POLARIZATION OF ELECTRONS IN AN INHOMOGENEOUS MAGNETIC FIELD

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The possibility of obtaining spiral (polarized parallel or antiparallel to their momentum) electrons by means of an inhomogeneous magnetic field is investigated theoretically. The correlation between the spin and the momentum is calculated in the nonrelativistic limit. It is shown that electrons in passing through an inhomogeneous magnetic field with a gradient along the lines of force acquire a certain longitudinal polarization.

SEVERAL methods of obtaining polarized electrons [1] and several methods for measuring their polarization [2] are known. We shall discuss here the possibility of obtaining spiral electrons (i.e., electrons polarized parallel or antiparallel to the direction of their motion) by means of an inhomogeneous magnetic field. The method under consideration may turn out to be effective at low electron velocities $v \ll c$ (c is the velocity of light). Therefore we shall consider the problem of the correlation of the momentum and the spin of the electron in an homogeneous magnetic field in the nonrelativistic limit. For this purpose we write the Hamiltonian for the problem under consideration in the following form

$$\mathcal{H} = \frac{\pi^2}{2m} - g\frac{e\hbar}{2mc}\boldsymbol{\sigma}\mathbf{H}(x),$$
$$\boldsymbol{\pi} = \frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}(x), \qquad g = 1 + \frac{\alpha}{2\pi} \tag{1}$$

(α is the fine structure constant), where $\operatorname{div}\mathbf{H} = 0$ and we are considering the region in which $\operatorname{curl}\mathbf{H} = 0$. Heisenberg's equations for the momentum and for the spin give us

$$\dot{\boldsymbol{\pi}} = -\frac{i}{\hbar}[\boldsymbol{\pi}, \mathcal{H}] = \frac{e}{mc}[\boldsymbol{\pi}\mathbf{H}] + g\frac{e\hbar}{2mc}(\boldsymbol{\sigma}\nabla)\mathbf{H},$$
$$\dot{\boldsymbol{\sigma}} = -\frac{i}{\hbar}[\boldsymbol{\sigma}, \mathcal{H}] = g\frac{e}{mc}[\boldsymbol{\sigma}\mathbf{H}]. \qquad (2)^*$$

In future we shall be interested in the correlation between the spin and the momentum $\sigma_{\mathbf{n}} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$, where the operator $\hat{\mathbf{n}}$ is defined in the following manner:

$$\hat{\mathbf{n}} = \sum_{\mathbf{p},\xi} |\mathbf{p}, \xi\rangle \frac{\langle \mathbf{p}, \xi | \boldsymbol{\pi} | \mathbf{p}, \xi\rangle}{\langle \mathbf{p}, \xi | \pi | \mathbf{p}, \xi\rangle} \langle \mathbf{p}, \xi |, \tag{3}$$

where $|\mathbf{p}, \xi\rangle$ is the quantum mechanical state of the electron of momentum $\mathbf{p}$ and spin ξ: $\mathbf{p} = \langle\boldsymbol{\pi}\rangle$, $\xi = \langle\boldsymbol{\sigma}\rangle$, while the operator $\pi = \sqrt{\pi^2}$. The expression (3) for $\hat{\mathbf{n}}$ is written taking into account the fact that the spin and the momentum taken at any given instant of time are not correlated.

*$[\boldsymbol{\pi}\mathbf{H}] = \boldsymbol{\pi}\times\mathbf{H}$, $(\boldsymbol{\sigma}\nabla) = \boldsymbol{\sigma}\cdot\nabla$.

Having in mind the definition (3) and the equations (2) we can easily obtain the equation[1)] for $\xi_{\mathbf{n}} = \langle\sigma_{\mathbf{n}}\rangle$:

$$\dot{\xi}_{\mathbf{n}} = (g-1)\frac{e}{mc}[\mathbf{H}\mathbf{n}]\,\boldsymbol{\xi} + g\frac{e\hbar}{2mc}p^{-1}\{(\boldsymbol{\xi}\nabla) - (\mathbf{n}\boldsymbol{\xi})(\mathbf{n}\nabla)\}(\mathbf{H}\boldsymbol{\xi}) \tag{4}$$

($\mathbf{n}$ is the unit vector in the direction of the momentum $\mathbf{p}$). We note that Eq. (4) is nonlinear with respect to $\boldsymbol{\xi}$.

We now estimate the order of magnitude of the quantities appearing in (4). The characteristic frequencies $\omega_1 = \alpha eH/2\pi mc$ and $\omega_2 = e\hbar\,|\nabla H|/2m^2c^2\beta$ ($\beta = v/c$) under the condition that $H \sim 10^5$ oe, $|\nabla H| \sim 10^5$ oe/cm, and $\beta \sim 10^{-3}$ (at the same time $R_L\,|\nabla H|/H \ll 1$, R_L is the Larmor radius of the electron) are of the order of magnitude $\omega_1 \sim 10^9\ \text{sec}^{-1}$ and $\omega_2 \sim 10^5\ \text{sec}^{-1}$, i.e., $\omega_1 \gg \omega_2$.

