [\mathbf{p}^2 + m^2 (1 + \xi^2) + \lambda^2]^{-1/2}. \quad (5)$$

The matrix element corresponding to the emission of a photon $(\mathbf{k}, k_z, \omega)$ is

$$(1; \mathbf{p}\lambda | H_{int} | 0; \mathbf{p}'\lambda') = e\sqrt{2\pi/\omega} \exp \{i (\lambda - \lambda' + \omega - k_z)\, t\}\, v^+ M v', \quad (6)$$

$$v^+ M v' = \int \psi^+_{\mathbf{p}\lambda} (\mathbf{e}\boldsymbol{\alpha} + e_z \alpha_z)\, \psi_{\mathbf{p}'\lambda'} \exp \{- i (\mathbf{kr} + k_z \tau)\}\, d\mathbf{r} d\tau, \quad (7)$$

where $(\mathbf{e}, e_z)$ is the polarization vector of the photon; the primed quantities refer to the initial state of the electron, and $\boldsymbol{\alpha}$, α_z are Dirac matrices.

In the formula for the probability per unit time for emission of a photon

$$dw = 2\pi | (1\mathbf{p}\lambda | H_{int} | 0\mathbf{p}'\lambda') |^2 \frac{\omega^2 do}{(2\pi)^3} \frac{d\omega}{d (\omega - k_z + \lambda)} \quad (8)$$

the last factor is somewhat unusual (cf. the time dependence of the matrix element). This is due to the fact that the unperturbed problem—an electron in the field of a plane wave—is not stationary.

Integration over $\mathbf{r}$ in Eq. (7) gives a twodimensional δ function, so that the matrix M defined in Eq. (7) is of the form

$$M = 2cc'\delta_{\mathbf{p}',\,\mathbf{p}+\mathbf{k}} \int (b_0 + i\sigma_i b_i) \exp\{-i(S - S' + k_z\tau)\}\, d\tau; \quad (9)$$

$$b_0 = \mathbf{e}\,(\boldsymbol{\pi}\lambda' + \boldsymbol{\pi}'\lambda) + e_z\,(m^2 - \lambda'\lambda + \boldsymbol{\pi}'\boldsymbol{\pi}),$$
$$b_1 = e_y m\,(\lambda - \lambda') - e_z m\,(\pi'_y - \pi_y),$$
$$b_2 = - e_x m\,(\lambda - \lambda') + e_z m\,(\pi'_x - \pi_x),$$
$$b_3 = e_x\,(\lambda\pi'_y - \lambda'\pi_y) - e_y\,(\lambda\pi'_x - \lambda'\pi_x) + e_z\,(\pi_x\pi'_y - \pi'_x\pi_y). \quad (10)$$

The coefficient of the exponential in the integrand in Eq. (7), regarded as a function of τ, is a linear combination of **A** and $\mathbf{A}^2$. We introduce the following notation for the integrals involved:

$$[I, F, G, H] = \int_0^L \exp\{-i(S - S' + k_z\tau)\} \times \left[1,\ -\frac{e}{m}A_x,\ -\frac{e}{m}A_y,\ \frac{e^2}{m^2}(\mathbf{A}^2 - \overline{\mathbf{A}^2})\right] d\tau. \quad (11)$$

We now stipulate that the electron's initial motion was along the z axis and that the momentum of the emitted photon is in the xz plane:

$$\mathbf{p}' = 0, \qquad -p_x = k_x = \omega\sin\theta, \qquad k_z = \omega\cos\theta,$$
$$\lambda' = \varepsilon' - p_z'. \quad (12)$$

We note, moreover, that from Eq. (6) we have the "conservation law"

$$\lambda = \lambda' - \omega\,(1 - \cos\theta). \quad (13)$$

Integration over τ in Eq. (9) gives

$$M = 2cc'\delta_{\mathbf{p}',\mathbf{p}+\mathbf{k}}\,(B_0 + iB_i\sigma_i), \quad (14)$$

with

$$B_0 = - e_x\,[\lambda' p_x I + m\,(\lambda + \lambda')\,F] + e_y m\,(\lambda + \lambda')\,G + e_z\,\{[m^2(1 + \xi^2) - \lambda\lambda']\,I + m p_x F + m^2 H\},$$
$$B_1 = e_y m\,(\lambda - \lambda')\,I,$$
$$B_2 = - e_x m\,(\lambda - \lambda')\,I - e_z m p_x I,$$
$$B_3 = e_x m\,(\lambda - \lambda')\,G + e_y\,[\lambda' p_x I - m\,(\lambda - \lambda')\,F] + e_z m p_x G. \quad (15)$$

We sum over the final and average over the initial states of polarization of the electrons:

$$\frac{1}{2}\sum_{\alpha,\beta} |v_\alpha^+ M v'_\beta|^2 = \frac{1}{2}\,\mathrm{Sp}\,M^+M = 4c^2c'^2\delta_{\mathbf{p}',\mathbf{p}+\mathbf{k}}\sum_{i=0}^{3}|B_i|^2 \quad (16)$$

and consider the polarization states of the emitted photon. For the first state we take $e_x^{(1)} = e_z^{(1)} = 0$, $e_y^{(1)} = 1$, and for the state orthogonal to this $e_x^{(2)} = \cos\theta$, $e_y^{(2)} = 0$, $e_z^{(2)} = -\sin\theta$. Using Eqs. (12) and (13), we find for the first polarization of the photon

$$\sum |B_i^{(1)}|^2 = m^2(\lambda' - \lambda)^2\,(|I|^2 + |\rho I - F|^2) + m^2(\lambda' + \lambda)^2\,|G|^2, \quad (17)$$

where $\rho = (\lambda'/m)\cot(\theta/2)$.

