COMPLEX ASYMPTOTIC EXPRESSIONS FOR THE AMPLITUDES OF INELASTIC PROCESSES AND SINGULARITIES IN THE ANGULAR MOMENTUM PLANE

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The total cross sections for the production of n particles (or of n groups of particles with low energies in the c.m.s. of each group) and the energy distribution of the particles in high energy inelastic collisions are calculated by the technique of momentum integration developed previously.[1,2] The asymptotic amplitudes are described by the values obtained previously for the most important cases of "truly inelastic" collisions and corresponding to the contribution of an isolated vacuum Regge pole. In order to avoid unnecessary complications, it is assumed throughout that all particles are identical and have no spin or isospin. It is shown that for any inelastic process there exist very definite configurations of momenta of the produced particles which make the most important contribution to the amplitude. The distributions of the produced particles with respect to the logarithms of their momenta are found. The shapes of these distributions depend significantly on the behavior of the vertex functions (corresponding to the emission of a particle or a group of particles by a reggeon). It is shown that a contradiction arises in the unitarity condition in the s channel for the amplitude of elastic scattering at zero angle if these vertex functions do not decrease with the decrease of the squares of the reggeon momenta. The dependence of both sides of the unitarity condition in the s channel for the amplitude of elastic scattering into nonzero angles on the magnitude of the momentum transfer κ^2 is investigated. It is shown that the asymptotic Regge amplitude corresponding to the contribution of the vacuum pole is not reproduced by the right-hand side of the unitarity condition when all terms corresponding to the formation of an arbitrary number of particles are taken into account. The dependence on κ^2 can be reproduced only if all amplitudes—elastic as well as inelastic—are described by asymptotic expressions corresponding to the contribution not of an isolated pole but a whole set of singularities of the branch point type in the j plane located to the right of the vacuum pole and condensing at the point $j = 1$.

In previous papers,[1,2] the asymptotic expressions for the amplitudes of truly inelastic processes[3-5] were used in the analysis of the simplest reactions: the formation of three, four, or five particles. It will be shown below that the method of momentum integration developed in these papers can be used to calculate the asymptotic values of the total cross sections for the formation of an arbitrary number of particles.

Summing these cross sections, one obtains the total cross section for the interaction of the primary particles a and b for $s = s_{ab} \to \infty$. According to the optical theorem (or the unitarity condition in the s channel for the forward elastic scattering amplitude), the total cross section must be equal to the imaginary part of the elastic scattering amplitude divided by s. As is well known, the elastic scattering cross section has a constant value independent of s if only one vacuum pole in the j plane is taken into account. However, it will be shown below that one obtains for the sum of the cross sections for all processes a value which increases asymptotically as $s \to \infty$. This contradiction with the unitarity condition in the s channel for the elastic forward scattering amplitude can possibly be removed by assuming that all vertex parts corresponding to the emission of particles by the "reggeon," $\gamma(t_i, t_k)$, vanish for $t_i = t_k = 0$.

Possessing a system of amplitudes for all inelastic processes, we can ask a wider question: do these amplitudes satisfy the general unitarity condition in the s channel (not only for the elastic scattering amplitude and not only for vanishing angle); does the right-hand side of the unitarity condition reproduce the dependence of the imaginary parts of the amplitudes on the energy and momentum transfers? The present paper contains only a preliminary study of this problem. It is shown, in particular, that this dependence can in principle only be reproduced if the singularity farthest to the right in the j plane is a condensation of branch points near $j = 1$. Here the position of the q-th branch point ($q = 2, 3, 4, \ldots$) at small values of the momentum transfer $\kappa^2 = -t$ must be given by the relation[6,7]

$$j_q(t) = 1 - j_0''\varkappa^2/q, \tag{I}$$

where $j_0(t) = 1 - j_0'\kappa^2$ is the position of the vacuum pole.

Therefore, it evidently follows from the unitarity condition in the s channel (in the region $s \to \infty$) that the vacuum pole must be "accompanied" by a system of branch points located between $j = j_0(t)$ and $j = 1$ and condensing at $j = 1$.

1. ASYMPTOTIC EXPRESSIONS FOR THE CROSS SECTIONS OF TRULY INELASTIC PROCESSES

The asymptotic form[5] of the amplitude for the formation of n particles,

$$a + b \to 1 + 2 + 3 + \ldots + n$$

in a truly inelastic collision represented by the graph of Fig. 1 is given by

$$A(n \leftarrow 2) = a_n(\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1}) \left(\frac{s_{12}}{m^2}\right)^{j_0(t_1)} \left(\frac{s_{23}}{m^2}\right)^{j_0(t_2)} \ldots \left(\frac{s_{n-1,\,n}}{m^2}\right)^{j_0(t_{n-1})},$$

$$(t_i = (p_a - p_1 - p_2 - \ldots - p_i)^2, \quad s_{i,\,i+1} = (p_i + p_{i+1})^2, \quad i = 1, 2, 3, \ldots, n-1) \tag{1}$$

and is large if the energies $s_{i,i+1}$ of all pairs of produced particles are large and the momentum transfers t_i of all reggeons are small for $s = s_{ab} \to \infty$.

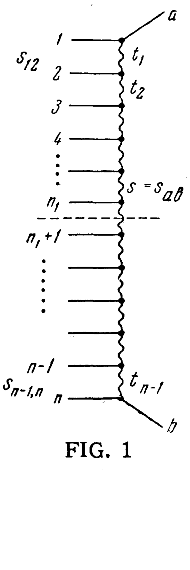

FIG. 1

The kinematic analysis[1,2] has shown that there are $n-1$ different configurations of momenta for which these conditions are fulfilled. In these configurations the produced particles must, in the c.m.s., be emitted into a narrow angle around the directions of the colliding particles, and hence all perpendicular components $\kappa_1, \kappa_2, \ldots, \kappa_n$ of their momenta must be small. Each of these configurations can be obtained by dividing the graph of Fig. 1 into two parts by a horizontal line (which intersects the propagation line of one of the $n-1$ reggeons, as indicated by the dotted line in Fig. 1) and assuming that the momenta of all n_1 particles whose lines are above the divison line are almost parallel to $\mathbf{p}_a$ and all momenta of the $n_2 = n - n_1$ particles whose lines are below are almost parallel to $\mathbf{p}_b = -\mathbf{p}_a$. Here $\mathbf{p}_a$ and $\mathbf{p}_b$ are the momenta of the colliding primary particles.

It is further necessary that the momenta of all particles be ultrarelativistic for $s \to \infty$ and that in each of the two groups of particles the outermost particles in Fig. 1 (i.e., the first and the n-th particle) have the largest momentum, with each particle having a slightly smaller momentum than the preceding one as one goes toward the center of the graph.

