SOVIET PHYSICS JETP VOLUME 19, NUMBER 4 OCTOBER, 1964

# *ON THE LOWER CRITICAL FIELD OF THIN LAYERS OF SUPERCONDUCTORS OF THE SECOND GROUP*

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Submitted to JETP editor November 13, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 1464-1469 (April, 1964)

The values of the external field, at which quantized magnetic flux filaments first appear in thin layers of superconductors of the second group, are found. It is shown that at this value the curve $M(H_0)$ exhibits a break and possesses a vertical tangent.

As is well known (see [1,2]) thin superconducting films which are condensed on a substrate kept at helium temperature and not subject to annealing, display a behavior which is characteristic of superconductors of the second group. The author has calculated earlier [3] the dependence of the critical field of such films on the film thickness. Khukhareva [2] measured the critical field of films simultaneously with measuring their electric conductivity in the normal state. This makes it possible to determine the depth of penetration $\delta(T)$ for such films by means of a formula derived by the author with Gor'kov [4], and then compare the experimental data with the results obtained in [3] without refitting any parameters whatever. The agreement between theory and experiment turns out to be fully satisfactory (see [2]).

It was shown earlier [5] that the transition of a bulky superconductor of the second group from the superconducting state into the normal state occurs not abruptly but gradually, stretching over an entire interval of fields. In this interval there exists the so-called mixed state, in which the superconductor is gradually filled with quantized filaments of magnetic flux. The value of the lower critical field $H_{c1}$ and the form of the $B(H_0)$ curve in the mixed-state region, determined in [5], have been experimentally confirmed many times (see [6,7]).

It is quite obvious that the same character of transition is retained also for layers of superconductors of the second group which are not too thin. In the present paper we shall find the field $H_{c1}(d)$ corresponding to the field $H_{c1}$ for a bulky specimen. At this value of the external field there penetrate into a film of thickness d, at first, quantized filaments of flux. We shall consider also how the penetration of the filaments affects the dependence of the magnetic moment of the film on the field.

As in [5], we confine ourselves to the case $T_c - T \ll T_c$, and employ the equations of Ginzburg and Landau [8]. In the case when $\kappa \gg 1$ and the distance between the vortices is large, as shown in [5], the equation for the magnetic field can be written in the form [1)]

$$\Delta H - H = -\frac{2\pi}{\varkappa}\sum_i \delta(\mathbf{r}-\mathbf{r}_i), \tag{1}$$

where $\mathbf{r}_i$ —coordinates of the centers of the filaments (we use the same units for H and $\mathbf{r}$ as in [5]). We assume that the film is perpendicular to the x axis and occupies a region of space x = −d/2 to d/2, and that the magnetic field is directed along the z axis. The plate thickness d is assumed large compared with $1/\kappa$.

In the absence of filaments, the solution of equation (1) is of the form

$$H^{(0)} = H_0\frac{\operatorname{ch} x}{\operatorname{ch}(d/2)}, \qquad (2)^*$$

where $H_0$ is the external field. We now assume that flux filaments have penetrated into the film. From symmetry considerations it follows that when the distance between filaments is large,

1) Actually, an equation of this type in the case of $\kappa \gg 1$ is valid for all temperatures. It is more conveniently written in the usual units

$$\Delta H - \frac{1}{\delta^2}H = -\frac{\Phi_0}{\delta^2}\sum_i \delta(\mathbf{r}-\mathbf{r}_i),$$

where $\Phi_0 = \pi ch/e$ is the magnetic-flux quantum and $\delta$ is the temperature-dependent depth of penetration. The value of the critical field obtained in this manner is valid with logarithmic accuracy, that is, when $\ln\kappa \gg 1$ (we recall that $\kappa$ changes little with temperature).

*ch = cosh.

their centers will be located in the plane x = 0 and will be periodic in y, that is, y = ma, where m is an integer and a is the period, which we shall assume to be large compared with d. In view of the linearity of (1), the solution has the form of the sum of the field $H^{(0)}$ and the fields of the individual filaments, obtained under the condition H = 0 on the boundaries.