Similarly, for the second state of polarization

$$\sum_i |B_i^{(2)}|^2 = m^2(\lambda' - \lambda)\,(|I|^2 + |G|^2) + m^2|\,(\lambda' + \lambda)\,(\rho I - F). \quad (18)$$

We now make use of the fact that the incident electromagnetic wave **A** is monochromatic with the frequency ω'. When τ is increased by $2\pi/\omega'$ the argument $S - S' + \omega$ of the exponential in Eq. (11) increases by the amount

$$\Delta = \pi\,[\mathbf{p}^2 + m^2(1 + \xi^2) - \lambda^2]/\lambda\omega' - \pi\,[m^2(1 + \xi^2) - \lambda'^2]/\lambda'\omega' + 2\pi\omega\cos\theta/\omega', \quad (19)$$

and since the coefficient of the exponential in Eq. (11) is periodic in τ with the period $2\pi/\omega'$ the increment (19) must be equal to $2\pi n$.[1)]

We solve the resulting equation for ω, eliminating λ and **p** by means of Eqs. (12) and (13):

$$\omega = \frac{2n\lambda'\omega'}{\lambda'(1 + \cos\theta) + [2n\omega' + m^2(1 + \xi^2)/\lambda'](1 - \cos\theta)}. \quad (20)$$

This expression for the frequency of the emitted photon in terms of the angle θ and the harmonic number n = 1, 2, ... goes over into the Compton formula if we set n = 1 and $\xi \ll 1$. (For an electron initially at rest we set $\lambda' = m$.) It can be seen from Eq. (20) that ω increases with increasing n, but does not exceed the value ($n \to \infty$)

$$\omega_{max} = \lambda'/(1 - \cos\theta). \quad (21)$$

We write the expression for the field of an elliptically polarized monochromatic electromagnetic wave in the form

$$-eA_x = \sqrt{2}\xi m\sin\alpha\cos\omega'\tau,$$
$$-eA_y = \sqrt{2}\xi m\cos\alpha\cos(\omega'\tau + \beta). \quad (22)$$

The angles α, β are connected with the Stokes parameters ξ_i by the following relations:

$$\xi_1 = \sin 2\alpha\cos\beta, \qquad \xi_2 = \sin 2\alpha\sin\beta, \quad \xi_3 = \cos 2\alpha, \quad (23)$$

[1)]The contribution to the integral (11) from the m-th period is proportional to $e^{-im\Delta}$, so that the square of the absolute value of the sum over all the periods contains the factor

$$\frac{\sin^2(f\Delta/2)}{\sin^2(\Delta/2)} \approx 2\pi f \sum_{n=-\infty}^{\infty} \delta(\Delta - 2\pi n),$$

where $f \gg 1$ is the number of periods in the length L.

and the parameter ξ_2 characterizes the degree of circular polarization, ξ_3 the linear polarization with the electric vector perpendicular to the plane of the scattering (along the y axis), and ξ_1 that with the vector at angle $\pi/4$ with this plane. For a circularly polarized wave we must set $\alpha = \pi/4$ and $\beta = \pi/2$ or $3\pi/2$. A wave polarized linearly at angle α with the y axis is obtained for $\beta = 0$.

We carry out the integration in Eq. (2):

$$S(\tau) = [p_x^2 + m^2(1+\xi^2) - \lambda^2]\frac{\tau}{2\lambda} + \frac{\sqrt{2}\,\xi p_x m \sin\alpha}{\lambda\omega'}\sin\omega'\tau + \frac{(\sin^2\alpha + \cos^2\alpha\cos 2\beta)\,\xi^2 m^2}{4\lambda\omega'}\sin 2\omega'\tau + \frac{\cos^2\alpha\sin 2\beta\,\xi^2 m^2}{4\lambda\omega'}\cos 2\omega'\tau. \quad (24)$$

The expression (11) for the integral I takes the form

$$I_n(s, q, r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\{-i(n - s\sin x + q\sin 2x + r\cos 2x)\}\,dx; \quad (25)$$

$$s = \frac{2\sqrt{2}\,n\rho\xi\sin\alpha}{1+\xi^2+\rho^2}, \quad q = \frac{n\xi^2(\sin^2\alpha + \cos^2\alpha\cos 2\beta)}{2(1+\xi^2+\rho^2)}, \quad r = \frac{n\xi^2\cos^2\alpha\sin 2\beta}{2(1+\xi^2+\rho^2)}. \quad (26)$$