In other words, if the momenta of the produced particles $\mathbf{p}_i = (k_i, \kappa_i)$ are given in terms of their longitudinal (k_i) and transverse (κ_i) projections with respect to the direction $\mathbf{p}_a = -\mathbf{p}_b$, then $k_i \gg \kappa_i$ and

$$p_i = \sqrt{k_i^2 + \varkappa_i^2} \approx k_i + \varkappa_i^2/2k_i, \quad \varepsilon_i = \sqrt{\mathbf{p}_i^2 + m_i^2} \approx k_i + (m_i^2 + \varkappa_i^2)/2k_i. \tag{2}$$

Here [1] it is necessary that

$$k_1 \gg k_2 \gg \ldots \gg k_{n_1} \gg m,$$
$$k_n \gg k_{n-1} \gg \ldots \gg k_{n_1+1} \gg m, \tag{3}$$

where m is a quantity of the order of the mass of the i-th particle. One of these $n-1$ configurations is shown in Fig. 2. In all other cases either the energies $s_{i,i+1}$ are not large or the $|t_i|$ are not small. If, for example, the momentum $\mathbf{p}_i$ of the i-th particle of the "lower" group of Fig. 1 or Fig. 2 is reversed, the magnitude of $|t_i|$ becomes at once of the order of $s_{i,i+1}$. Thus the momenta of all produced particles in Fig. 1 are strictly ordered in those configurations where the asymptotic expression (1) is large.

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[1] We assume that all k_i are positive. This means that the quantities k_i are the projections of the momenta of the particles of the upper group on the direction of $\mathbf{p}_a$ and of the lower group on the direction of $\mathbf{p}_b = -\mathbf{p}_a$.

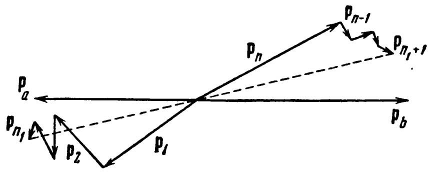

FIG. 2

It clearly follows from the conservation laws that

$$k_1 + k_2 + \ldots + k_{n_1} = k_{n_1+1} + k_{n_1+2} + \ldots + k_n \approx \sqrt{s}/2,$$

$$\varkappa_1 + \varkappa_2 + \ldots + \varkappa_n = 0. \qquad (4)$$

Moreover, as shown in [2],

$$k_1 \approx k_n \approx p_a \approx p_b \approx \sqrt{s}/2 \gg m.$$

Taking into account the inequalities (3) and (2) and (4), we obtain for the graph of Fig. 2 the following simple expressions for the energies $s_{i,i+1}$ and momentum transfers:

$$s_{12} = \frac{k_1}{k_2}(m_2^2 + \varkappa_2^2), \quad s_{23} = \frac{k_2}{k_3}(m_3^2 + \varkappa_3^2), \ldots,$$

$$s_{n_1-1,\, n_1} = \frac{k_{n_1-1}}{k_{n_1}}(m_{n_1}^2 + \varkappa_{n_1}^2),$$

$$s_{n_1+1,\, n_1+2} = \frac{k_{n_1+2}}{k_{n_1+1}}(m_{n_1+1}^2 + \varkappa_{n_1+1}^2), \ldots,$$

$$s_{n-1,n} = \frac{k_n}{k_{n-1}}(m_{n-1}^2 + \varkappa_{n-1}^2), \qquad (5)$$

$$s_{n_1,\, n_1+1} = 4k_{n_1}k_{n_1+1};$$

$$t_1 = -\varkappa_1^2, \quad t_2 = -\varkappa_2'^2, \ldots, \quad t_{n-1} = -\varkappa_{n-1}'^2, \qquad (6)$$

where

$$\varkappa_1' = \varkappa_1, \quad \varkappa_2' = \varkappa_2 + \varkappa_1', \quad \varkappa_3' = \varkappa_3 + \varkappa_2'$$

etc. (with $\kappa_{n-1}' \equiv -\kappa_n$).

It is clear that, if all κ_i are small and the inequalities (3) are fulfilled, all $s_{i,i+1}$ are large (compared with the squares of the masses of the particles) and all t_i are small. Multiplying all the values (5) of the quantities $s_{i,i+1}$, we note that they satisfy the condition

$$s_{12}s_{23} \ldots s_{n-1,n}$$
$$= (m_2^2 + \varkappa_2^2)(m_3^2 + \varkappa_3^2) \ldots (m_{n-1}^2 + \varkappa_{n-1}^2)\, s. \qquad (7)$$

It is also easy to see that there still exist a whole series of relations between the values of $s_{i,i+1}$, $s_{i,i+2}$, $s_{i,i+3}$, etc., in configurations of the type of Fig. 2. For example, $s_{i,\,i+1}s_{i+1,i+2} = (m_{i+1}^2 + \varkappa_{i+1}^2)\, s_{i,\,i+2}$ or $s_{-i1,\,i}s_{i,\,i+1}s_{i+1,\,i+2} = (m_i^2 + \varkappa_i^2)\,(m_{i+1}^2 + \varkappa_{i+1}^2)\, s_{i-1,\,i+2}$, if all particles $i-1$, i, and $i+2$ belong to the same group in Fig. 2.

Substituting formulas (1) to (7) in expression (7) of the previous paper, we find for the differential cross section $d\sigma_{n_1,n}$ for the formation of n particles in the configuration of Fig. 2

$$d\sigma_{n_1,\,n} = \lambda_n(\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1}) \exp\left[-2j_0'\left(\sum_{i=1}^{n-1} \varkappa_i'^2 \xi_{i,i+1}\right)\right]$$
$$\times \frac{d^2\varkappa_1'}{\pi}\frac{d^2\varkappa_2'}{\pi} \ldots \frac{d^2\varkappa_{n-1}'}{\pi}\, d\xi_2 d\xi_3 \ldots d\xi_{n-1}, \qquad (8)$$

where

$$\xi_{i,i+1} = \ln(s_{i,i+1}/m^2), \qquad \xi_i = \ln(k_i/m),$$
$$\lambda_n = 2\pi\,(4\pi)^{2-2n}(m_2^2 + \varkappa_2^2)(m_3^2 + \varkappa_3^2)$$
$$\ldots (m_{n-1}^2 + \varkappa_{n-1}^2)(m^2)^{2-n}\,|a_n(\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1})|^2, \qquad (9)$$

where the "pole" graph of Fig. 1 corresponds to an expression of the coefficient a_n of the form of a product of vertex parts:

$$a_n = g_1(\varkappa_1)\,\gamma_2(\varkappa_1', \varkappa_2')\,\gamma_3(\varkappa_2', \varkappa_3')$$
$$\ldots \gamma_{n-1}(\varkappa_{n-2}', \varkappa_{n-1}')\, g_n(\varkappa_n), \qquad (10)$$

where $\kappa_i' = \kappa_i + \kappa_{i-1}'$ are the transverse components of the momenta of the reggeons in Fig. 1.

Neglecting the logarithms of quantities of order unity, we find from (5) the following values for $\xi_{i,i+1}$ in momentum configurations of the type of Fig. 2:

$$\xi_{1,2} = \xi_1 - \xi_2, \quad \xi_{2,3} = \xi_2 - \xi_3, \ldots, \quad \xi_{n_1-1,n_1} = \xi_{n_1-1} - \xi_{n_1},$$
$$\xi_{n_1,\,n_1+1} = \xi_{n_1} + \xi_{n_1+1}, \quad \xi_{n_1+1,\,n_1+2} = \xi_{n_1+2} - \xi_{n_1-1}, \ldots,$$
$$\xi_{n-1,n} = \xi_n - \xi_{n-1}, \qquad (11)$$

where $\xi_n = \xi_1 = \xi = \ln(\sqrt{s}/m) \gg 1$. Formula (8) is valid under the condition that all these quantities are large compared to unity.