Let us find the field of an individual filament with center coordinates x = 0 and y = ma. To this end we expand the field H in a Fourier integral with respect to the coordinate y:

$$H_m^{\text{H}} = \int_{-\infty}^{\infty} H_k(x)\, e^{ik(y-ma)} \frac{dk}{2\pi}. \quad (3)$$

From (1) we obtain

$$\frac{d^2 H_k}{dx^2} - (k^2+1) H_k = -\frac{2\pi}{\varkappa} \delta(x). \quad (4)$$

The solution of this equation, which vanishes at x = ±d/2, is of the form

$$H_k(x) = \frac{\pi}{\varkappa} \frac{\operatorname{ch}[\sqrt{k^2+1}(d-|x|)] - \operatorname{ch}[\sqrt{k^2+1}\,x]}{\sqrt{k^2+1}\operatorname{sh}[\sqrt{k^2+1}\,d]}. \quad (5)^*$$

The over-all magnetic field is

$$H = H^{(0)} + \sum_m H_m^{\text{H}}. \quad (6)$$

We now calculate the energy. Inasmuch as the distances between the filaments themselves and the filaments and the boundaries are much larger than $1/\kappa$, vicinities of order $1/\kappa$ about the centers make the same contribution to the energy as in the case of isolated filaments. In view of this, it is convenient to separate that part of the energy which corresponds to the isolated filaments. In full agreement with [5], we obtain for the energy per unit volume

$$F_B = \frac{\varepsilon(\infty)}{ad} + \frac{1}{d} \lim_{Y\to\infty} \left\{ \frac{1}{Y} \int_{-Y/2}^{Y/2} dy \int_{-d/2}^{d/2} dx\, [H_d^2 + (\nabla H_d)^2] - \frac{1}{Y} \int_{-Y/2}^{Y/2} dy \int_{-\infty}^{\infty} dx\, [H_\infty^2 + (\nabla H_\infty)^2] \right\}, \quad (7)$$

where $H_d$ and $H_\infty$ are respectively the fields in a plate of thickness d and in infinite space as $a \to \infty$ and $\epsilon(\infty)$ is the energy per unit length of the isolated filament. Substituting (6) and integrating by parts, we get

$$F_B = \frac{\varepsilon(\infty)}{ad} + \frac{H_0}{d} \left\{ \left[ \frac{\partial H^{(0)}}{\partial x} + \frac{1}{a} \int_{-\infty}^{\infty} dy \frac{\partial H_{m=0,d}^{\text{H}}}{\partial x} \right]_{x=d/2} - \left[ \frac{\partial H^{(0)}}{\partial x} + \frac{1}{a} \int_{-\infty}^{\infty} dy \frac{\partial H_{m=0}^{\text{H}}}{\partial x} \right]_{x=-d/2} \right\}$$

$$+ \frac{2\pi}{\varkappa d} \left\{ H^{(0)}(0) + \frac{1}{a} [H_{m=0,d}^{\text{H}} - H_{m=0,\infty}^{\text{H}}]_{x=y=0} + \frac{1}{a} \sum_{m\neq 0} [H_{m=0,d}^{\text{H}}]_{x=0,y=ma} \right\}. \quad (8)$$

Substituting (2), (3), and (5) in (8) we get

$$F_B = \frac{2H_0^2}{d} \operatorname{th}\frac{d}{2} + \frac{\varepsilon(\infty)}{ad} - \frac{2\pi}{\varkappa^2 ad} \int_0^{\infty} \left(1 - \operatorname{th}\left(\sqrt{k^2+1}\,\frac{d}{2}\right)\right) \frac{dk}{\sqrt{k^2+1}} + \frac{\pi}{\varkappa^2 ad} \int_{-\infty}^{\infty} \operatorname{th}\left(\sqrt{k^2+1}\,\frac{d}{2}\right) \sum_{m\neq 0} e^{ikma} \frac{dk}{\sqrt{k^2+1}}. \quad (9)^*$$

The first term of this expression corresponds to the energy in the absence of the filaments, the next two terms represent the energy of the non-interacting filaments in the plate, and the last term describes the interaction between the filaments. When $a \gg d$ this term decreases exponentially.