The remaining integrals (11) can be expressed in terms of the I_n in the following way:

$$F_n = 2^{-1/2}\,\xi\sin\alpha\,(I_{n-1} + I_{n+1}),$$

$$G_n = 2^{-1/2}\,\xi\cos\alpha\,(e^{i\beta}I_{n-1} + e^{-i\beta}I_{n+1}),$$

$$H_n = \tfrac{1}{2}\xi^2(\sin^2\alpha + \cos^2\alpha\, e^{2i\beta})\,I_{n-2} + \tfrac{1}{2}\xi^2(\sin^2\alpha + \cos^2\alpha\, e^{-2i\beta})\,I_{n+2}. \quad (27)$$

We here consider the case of a circularly polarized wave ($\alpha = \pi/4$, $\beta = \pi/2$), which leads to closed formulas. For this case

$$q = r = 0, \quad s = 2n\rho\xi/(1+\xi^2+\rho^2),$$

and I_n and the other integrals can be expressed in terms of Bessel functions:

$$I_n(s, 0, 0) = J_n(s), \quad F_n = [(1+\xi^2+\rho^2)/2\rho]\,J_n(s), \quad G_n = i\xi J_n'(s), \quad H_n = 0. \quad (28)$$

Substitution of these values in Eqs. (17) and (18) gives

$$\sum_{\alpha, i}|B_i^{(\alpha)}|^2 = 2m^2\xi^2(\lambda^2 + \lambda'^2)\,[J_n'^2 + (n^2/s^2 - 1)\,J_n^2] - 4m^2\lambda\lambda' J_n^2. \quad (29)$$

To determine the probability we have still to calculate the derivative $d\lambda/d\omega$ which occurs in Eq. (8). From the condition that the increment of $S' - S + \omega\tau\cos\theta$ in the length L is constant we find

$$\frac{d\lambda}{d\omega} = \frac{2\lambda(\omega\sin^2\theta + \lambda\cos\theta)}{\omega^2\sin^2\theta + \lambda^2 + m^2(1+\xi^2)}. \quad (30)$$

Collecting the other factors, we find the differential probability

$$dw = \frac{e^2(m^2p^2 + \lambda'^2)\,|B_i|^2\,\omega\,do}{4\pi m^2\lambda'^2\,[\lambda'^2 + m^2(1+\xi^2)]\,(1+\xi^2+\rho^2)}. \quad (31)$$

To obtain the effective cross section we divide dw by the flux density

$$N(1 - v') = \xi^2 m^2\omega'(1 - v')/4\pi e^2, \quad (32)$$

where $v' = v_z'$ is the initial speed of the electron. Using Eq. (29), we finally find for circular polarization of the incident photons the result

$$\frac{d\sigma}{do} = \frac{2r_0^2(\omega/n\omega')^2}{(1-v')\,[\lambda'^2 + m^2(1+\xi^2)]} \times \left\{\left(\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda}\right)\left[J_n'^2(s) + \left(\frac{n^2}{s^2} - 1\right)J_n^2(s)\right] - 2\xi^{-2}J_n^2(s)\right\},$$

$$r_0 = e^2/mc^2, \quad \lambda' = m\sqrt{(1-v)/(1+v')}. \quad (33)$$

An analysis of this formula shows that for $\xi \gg 1$ the most probable number of quanta absorbed in one act is $n \sim \xi^3$, and the angle θ_0 of the scattered photon is given by the condition $\cot(\theta_0/2) \approx m(1+\xi^2)^{1/2}/\lambda'$. For $\xi \ll 1$ the main term in the cross section for absorption of n quanta with the emission of one quantum of higher energy is of the form

$$\frac{d\sigma}{do} = \frac{r_0^2}{2}\left(\frac{\omega}{n\omega'}\right)^2\left(\frac{\omega}{n\omega'} + \frac{n\omega'}{\omega} - \sin^2\theta\right)\Phi_n, \quad (34)$$

where

$$\Phi_n = \frac{n^{2n+1}}{n!^2}\left(\frac{\xi\sin\theta}{2}\right)^{2n-2}, \quad (35)$$

and it is assumed that the electron was initially at rest. The exact connection between ω and ω' is given by the relation

$$\omega = \frac{n\omega'}{1 + [n\omega'/m + \xi^2/2]\,(1 - \cos\theta)}. \quad (36)$$

For the first harmonic one gets the Klein-Nishina formula. For n = 2 we have $\Phi_2 = 2\xi^2 \times \sin^2\theta$, and integration over the angles gives for the total cross section for absorption of two photons with emission of one harder photon the result:

$$\sigma_2 = 4\pi r_0^2\xi\left[\frac{(1+\gamma)(2+4\gamma-\gamma^2)}{\gamma^5}\ln(1+2\gamma) + \frac{2}{1+2\gamma} - \frac{2(\sigma+12\gamma-\gamma^2)}{3\gamma^4}\right], \quad \gamma = 2\omega'/m. \quad (37)$$

For the most powerful laser devices developed at present we have $\xi \sim 10^{-4}$ and $\sigma_2 \sim 10^{-23}$ cm^2. A further increase in ξ can be achieved by focusing the laser beam, which will lead to increasing ξ by three orders of magnitude.

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