Introducing a certain number $\lambda > 1$, it is convenient to define the region of validity of (8) by the condition $\xi_{i,i+1} \geq \lambda$. According to (11), this implies

$$\xi_1 \geqslant \lambda + \xi_2, \quad \xi_2 \geqslant \lambda + \xi_3, \ldots, \xi_{n_1} > \lambda,$$
$$\xi_n \geqslant \lambda + \xi_{n-1}, \quad \xi_{n-1} \geqslant \lambda + \xi_{n-2}, \ldots, \quad \xi_{n_1+1} > \lambda. \quad (12)$$

As in the case of the formation of three, four, or five particles, [1,2] this region makes the largest contribution to the total cross section for $\xi \to \infty$.

Let us obtain from (8) the value of the total cross section. For $\xi_{i,i+1} \gg 1$ the integration over the transverse components κ_i' can be extended to infinity, and since the region of small values of $\kappa_i'^2$ (of order $1/2j_0'\xi_{i,\,i+1} \leqslant m^2/\lambda$),) makes the most important contribution, the function λ can be pulled out in front of the integral sign at the point $\varkappa_1' = \varkappa_2' = \varkappa_3' = \ldots = \varkappa_{n-1}' = 0$. Therefore

$$\int \lambda_n \exp\left[-2j_0'\left(\sum_{i=1}^{n-1} \varkappa_i'^2 \xi_{i,\,i+1}\right)\right] \frac{d^2\varkappa_1'}{\pi}\frac{d^2\varkappa_2'}{\pi} \ldots \frac{d^2\varkappa_{n-1}'}{\pi}$$
$$= \frac{2\sigma_n^{(0)}}{\xi_{1,2}\xi_{2,3} \ldots \xi_{n-1,n}}, \qquad (13)$$

where, according to (9) and (10),

$$\sigma_n^{(0)} = (2j_0')^{-n+1}\,[\lambda_n(\varkappa_1', \varkappa_2', \varkappa_3', \ldots, \varkappa_{n-1}')]_{\varkappa_i'=0} = \sigma_n^0\beta_2\beta_3\ldots\beta_{n-1},$$

$$\sigma_n^0 = \frac{\pi}{m^2}\frac{1}{2j_0'm^2}\frac{g_{10}^2}{4\pi}\frac{g_{n0}^2}{4\pi}, \qquad \beta_i = \frac{1}{2j_0'm^2}\left(\frac{\gamma_{i0}m_i}{4\pi}\right)^2, \quad (14)$$

with $g_{10} = g_1(0)$, $g_{n0} = g_n(0)$, $\gamma_{i0} = [\gamma_i(\varkappa_{i-1}', \varkappa_i')]_{\varkappa_{i-1}'=\varkappa_i'=0}$.

The result (13) is changed in an essential way if it is not assumed that the vertex parts

$$\gamma_i(\varkappa_{i-1}', \varkappa_i') = \gamma_i(t_{i-1}, t_i, s_{i-1,i}s_{i,i+1}/s_{i-1,i+1})$$

remain finite for $\kappa_{i-1}' = \kappa_i' = 0$, i.e., for $t_{i-1} = t_i = 0$. The quantities γ_i are (by definition) symmetric functions of the variables t_{i-1} and t_i, and the points $t_i = 0$ or t_{i-1} evidently cannot be singular points of these functions. However, γ_i can, in general, go to zero proportionally to an arbitrary power of the quantities $t_{i-1}t_i$ or $t_{i+1} + t_i$. If, for example,

$$\gamma_i = t_{i-1}t_i\gamma_{i0}' \quad \text{as} \quad t_i \to 0, \quad t_{i-1} \to 0, \qquad (15)$$

then we obtain on the right-hand side of (13)

$$\frac{2\sigma_{n_1}^{(0)}}{\xi_{1,2}^3\xi_{2,3}^5\ldots\xi_{n-2,\,n-1}^5\xi_{n-1,n}^3}, \qquad \sigma_{n_1}^{(0)} = \sigma_n^{(0)}\beta_2'\beta_3'\ldots\beta_{n-1}',$$

$$\beta_i' = \frac{1}{2j_0'm^2}\left(\frac{\gamma_{i0}'m_i^2}{4\pi}\right)^2. \qquad (16)$$

This changes all further results considerably. Let us therefore establish at the outset what happens in the case when γ_i does not vanish for $\kappa_i' = \kappa_{i-1}' = 0$. Then we obtain from (8) and (11), (13)

$$\sigma_{n_1,\,n}(\xi) = \sigma_n^0\int\limits_{(n_1-1)\lambda}^{\xi-\lambda} d\xi_2 \int\limits_{(n_1-2)\lambda}^{\xi_2-\lambda} d\xi_3 \ldots \int\limits_{\lambda}^{\xi_{n_1-1}-\lambda} d\xi_{n_1} \times \int\limits_{(n_2-1)\lambda}^{\xi-\lambda} d\xi_{n-1}\ldots\int\limits_{\lambda}^{\xi_{n_1+1}-\lambda} d\xi_{n_1+1}\, f \times (\xi, \xi_2, \xi_3, \ldots, \xi_{n-1}), \qquad (17)$$

$$f(\xi, \xi_2, \xi_3, \ldots, \xi_{n-1}) = 2\{(\xi-\xi_2)(\xi_2-\xi_3)\ldots(\xi_{n_1-1}-\xi_{n_1})(\xi_{n_1}+\xi_{n_1+1})(\xi_{n_1+1}-\xi_{n_1+2})\ldots(\xi_n-\xi_{n-1})\}^{-1}, \qquad (18)$$

where $n_2 = n - n_1$. The cross section for the formation of all $n-1$ configurations is obtained by summing this value $\sigma_{n_1,n}$ over n_1:

$$\sigma_n(\xi) = \sum_{n_1=1}^{n-1}\sigma_{n_1,\,n}(\xi). \qquad (19)$$

Actually, the value (17), (14), (19) of the total cross section is correct only if all produced particles are identical.[2] In this case one must take $\gamma_1 = \gamma_2 = \ldots = \gamma_{n-1}$, $m_1 = m_2 = \ldots = m_n = m$. For the coefficient (14) we obtain in this case

$$\sigma_n^{(0)} = \sigma_0\beta^{n-2}, \qquad \beta = (\gamma_0/4\pi)^2/2j_0'. \qquad (20)$$

For n = 3, 4, and 5 the formulas (17) to (20) yield values of the total cross sections which agree with those found earlier.