It will be convenient in what follows to change over to the energy in a specified field $H_0$. Its value is $F_H = F_B - 2BH_0$, where B is the induction. The average value of the field (6) is obtained in the form

$$B = \bar{H} = \frac{2H_0}{d} \operatorname{th}\frac{d}{2} + \frac{2\pi}{\varkappa ad}\left(1 - \operatorname{ch}^{-1}\frac{d}{2}\right). \quad (10)$$

It follows therefore that

$$F_H = -\frac{2H_0^2}{d} \operatorname{th}\frac{d}{2} - \frac{4\pi H_0}{\varkappa ad}\left(1 - \operatorname{ch}^{-1}\frac{d}{2}\right) + \frac{\varepsilon(\infty)}{ad} - \frac{2\pi}{\varkappa^2 ad} \int_0^{\infty} \left[1 - \operatorname{th}\left(\frac{d}{2}\sqrt{k^2+1}\right)\right] \frac{dk}{\sqrt{k^2+1}} + \frac{\pi}{\varkappa^2 ad} \int_{-\infty}^{\infty} \operatorname{th}\left(\frac{d}{2}\sqrt{k^2+1}\right) \sum_{m\neq 0} e^{ikma} \frac{dk}{\sqrt{k^2+1}}. \quad (11)$$

To determine the first critical field we put $a \to \infty$. The interaction between the filaments [that is, the last term in (11)] can then be neglected. The addition to the energy due to the isolated filaments is proportional to 1/a. Equating to zero the sum of the terms proportional to 1/a, we obtain the value of the field $H_{c1}$, starting with which the appearance of the filaments becomes energetically convenient. It is found to be [2)]

*sh = sinh.

*th = tanh.

[2)]As was already noted earlier (footnote 1) this formula, together with all the following ones, is valid in the case $\kappa \gg 1$ with logarithmic accuracy at all temperatures. In order to have an expression in ordinary units, it is necessary to replace in the formulas the common coefficient $1/2\kappa$ by $c\hbar/4e\delta^2(T)$, d by $d/\delta(T)$, and leave $\kappa$ under the logarithm sign the same as before. In the case of thin films, the condition for the applicability of the formula will be $\ln[\kappa d/\delta(T)] \gg 1$.

$$H_{c1}(d) = \left\{ H_{c1}(\infty) - \frac{1}{2\varkappa} \int_0^\infty \left[ 1 - \operatorname{th}\left( \frac{d}{2} \sqrt{k^2+1} \right) \right] \frac{dk}{\sqrt{k^2+1}} \right\} \Big/ \left( 1 - \operatorname{ch}^{-1} \frac{d}{2} \right). \tag{12}$$

In this expression the integral cannot be calculated exactly. It can be represented, however, in the form of infinite series:

$$H_{c1}(d) = \left\{ H_{c1}(\infty) - \frac{1}{\varkappa} \sum_{n=1}^{\infty} (-1)^{n+1} K_0(nd) \right\} \times \left( 1 - \operatorname{ch}^{-1} \frac{d}{2} \right)^{-1}, \tag{12'}$$

$$H_{c1}(d) = \frac{1}{2\varkappa} \left\{ \ln \frac{\gamma \varkappa d}{4\pi} + 0.081 + \sum_{n=1}^{\infty} \left[ \left[ \left( \frac{d}{2\pi} \right)^2 + \left( n - \frac{1}{2} \right)^2 \right]^{-1/2} - \frac{1}{n} \right] \right\} \left( 1 - \operatorname{ch}^{-1} \frac{d}{2} \right)^{-1}, \tag{12''}$$

where $\gamma = e^C = 1.78$; $K_0$ is the Hankel function of imaginary argument; we have put in accordance with [5] $H_{c1}(\infty) = (\ln \kappa + 0.081)/2\kappa$.

Formulas (12′) and (12″) enable us to write down immediately the limiting relations

$$H_{c1}(d) = H_{c1}(\infty)\,(1 + 2e^{-d/2}), \qquad d \gg 1;$$

$$H_{c1}(d) = \frac{4}{\varkappa d^2} \left( \ln \frac{\gamma \varkappa d}{\pi} + 0.081 \right), \qquad d \ll 1. \tag{13}$$

An interesting result is that although the magnetic field of each filament attenuates at a distance on the order of unity ($\delta$ in the ordinary units), in the case of $\kappa \gg 1$ the filaments penetrate into a plate of thickness $d \ll 1$. According to (13), with increasing plate thickness the function $H_{c1}(d)$ increases essentially in proportion to $1/d^2$. At the same time, according to [3], when $d \gg 1/\kappa$, $H_{c2}(d)$ is practically independent of the thickness and is equal to $\kappa$. Both fields become comparable, as expected, at $d \sim 1/\kappa$. This means that the filaments cannot penetrate in thinner plates, and the transition from the superconducting state into the normal state proceeds like a second-order phase transition without formation of a mixed phase.