To calculate the asymptotic value of the total cross section (17) to (19) it is convenient to rewrite the expression (18) for the function f in the form

$$f = \frac{2}{\xi+\xi_n}\left(\frac{1}{\xi-\xi_2} - \frac{1}{\xi_2+\xi_{n_1+1}}\right)\left(\frac{1}{\xi_2-\xi_3} + \frac{1}{\xi_3+\xi_{n_1+1}}\right)\ldots\left(\frac{1}{\xi_{n_1-1}-\xi_{n_1}} + \frac{1}{\xi_{n_1}+\xi_{n_1+1}}\right) \times\left(\frac{1}{\xi_{n_1+2}-\xi_{n_1+1}} + \frac{1}{\xi+\xi_{n_1+1}}\right) \times\left(\frac{1}{\xi_{n_1+3}-\xi_{n_1+2}} + \frac{1}{\xi_1+\xi_{n_1+2}}\right)\ldots\left(\frac{1}{\xi_n-\xi_{n-1}} + \frac{1}{\xi+\xi_{n-1}}\right). \qquad (21)$$

For $\xi = \xi_n \to \infty$ the second terms in each parenthesis of (21) give small contributions compared to the first terms under substitution in (17). Indeed, the large values of all variables ξ_i are the most important in (17) as $\xi \to \infty$. But near the upper limit in (17) there exists a large region of values ξ_i in which the first terms in the parentheses in (21) have the constant value $1/\lambda$, which does not decrease as ξ increases. The values of the second terms decrease with growing ξ in this region. Neglecting the contribution of all second terms in (21), we obtain

$$\sigma_{n_1,n}(\xi) = \sigma_n^{(0)} I_{n_1}(\xi)\, I_{n_2}(\xi)\,\xi^{-1}, \qquad (22)$$

$$I_{n_1}(\xi) = \int\limits_{(n_1-1)\lambda}^{\xi-\lambda}\frac{d\xi_2}{\xi-\xi_2}\int\limits_{(n_1-2)\lambda}^{\xi_2-\lambda}\frac{d\xi_3}{\xi_2-\xi_3}\ldots\int\limits_{\lambda}^{\xi_{n_1-1}-\lambda}\frac{d\xi_{n_1}}{\xi_{n_1-1}-\xi_{n_1}}. \qquad (23)$$

The quantity $I_{n_2}(\xi)$ has exactly the same value as (23) (symmetric with respect to the dotted line in Fig. 1). The asymptotic form (22) of the integral (17) is correct with an accuracy up to terms of the order $[\ln(\xi/\lambda - n_1)]^{-1}$, which are small compared to unity.

For $\xi \to \infty$ the most important region in the integral $I_{n_1}(\xi)$ is near the upper limit [where the denominators in (23) have the smallest values].

[2]If the particles are distinguishable, this value [or rather, the coefficient (13)] must be summed over all n! permutations of the produced particles in the graph of Fig. 1.

Therefore the logarithm appearing in the integration over each of the variables can be taken outside the integral at the upper limit, and the contribution of the lower limit in each integration can be neglected. As a result we obtain

$$I_{n_1}(\xi) \approx [\ln(\xi/\lambda - (n_1 - 1))]^{n_1 - 1}. \tag{24}$$

As is easily verified, all approximations made in the calculation of $I_{n_1}(\xi)$ could only lead to an increase in its value. Therefore (24) is actually an upper estimate of the asymptotic form of the integral (23).

By a slightly more complicated calculation (given in the Appendix at the end of the paper) one can obtain a lower estimate of the asymptotic form of the integral (23). It agrees with (24). Therefore (24) is the exact asymptotic form. It is noted that, just as (22), the asymptotic form (24) is correct with an accuracy up to doubly logarithmic terms of the order $[\ln(\xi/\lambda - n_1 + 1)]^{-1}$, which are small compared to unity. As a result we obtain for the total cross section (19) for the formation of particles in truly inelastic collisions of the type of Fig. 1

$$\sigma_n(\xi) = \frac{\sigma_n^{(0)}}{\xi} \sum_{n_1=1}^{n-1} \left[\ln\left(\frac{\xi}{\lambda} - n_1 + 1\right)\right]^{n_1 - 1} \times \left[\ln\left(\frac{\xi}{\lambda} - n + n_1 + 1\right)\right]^{n - n_1 + 1} \tag{25}$$

All relations obtained above are valid if $n < n_0$, where $n_0 = \xi/\lambda$. This follows from the values (11) of the quantities $\xi_{i,i+1}$, each of which must be larger than λ.[3] Assuming that ξ is very large and $n_0 = \xi/\lambda \gg 1$, we consider the case of the formation of a number of particles small compared to n_0. Let

$$n < n_0/C = \xi/C\lambda,$$

where C is some large number $C > 1$, but n_0 is assumed so large that $n_0/C \gg 1$. We have for all values n restricted by this inequality

$$\sigma_n(\xi) = (n-1)\frac{\sigma_n^{(0)}}{\xi} \ln^{n-2}\frac{\xi}{\lambda}$$

or, if we take account of (20),

$$\sigma_n(\xi) = (n-1)\frac{\sigma_0}{\xi}\left(\beta \cdot \ln\frac{\xi}{\lambda}\right)^{n-2}.$$

Let us obtain the energy dependence of the total cross section for the formation of an arbitrary number of particles $n \leq n_0/C$ on the basis of the mechanism of Fig. 1. This cross section $\sigma'_{tot}(\xi)$ is only a small part of the total cross section $\sigma_{tot}(\xi)$ for the interaction of the colliding particles (see below). If ξ is so large that

$$\beta \ln \xi \gg 1,$$

then the terms of the sum over n increase rapidly with increasing n, i.e.,

$$\sigma'_{tot}(\xi) = \sum_{n=2}^{n_0/C} \sigma_n(\xi) = \sigma_{n_0/C}(\xi) \approx \frac{\sigma_0}{C\lambda}(\beta \ln \xi)^{\xi/C\lambda - 2} = \frac{\sigma_0}{C\lambda\beta^2 \ln^2 \xi}\left(\frac{s}{m^2}\right)^{\ln(\xi/\lambda)\,\beta/2C\lambda}. \tag{26}$$

This quantity increases with s more rapidly than an arbitrary power of s.

Since the contribution of the remaining inelastic processes to the total cross section is positive, $\sigma_{tot}(\xi)$ must evidently be larger than $\sigma'_{tot}(\xi)$. Therefore

$$\sigma_{tot}(\xi) > \frac{A}{\ln^2 \xi}\left(\frac{s}{m^2}\right)^{B_0 \ln \xi - B_1},$$

where A, B_0, and B_1 are constants. This inequality is in clear contradiction with the unitarity condition, for the imaginary part of the forward elastic scattering amplitude for particles a and b, divided by s, is a constant.

2. DIFFERENT FORMS OF INELASTIC PROCESSES IN THE REGION $s \to \infty$

This result indicates that a) either the assumption that the singularity of the amplitude farthest to the right in the j plane is an isolated pole is incorrect, or b) our assumption that the vertex functions $\gamma_i(t_{i-1}, t_i, x)$ are finite for $t_i = t_{i-1} = 0$ is not true. It will be shown below that the first assumption is indeed incorrect: when the vacuum pole is substituted in the unitarity condition it induces in the j plane an infinite number of singularities of the type of moving branch points located between this pole and the point $j = 1$. As a result the point $j = 1$ is a point of condensation of the branch points. This picture of the right singularities in the j plane has recently also been obtained in a completely different fashion.[7]

However, even with a more complicated right singularity in the j plane, the difficulty with the increase of expression (26) for $\sigma_{tot}(\xi)$ for $\xi \to \infty$

[3]Therefore the sum of the quantities $\xi_{i,i+1}$ in the upper row of (11), equal to $\xi - \xi_{n_1} \leq \xi - \lambda$, must evidently be larger than $(n_1 - 1)\lambda$. Then $n_1 \leq \xi/\lambda$. This also follows from (23), since for $n_1 = \xi/\lambda$ the region of integration over ξ_2 in (23) vanishes. But the number n_1 for given n can be equal to $n - 1$. Therefore $n - 1 \leq \xi/\lambda$. We note that one can obtain a less rigorous restriction, $n \leq 2\xi/\lambda$, by summing all terms in (11) or taking the logarithm of (7). If $n > \xi/\lambda$ but $n < 2\xi/\lambda$, then of all $n - 1$ configurations of the type of Fig. 2 only those give a contribution for which n_1 and n_2 are less than ξ/λ.

may not yet be taken care of. The contribution from the branch points to the total cross sections for the elastic as well as inelastic processes contains an extra factor of the form $1/\xi^n$ as compared to the contribution from the vacuum pole, where the number n depends on the character of the branch point. Therefore, it is entirely possible that taking account of the contribution from the branch points does not essentially alter the function (26) in the region $\xi \to \infty$. In order to avoid contradictions[4] it is then necessary to assume that $\gamma_i \to 0$ for $t_i \to 0$ and $t_{i-1} \to 0$. As shown above, already the simplest assumption (15) of a linear decrease of γ_i for $t_i \to 0$, $t_{i-1} \to 0$ leads to a considerable lowering of the value of the integral (13) over the transverse components of the momenta and hence to a decrease like ξ^{7-4n} of the total cross sections for the formation of n particles, $\sigma_n(\xi)$. In this case the sum of the cross sections $\sigma_n(\xi)$ of all processes of the type of Fig. 1 also decreases with increasing ξ, and the above-mentioned contradiction is removed.

It must be noted that the values $n \le \xi/\lambda$ of the number of particles for which the relations obtained above are valid are vanishingly small as compared to the number $N_0 = \sqrt{s}/m$ of particles whose formation is allowed by energy conservation. Therefore, the contribution of the graphs of the type of Fig. 1 to the cross section for multiple production of particles is very small for $s \to \infty$ (unless, of course, they lead to a catastrophic increase of σ_{tot}).

What then is the main mechanism for the multiple production in this case? A very important (and, possibly, the basic) mechanism could be the formation of a string of beams or groups of particles with small energies in the c.m.s. of each group on account of a mixed truly inelastic and "almost elastic" process of the type shown in Fig. 3. The graph of Fig. 3 differs from the graph of Fig. 1 only in that from each of its knots (for example, the i-th) issues not a line of a single particle, but of a group of ν_i particles. Let us denote the total momentum of all particles of this group by $p_i = (k_i, \kappa_i)$ and their remaining quantum numbers by ζ_i' $(i = 1, 2, \dots)$. The latter include, in particular, the square of the total energy of the particles of the i-th group.[5] With these notations, the asymptotic form of the amplitude corresponding to the graph of Fig. 3 is given by the same formula (1) as for the case of Fig. 1, with the only difference that the vertex parts in (10) will depend also on ζ_i', i.e., the coefficient a_n in (1) must be replaced by

$$a_n' = G_{\nu_1}(\zeta_1', \varkappa_1')\,\Gamma_{\nu_2}(\zeta_2', \varkappa_1, \varkappa_2')\,\Gamma_{\nu_3}(\zeta_3', \varkappa_2', \varkappa_3) \dots \Gamma_{\nu_{n-1}}(\zeta_{n-1}', \varkappa_{n-2}', \varkappa_{n-1}')\,G_{\nu_n}(\zeta_n', \varkappa_n), \qquad (27)$$

where G_{ν_i} and Γ_{ν_i} are the vertex parts of the graph of Fig. 3 corresponding to the formation of ν_i particles.

For the differential cross section for the formation of n groups according to Fig. 3 we obtain the same formulas (8) and (9) as in the case of Fig. 1, but the quantity $|a_n|^2$ in (9) must now be replaced by $|a_n'|^2 d\zeta_1' d\zeta_2' \dots d\zeta_n'$. In exactly the same way, the total cross sections for the formation of groups with an arbitrary number ν_i of particles in each group not exceeding some given number ν_0' and with an energy s_i' not larger than s_0' (where ν_0' and s_0' are fixed and do not increase for $s \to \infty$) are given by the same formulas as in the case of Fig. 1, but with the vertex parts $g_{i0}^2 = g_i^2(0)$ and $\gamma_i(0, 0) = \gamma_{i0}$ replaced respectively by

$$\sum_{\nu_i'=1}^{\nu_0} \int_{s_i' \le s_0'} |G_{\nu_i'}(\zeta_i', 0)|^2 d\zeta_i, \quad \sum_{\nu_i'=1}^{\nu_0} \int_{s_i' \le s_0'} |\Gamma_{\nu_i'}(\zeta_i', 0, 0)|^2 d\zeta_i^2 \qquad (28)$$

or, in the case of a linear decrease (15) of the vertices $\Gamma\nu_i$, where $\Gamma\nu_i \approx \kappa_{i-1}'^2 \kappa_i'^2 \Gamma'\nu_i(\zeta_i')$, with the quantities γ_{i0}^2 in (16) replaced by

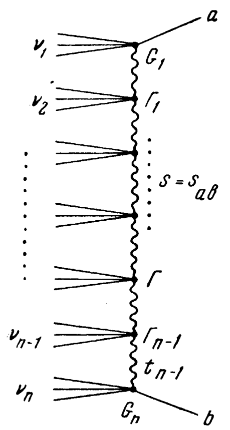

FIG. 3

[4]The account of the contribution from the branch points located between the vacuum pole and the point $j = 1$ does not alter the asymptotic form of the total cross section σ_{tot} = const (i.e., the asymptotic form of the imaginary part of the forward elastic scattering amplitude).

[5]For example, if the group consists of two particles, we can take for ζ_i' the momentum $\mathbf{k}_i$ of the relative motion of these particles in their c.m.s. (where s_i will be related to the quantity $k_i' = |\mathbf{k}_i'|$).

$$\sum_{\nu=1}^{\nu_0} \int\limits_{s_i' \leqslant s_0} |\Gamma'_{\nu_i}(\zeta_i')|^2 \, d\zeta_i'. \tag{29}$$

Therefore, all results of the preceding section refer not only to the truly inelastic processes of the type of Fig. 1, but also to the case of Fig. 3—the formation of n groups with few particles and small energies $s_i' \le s_0$.