We now consider the question of the manifestation of the field $H_{c1}(d)$ in experiments. To this end we find the form of the magnetic moment curve $M(H_0)$ in the vicinity of $H_0 = H_{c1}(d)$. It is first necessary to calculate the dependence of the distance between the filaments $a$ on the magnetic field $H_0$. This can be done with the aid of the relation

$$-{}^1\!/_2 \partial F_H / \partial H_0 = B. \tag{14}$$

Since an appreciable role is played in this case by the interaction between the filaments, we consider the last term in (11). Going over to a contour integral with respect to k, we obtain in place of the integral the sum of the series

$$\frac{8\pi^2}{\varkappa^2 a d^2} \sum_{n=1}^{\infty} \frac{\exp\left[ -a \sqrt{[\pi(2n-1)/d]^2 + 1} \right]}{\sqrt{[\pi(2n-1)/d]^2 + 1}}. \tag{15}$$

Inasmuch as the distance between the filaments is large in the vicinity of $H_{c1}(d)$, it is sufficient to take the limiting expression for large $a$. It is easy to see that in the cases $a \gg d^2 \gg 1$ and $a \gg d$, $d \ll 1$, the most important is the first term in the sum (15). In the case when $d^2 \gg a \gg d \gg 1$ the terms of the sum differ little, so that this sum can be replaced by an integral. As a result we get

$$\frac{8\pi^2}{\varkappa^2 a d^2} \frac{\exp\left[ -a \sqrt{\pi^2/d^2 + 1} \right]}{\sqrt{\pi^2/d^2 + 1}}, \qquad a \gg \max(d, d^2); \tag{16'}$$

$$\frac{(2\pi)^{3/2}}{\varkappa^2 a^{3/2} d} e^{-a}. \qquad d^2 \gg a \gg d \gg 1. \tag{16''}$$

We now replace the last term in (11) by (16′) or (16″), and substitute $F_H$ and B (10) in (14). In the differentiation of $F_H$ we shall assume that $a$ is a function of $H_0$. As a result we obtain an equation for the determination of $a$:

$$H_0 - H_{c1}(d) = \frac{2\pi a \exp\left[ -a \sqrt{\pi^2/d^2 + 1} \right]}{\varkappa d \left[ 1 - \operatorname{ch}^{-1}(d/2) \right]}, \qquad a \gg \max(d, d^2); \tag{17'}$$

$$H_0 - H_{c1}(d) = \frac{1}{\varkappa} \sqrt{\frac{\pi a}{2}}\, e^{-a}, \qquad d^2 \gg a \gg d \gg 1. \tag{17''}$$

Expressing $a$ in terms of B with the aid of (10), we obtain in implicit form the dependence $B(H_0)$ in the vicinity $a \gg d$, that is, $0 < H_0 - H_{c1}(d) \ll H_{c1}(d)$:

$$H_0 - H_{c1}(d) = \frac{4\pi^2}{(\varkappa d)^2} \left( B - \frac{2H_0}{d} \operatorname{th} \frac{d}{2} \right)^{-1} \times \exp\left\{ -\frac{2\pi}{\varkappa d} \left( 1 - \operatorname{ch}^{-1} \frac{d}{2} \right) \sqrt{\frac{\pi^2}{d^2} + 1} \left( B - \frac{2H_0}{d} \operatorname{th} \frac{d}{2} \right)^{-1} \right\}$$

$$\text{for } a \gg \max(d, d^2); \tag{18'}$$

$$H_0 - H_{c1}(d) = \frac{\pi}{\varkappa^{3/2} d^{1/2}} \left( B - \frac{2H_0}{d} \right)^{-1} \exp\left[ -\frac{2\pi}{\varkappa d} \Big/ \left( B - \frac{2H_0}{d} \right) \right]$$

$$\text{for } d^2 \gg a \gg d \gg 1. \tag{18''}$$

It follows from (18′) that as $H_0 \to H_{c1}(d) + 0$

$$B \to \frac{2H_0}{d} \operatorname{th} \frac{d}{2} + 0, \qquad \frac{dH_0}{dB} \to +0.$$