An essential difference appears in the case when the numbers ν_i' of the particles in the groups and the energies (masses) s_i' are large. In particular, in computing the total cross sections for the "jet" processes of Fig. 3 one must integrate over ν_i' and s_i' up to the highest values (for given s). Near the upper limits of these integrations [of the type (28) or (29)], where s_i' is large, we must here take account of the dependence of the transverse momenta (6) on $\kappa_i'^2$ as well as on the masses s_i' of the jets. The corresponding terms are not included in (6). The integrals of the type (28) or (29) over ν_i' and s_i' [with an upper limit depending on s, not up to ν_0' and s_0', as indicated in (28) and (29)] may grow in some manner with increasing s, but this growth may be compensated by the logarithmic decrease (26) of the total cross sections of the "jet" processes. It is entirely possible that by an appropriate choice of the s_i' and ν_i' dependence of the functions $G_{\nu_i'}$ and $\Gamma_{\nu_i'}$ in (28) and (29), one will be able to "balance out" the s (or ξ) dependence of both parts of the unitarity condition so as to conform with the elastic forward scattering amplitude.

3. DEPENDENCE OF THE UNITARITY CONDITION ON THE TRANSVERSE MOMENTUM AND CONDENSATION OF SINGULARITIES IN THE j PLANE

The imaginary part of the amplitude $A(2 \leftarrow 2) = A(\mathbf{p}', \mathbf{p})$ for elastic scattering into non-forward angles (Fig. 4) satisfies the unitarity condition in the s channel

$$\frac{1}{s} \operatorname{Im} A_2(\mathbf{p}', \mathbf{p}) = \sum_{n=2}^{N_0(s)} \sigma_n(\mathbf{p}', \mathbf{p}), \tag{30}$$

where the right-hand side contains the terms

$$\sigma_n(\mathbf{p}', \mathbf{p}) = \frac{1}{2s} \int A_n^*(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; \mathbf{p}') A_n \times (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots, \mathbf{p}_n; \mathbf{p})\, d\tau_n, \tag{31}$$

which agree for $\mathbf{p}' = \mathbf{p}$ with the total cross sections $\sigma_n(\xi)$ for the formation of n groups of particles [cf. formula (4) of the previous paper [2]]. Here $A_n = A(n \leftarrow 2)$ is the amplitude for the formation of n groups of particles, $d\tau_n$ is the statistical weight of all their states, and $\mathbf{p}$ and $\mathbf{p}'$ are the

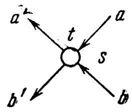

FIG. 4

initial and final momenta of the particles for elastic scattering in the c.m.s. (Fig. 4, $\mathbf{p} = \mathbf{p}_a = -\mathbf{p}_b$, $\mathbf{p}' = \mathbf{p}_a' = -\mathbf{p}_b'$, where $|\mathbf{p}| = |\mathbf{p}'| = \sqrt{s}/2$).

Assuming that s is large, we use the asymptotic form (1) for A_n [with a_n replaced by a_n' defined by (27)]. Then (31) should be compared with the graph of Fig. 5. The momenta of all particles are conveniently defined in a coordinate system whose OZ axis is located symmetrically with respect to $\mathbf{p}$ and $\mathbf{p}'$, as shown in Fig. 6. We shall give the momenta in terms of their projections on the OZ axis and on the XOY plane. Then the longitudinal components of $\mathbf{p}$ and $\mathbf{p}'$ will be identical: $k = k'$, and the transverse components will be equal in magnitude but opposite in direction. Let us denote the latter by $\boldsymbol{\kappa}/2$ and $-\boldsymbol{\kappa}/2$. Thus

$$\mathbf{p} = (k, \varkappa/2), \quad \mathbf{p}' = (k, -\varkappa/2), \quad \mathbf{p}_i = (k_i, \varkappa_i).$$

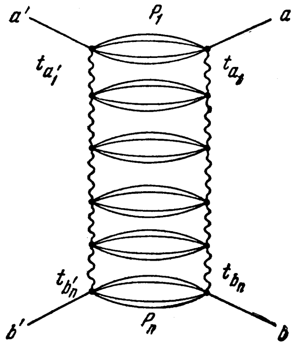

FIG. 5

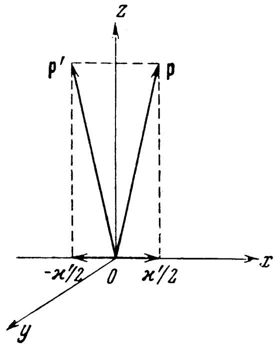

FIG. 6

The momentum transfer $t = (\mathbf{p} - \mathbf{p}')^2$ in elastic scattering will obviously be equal to $-\kappa'^2$, and the values $t_{a1}, t_{a12}, \ldots$ and $t_{a1}' = (\mathbf{p}' - \mathbf{p}_1)$, $t_{a12}' = (\mathbf{p}' - \mathbf{p}_1 - \mathbf{p}_2), \ldots$ of the squares of the momenta of the left and right reggeons of Fig. 5 are defined in complete analogy to (6):

$$t_{a1} = -(\varkappa_1' - \varkappa'/2)^2, \quad t_{a12} = -(\varkappa_2' - \varkappa'/2), \ldots,$$
$$t_{a1\ldots n-1} = -(\varkappa_n' - \varkappa'/2)^2, \quad t_{a1}' = -(\varkappa_1' + \varkappa'/2),$$
$$t_{a12}' = -(\varkappa_2' + \varkappa'/2)^2, \ldots, \quad t_{a1\ldots n-1}' = -(\varkappa_n' + \varkappa'/2)^2.$$

Substituting these values in the expression for the asymptotic amplitude (1) and in (31), we obtain in analogy to (8) and (19):

$$\sigma_n(p', p) = \sum_{n_1=1}^{n-1} \sigma_{n, n_1}(p', p); \tag{32}$$

$$d\sigma_{n_1,\,n}(p, p') = \lambda_n(\varkappa_1, \ldots, \varkappa_{n-1}) \exp\left\{-2j_0' \sum_{i=1}^{n}\left[\left(\varkappa_i - \frac{\varkappa'}{2}\right)^2 + \left(\varkappa_i + \frac{\varkappa'}{2}\right)^2\right] \xi_{i,\,i-1}\right\} \frac{d^2\varkappa_1'}{\pi}\frac{d^2\varkappa_2'}{\pi} \ldots \frac{d^2\varkappa_n'}{\pi} d\xi_2 d\xi_3 \ldots d\xi_{n-1}, \tag{33}$$

where λ_n is defined by (9) with $|a_n|^2$ replaced by $|a_n|^2 d\zeta_1' d\zeta_2' \ldots d\zeta_n'$.

Since the square bracket in the exponent of (33) contains the quantity $\kappa_1'^2 + \kappa'^2/2$ and, according to (11),

$$\sum_{i=1}^{n-1} \xi_{i,\,i-1} = 2\xi,$$

the entire dependence of (32) on κ'^2 is separated out in the form of a factor $\exp(-j_0'\kappa'^2\xi)$. As a result we obtain from (32) and (33)

$$\sigma_n(p', p) = \exp\left(-\tfrac{1}{2} j_0' \varkappa'^2 \,(2\xi)\right)\sigma_n(\xi),$$

where $\sigma_n(\xi)$ is the total cross section. Therefore the right-hand side of the unitarity condition (31) is equal to

$$C_1(\xi) \exp\left\{-\frac{1}{2} j_0' \varkappa'^2 (2\xi)\right\}, \qquad C_1(\xi) = \sum_{n=2}^{\infty} \sigma_n(\xi), \tag{34}$$

where $C_1(\xi)$ is the total cross section of all processes whatsoever, calculated with the help of the "pole-type" asymptotic amplitudes (1) for the inelastic processes.

The quantity $A_2(p, p')$, which is analogous to (1) and corresponds to the contribution of only one (vacuum) Regge pole, (I), can for small κ^2 be written in the form (Fig. 7)

$$\frac{1}{s} A_2(\mathbf{p}, \mathbf{p}') = i g_0^2 \exp\{-j_0' \varkappa'^2 (2\xi)\}, \tag{35}$$

where $\kappa^2 = -t$, $2\xi = \ln(s/m^2)$. According to (30), the imaginary part of this expression must be identically equal to (34) for all κ^2 and ξ.

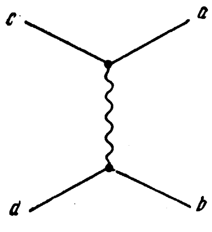

FIG. 7

It is clear that, even if the dependence of the vertex parts in (27) can be chosen such that $C_1'(\xi) = g_0^2 = \text{const.}$, the κ^2 dependence of (34) is quite different from that of (35) (because of the coefficient $\frac{1}{2}$ in the exponent).

In the general case, where $\sigma_t^{(1)}(\xi) \neq \text{const}$, (34) may be regarded as the contribution to the amplitude $A_2(p', p)$ from the singularities of the type of the branch pointed located at [6]

$$j_1 \approx 1 - j_0'\varkappa^2/2. \tag{36}$$

If we now substitute in the right-hand side of the unitarity condition the asymptotic form of all amplitudes corresponding not to the vacuum pole (I), but to the singularity (36) further to the right, we obtain a value which differs from (32), (33) only by the factor $\frac{1}{2}$ in the exponent of (33). As a result, we have on the right-hand side of the unitarity condition (30) $C_2(\xi)\exp\{-\frac{1}{4} j_0'\kappa^2(2\xi)\}$, which corresponds to the contribution to the asymptotic amplitude $A_2(p, p')$ from the following branch point

$$j_2(\varkappa^2) = 1 - j_0'\varkappa^2/4 \tag{37}$$

which lies more to the right than (36).

The singularity (36) corresponds to the graph of Fig. 5 with two "reggeons" in the t channel. The singularity (37) corresponding to the combination of two graphs similar to those of Figs. 5 and 7 corresponds to a four-reggeon exchange in the t channel. The general case of q reggeons (q = 1, 2, 3, ...) in the t channel gives rise to a singularity at the point

$$j_q(\varkappa^2) = 1 - j_0'\varkappa^2/q. \tag{38}$$

The positions of the vacuum pole (I) and all branch points (36) to (38) have been written down above under the assumption that $\kappa^2 = -t$ is small (since only small values of κ^2 were important in the discussion above). It follows from what has been said above that if all asymptotic amplitudes are defined on the basis of the assumption that the vacuum pole (I) in the j plane is accompanied by an infinite system of singularities (38) which condense towards j = 1, the right- as well as the left-hand sides of the unitarity condition (30) will have the form of a sum of terms of the type

$$\sum_{q=1}^{\infty} C_q(\xi) \exp\left\{-\frac{1}{q} j_0'\varkappa^2 (2\xi)\right\}. \tag{39}$$

The amplitudes of the inelastic processes will have a complicated form.[6)] For example, instead of (1) we obtain

6)The right-hand side of the unitarity condition of Fig. 5 contains only terms with q = 2,3,... The terms with q = 1, corresponding to the contribution of the Regge pole, can only be obtained if the contribution of graphs of the type of Fig. 1 is taken into account, but not with reggeons, but ordinary particles[6] in the intermediate state.

$$A\ (n \to 2) = \sum_{q_1, q_2, \ldots, q_{n-1}} C_{q_1, q_2, \ldots, q_{n-1}} \times \exp\left\{-j_0' \sum_{i=1}^{n-1} \frac{\varkappa_i^2}{q_i}\ \xi_{i,\,i+1}\right\}, \tag{40}$$

where $C_{q_1, q_2, \ldots, q_{n-1}}$ is a certain function of all κ_i and $\xi_{i,i+1}$; the dependence on $\xi_{i,i+1}$ is determined by the character of the q_i-th singularity.

In order to satisfy the unitarity condition (30) and (31) for the scattering amplitude (40), it suffices to include in (40) only the terms with $q_1 = q_2 = \ldots = q_{n-1}$, i.e., to substitute in the right-hand side of (31) the asymptotic form of the inelastic amplitudes

$$\frac{1}{s} A\ (n \leftarrow 2) = \sum_q C_q'\,(\xi_{12}, \xi_{23}, \ldots, \xi_{n-1,\,n}) \times \exp\left\{-\frac{j_0'}{q} \sum_{i=1}^{n-1} \varkappa_i^2\ \xi_{i,\,i+1}\right\}. \tag{41}$$

This quantity also defines the asymptotic amplitude for the formation of showers. The coefficients C_q' depend not only on the total energies of the showers $\xi_{i,i+1}$, but also on their other quantum numbers. This dependence is not shown explicitly in (41).

Let us verify that by substitution of the asymptotic form (41) in the right-hand side of the unitarity condition (31) we reproduce precisely the imaginary part of the asymptotic elastic scattering amplitude in the form (39). For this purpose we consider a single term of the form of Fig. 5 on the right-hand side of (31) which comes from the term in the asymptotic form (41) corresponding to some value $q = q_\alpha$ and from the analogous term in the asymptotic form of the complex conjugate amplitude $s^{-1}A^*(n \leftarrow 2)$, corresponding to $q = q_\beta$. After integration over all variables defining the properties of the jets (in the c.m.s. of the particles of each jet) we obtain for this term, in analogy to (32) and (33),

$$\sigma_n^{(\alpha,\,\beta)}\,(\mathbf{p}', \mathbf{p}) = \sum_{n_1=1}^{n-1} \sigma_{n,\,n_1}^{(\beta,\,\alpha)}\,(\mathbf{p}', \mathbf{p}), \tag{42}$$

$$d\sigma_{n,\,n_1}^{(\beta,\,\alpha)} = \lambda_n\,(\xi;\ \xi_{1,2}, \ldots, \xi_{n-1,\,n}) \times \exp\left\{-j_0' \sum_{i=1}^{n-1}\left[\frac{1}{q_\alpha}\left(\varkappa_i' + \frac{q_\alpha \varkappa'}{q_\alpha + q_\beta}\right)^2 + \frac{1}{q_\beta}\left(\varkappa_i' - \frac{q_\beta \varkappa'}{q_\alpha + q_\beta}\right)^2\right]\xi_{i,\,i+1}\right\} \frac{d^2\varkappa_1'}{\pi}\,\frac{d^2\varkappa_2}{\pi} \ldots \frac{d^2\varkappa_{n-1}'}{\pi}\,d\xi_2\,d\xi_3 \ldots d\xi_{n-1}. \tag{43}$$