All this indicates that when $H_0 = H_{c1}(d)$ the linearity of the magnetic moment $M(H_0)$, described by

$$4\pi M = H_0\left(\frac{2}{d}\,\mathrm{th}\,\frac{d}{2} - 1\right),$$

is violated. The curve has a break at the point $H_{c1}(d)$, the magnetic moment begins to decrease in absolute magnitude, and, the curve $M(H_0)$ has a vertical tangent on the side of the fields larger than $H_{c1}(d)$. Such a behavior of the $M(H_0)$ curve at the point $H_{c1}(d)$ is perfectly analogous to the case of a bulky superconductor and makes it possible to identify the field $H_{c1}(d)$ on the experimental curves.

It must be stated that the scheme described makes it possible in principle to determine the dependence $M(H_0)$ not only for $H_0$ close to $H_{c1}$, but also for the entire region of fields where the distance between filaments exceeds $1/\kappa$. Actually, however, this problem is made very difficult by the fact that when $a \sim d$ a great variety of arrangements of filaments is possible (bent surfaces, lattices, etc.). Each of these structures leads to cumbersome formulas, making the selection of a more convenient structure very difficult.

In the comparison of the formula for $H_{c1}(d)$ with the experimental data, it is necessary to bear in mind that their applicability is limited by the condition $\kappa \gg 1$. Inasmuch as experimentally, as a rule, there are observed not too large values of $\kappa$, for a better comparison it is sensible to carry out interpolations similar to that which was made for $H_{c1}$ in the bulky superconductor by Goodman[7]. In view of the complexity of the formula, in the case of arbitrary thickness we shall carry out such an interpolation only for the limiting cases $d \gg 1$ and $d \ll 1$.

We must start from the fact that when $\kappa \to 1/\sqrt{2}$ we should obtain a superconductor of the first group, in which the fields $H_{c1}(d)$ and $H_{c2}(d)$ coincide. According to [3] we have for $\kappa = 1/\sqrt{2}$

$$H_{c2}(d) = 1/\sqrt{2} + \sqrt{2/\pi}\, d e^{-d^2/8}, \qquad d \gg 1; \qquad (19')$$

$$H_{c2}(d) = 2\sqrt{3}/d, \qquad d \ll 1. \qquad (19'')$$

By way of interpolation between formulas (13) and (19) we can use the formulas

$$H_{c1}(d) = \frac{1}{2\kappa}\,[\ln(\kappa + 1.8) + 0.081] \times\left[1 + 2\left(1 - \frac{1}{\sqrt{2}\kappa}\right)e^{-d/2} + \sqrt{\frac{2}{\pi}}\, d e^{-d^2/8}\right], \qquad d \gg 1; \qquad (20')$$

$$H_{c1}(d) = \frac{4}{\kappa d^2}\,[\ln(0.62\kappa d + 1) + 0.18 d], \qquad d \ll 1. \qquad (20'')$$

Unfortunately, no measurements of the moment have been made as yet for thin low-temperature films. This makes it impossible to compare the obtained formulas with experiment.


[1] N. V. Zavaritskii, DAN SSSR **82**, 229 (1952) and **86**, 501 (1952). W. Buckel and K. Hilsch, Z. Phys. **138**, 109 (1954). L. A. Prozorova, JETP **34**, 14 (1958), Soviet Phys. JETP **7**, 9 (1958).

[2] N. S. Khukhareva, JETP **41**, 728 (1961), Soviet Phys. JETP **14**, 526 (1962).

[3] A. A. Abrikosov, DAN SSSR **86**, 489 (1952).

[4] A. A. Abrikosov and L. P. Gor'kov, JETP **36**, 319 (1959), Soviet Phys. JETP **9**, 220 (1959).

[5] A. A. Abrikosov, JETP **32**, 1442 (1957), Soviet Phys. JETP **5**, 1174 (1957).

[6] T. G. Berlincourt and R. R. Hake, Phys. Rev. **131**, 140 (1963). Kinsel, Lynton, and Serin, Phys. Lett. **3**, 30 (1962).

[7] B. B. Goodman, I.B.M. Journal, **6**, 63 (1962).

[8] V. L. Ginzburg and L. D. Landau, JETP **20**, 1064 (1950).


Translated by J. G. Adashko
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