In writing down this expression, we have chosen the coordinate system such that the transverse components of the momenta $\mathbf{p}$ and $\mathbf{p}'$ (particles a and a′ in Fig. 5) have the values

$$\varkappa_{a'} = \frac{q_\beta}{q_\alpha + q_\beta}\varkappa', \qquad \varkappa_a = \frac{-q_\alpha}{q_\alpha + q_\beta}\varkappa',$$

where κ' is the transverse component of the difference $\mathbf{p}' - \mathbf{p}$. For $q_\alpha = q_\beta$ this coordinate system coincides with the one in which formulas (32) and (33) are written

According to (41), the right-hand side of the unitarity condition (30) is equal to the sum of the quantities $\sigma_n^{(\beta,\alpha)}$ over all singularities of the reggeons in both parts of Fig. 5. As is seen immediately, the entire κ'^2 dependence of the right-hand side of (42) is separated out in the form of a factor $\exp\{-j_0'(q_\alpha + q_\beta)^{-1}\varkappa'^2\,(2\xi)\}$ exactly as in (32). In other words, the right-hand side of the unitarity condition (30) has precisely the form (39) with $q = q_\alpha + q_\beta$. The coefficient $C_q(\xi)$ in (39) is reproduced in the form of a sum of terms of the type

$$\sum_{q_\alpha,\,q_\beta} C_{q_\alpha,\,q_\beta}\,(\xi)\,\delta_{q,\ q_\alpha + q_\beta}.$$

It is entirely possible that all these coefficients can be matched on both sides of (30) by an appropriate choice of the vertex functions in (27). If this can be done, then, in particular, the dependence of the vertex functions in (27) on the total energies (or masses, i.e., numbers of particles) of the various jets is determined. Thus such an experimentally important characteristic of the inelastic processes as the average multiplicity of showers is determined. In any case, it is clear now that the κ^2 dependence of the two sides of the unitarity condition (30) can be "matched" only if the j plane contains, besides the vacuum pole, a system of singularities (38) condensing towards the point $j = 1$.

The further investigation of the unitarity condition in the s channel, in particular, not only for the elastic scattering amplitude as in (30), but also for the inelastic amplitudes, is of considerable interest.

APPENDIX

Let us obtain a lower estimate for the multiple integral (23):

$$I_{n_1}(\xi) = \int_{(n_1-1)\lambda}^{\xi-\lambda} \frac{d\xi_2}{\xi - \xi_2} \int_{(n_1-2)\lambda}^{\xi_2-\lambda} \frac{d\xi_3}{\xi_2 - \xi_3} \cdots \int_{\lambda}^{\xi_{n_1-1}-\lambda} \frac{d\xi_{n_1}}{\xi_{n_1-1} - \xi_{n_1}}. \tag{A.1}$$

Integrating (1) over ξ_{n_1}, we obtain the following integral over ξ_{n_1-1}:

$$\int_{2\lambda}^{\xi_{n_1-2}-\lambda} \frac{d\xi_{n_1-1}}{\xi_{n_1-2} - \xi_{n_1-1}} \ln\frac{\xi_{n_1-1} - \lambda}{\lambda}\,d\xi_{n_1-1}. \tag{A.2}$$

At the lower limit the integrand vanishes. Let us multiply and divide the integrand by $\xi_{n_1-1} - 2\lambda$. Since the factor $(\xi_{n_1-1} - 2\lambda)^{-1} \ln [(\xi_{n_1-1} - \lambda)/\lambda]$ decreases monotonically as one goes from the lower to the upper limit, it can be taken outside the integral at the upper limit. This only decreases the value of the integral (1). The remaining integral over ξ_{n_1-1} is easily evaluated, so that the integral over ξ_{n_1-2} has the form

$$\int_{3\lambda}^{\xi_{n_1-3}-\lambda} \frac{\ln(\xi_{n_1-2} - 2\lambda)}{(\xi_{n_1-3} - \xi_{n_1-2})(\xi_{n_1-2} - 3\lambda)} \times \left[(\xi_{n_1-2} - 2\lambda) \ln \frac{\xi_{n_1-2} - 2\lambda}{\lambda} - (\xi_{n_1-2} - 3\lambda) \right] d\xi_{n_1-2}. \quad (A.3)$$

This expression also vanishes at the lower limit. Therefore, before the integration over ξ_{n_1-2} we multiply and divide the integrand by $(\xi_{n_1-2} - 3\lambda)^2$ and take the quantity

$$\frac{\ln(\xi_{n_1-2} - 2\lambda)}{\xi_{n_1-2} - 3\lambda} \left[(\xi_{n_1-2} - 2\lambda) \ln \frac{\xi_{n_1-2} - 2\lambda}{\lambda} - (\xi_{n_1-2} - 3\lambda) \right] \quad (A.4)$$

outside the integral evaluated at the upper limit (at the minimum). Then we compute the integral over ξ_{n_1-2}.

We proceed analogously in the calculation of all further integrals in (1). As a result, we obtain for $I_{n_1}(\xi)$

$$I_{n_1}(\xi) > \frac{2}{[\xi - n_1\lambda](n_1-1)(n_1-2)} \ln \frac{\xi - (n_1-1)\lambda}{\lambda} \times \left[(\xi - (n_1-1)\lambda) \ln \frac{\xi - (n_1-1)\lambda}{\lambda} + I_1 \right] \times \left[(\xi - (n_1-1)\lambda)^2 \ln \frac{\xi - (n_1-1)\lambda}{\lambda} + I_r \right] \ldots [(\xi - (n_1-1)\lambda)^{n_1-2} + I_{n_1-2}], \quad (A.5)$$

where I_r is a polynomial of r-th degree in ξ.

It is easy to show that the coefficient of the largest term in ξ in I_r is

$$(-1)^r \sum_{m=0}^{r-1} (-1)^m \frac{C_m}{r-m}$$

a number which increases slowly in absolute value as r increases. For $\xi \gg 1$ and $\ln \xi \gg 1$, the polynomials I_r in the square brackets in (A.5) can be neglected. We then obtain for I_{n_1} the following lower estimate:

$$I_{n_1} > \left[\frac{\xi - (n_1-1)\lambda}{\xi - n_1\lambda} \right]^{(n_1-1)(n_1-2)/2} \times \ln^{n_1-1} \frac{\xi - (n_1-1)\lambda}{\lambda} > \ln^{n_1-1} \frac{\xi - (n_1-1)\lambda}{\lambda}.$$

Analogously, we obtain a lower estimate for the integral I_{n-n_1}:

$$I_{n_2} = I_{n-n_1} > \ln^{n-n_1-1} \frac{\xi - (n-n_1-1)\lambda}{\lambda}.$$

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