We can average (5) over the time interval $t \gg \omega_1^{-1}$, utilizing for this the method of averaging over the rapidly rotating phase [3]. As in [3], the quantities with the indices $\parallel$ and $\perp$ are the components of the corresponding quantities along the direction of the field $\mathbf{H}(x)$ and perpendicular to the field. We have then

$$\boldsymbol{\xi} = \xi_\parallel \boldsymbol{\tau}_0 + \xi_\perp(\boldsymbol{\tau}_1\cos\beta + \boldsymbol{\tau}_2\sin\beta),$$
$$\mathbf{p} = p_\parallel \boldsymbol{\tau}_0 + p_\perp(\boldsymbol{\tau}_1\cos\alpha + \boldsymbol{\tau}_2\sin\alpha),$$

where $\boldsymbol{\tau}_0 = \mathbf{H}/H = \boldsymbol{\tau}_1\times\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2\times\boldsymbol{\tau}_0$, $\boldsymbol{\tau}_2 = \boldsymbol{\tau}_0\times\boldsymbol{\tau}_1$, $\beta \sim g\omega_H t$, and $\alpha \sim \omega_H t$ ($\omega_H = eH/mc$). From (2) we can obtain equations for $\xi_\parallel$, $\xi_\perp$ and $p_\parallel$, $p_\perp$:

[1)]This equation can be obtained also by a classical analysis of the motion of a particle with an intrinsic angular momentum in an inhomogeneous magnetic field.

$$\dot{\xi}_{\parallel} = -\xi_{\perp}\{\tau_0 \cos\beta\,[p_{\parallel} m^{-1}\nabla_{\tau_0}\tau_1 + p_{\perp} m^{-1}(\nabla_{\tau_1}\tau_1 \cos\alpha + \nabla_{\tau_2}\tau_1 \sin\alpha)] + \tau_0 \sin\beta \times [p_{\parallel} m^{-1}\nabla_{\tau_0}\tau_2 + p_{\perp} m^{-1}(\nabla_{\tau_1}\tau_2 \cos\alpha + \nabla_{\tau_2}\tau_2 \sin\alpha)]\},$$

$$\dot{\xi}_{\perp} = -\xi_{\parallel}\{\tau_1 \cos\beta\,[p_{\parallel} m^{-1}\nabla_{\tau_0}\tau_0 + p_{\perp} m^{-1}(\nabla_{\tau_1}\tau_0\, \alpha \cos + \nabla_{\tau_2}\tau_0 \sin\alpha)] + \tau_2 \sin\beta \times [p_{\parallel} m^{-1}\nabla_{\tau_0}\tau_0 + p_{\perp} m^{-1}[\nabla_{\tau_1}\tau_0 \cos\alpha + \nabla_{\tau_2}\tau_0 \sin\alpha)]\},\tag{5}$$

$[\nabla_T = (\boldsymbol{\tau}\cdot\nabla)]$ while the equations for $p_{\parallel}$ and $p_{\perp}$ can be left in the form [3]

$$\dot{p}_{\parallel} = \frac{p_{\perp}^2}{2m}\operatorname{div}\tau_0 + \frac{p_{\parallel}p_{\perp}}{m}\{\tau_1(\nabla_{\tau_0}\tau_0)\cos\alpha + \tau_2(\nabla_{\tau_0}\tau_0)\sin\alpha\} + \frac{p_{\perp}^2}{2m}\{\tau_1(\nabla_{\tau_1}\tau_0) - \tau_2(\nabla_{\tau_2}\tau_0)\}\cos 2\alpha + \frac{p_{\perp}^2}{2m}\{\tau_1(\nabla_{\tau_2}\tau_0) + \tau_2(\nabla_{\tau_1}\tau_0)\}\sin 2\alpha,$$

$$\dot{p}_{\perp} = -\frac{p_{\parallel}p_{\perp}}{2m}\operatorname{div}\tau_0 - \frac{p_{\parallel}^2}{m}\tau_1(\nabla_{\tau_0}\tau_0)\cos\alpha - \frac{p_{\parallel}^2}{m}\tau_2(\nabla_{\tau_0}\tau_0)\sin\alpha - \frac{p_{\parallel}p_{\perp}}{2m}\{\tau_1(\nabla_{\tau_1}\tau_0) - \tau_2(\nabla_{\tau_2}\tau_0)\}\cos 2\alpha - \frac{p_{\parallel}p_{\perp}}{2m}\{\tau_1(\nabla_{\tau_2}\tau_0) + \tau_2(\nabla_{\tau_1}\tau_0)\}\sin 2\alpha\tag{6}$$

(here the equations for $p_{\parallel}$ and $p_{\perp}$ are given with an accuracy up to terms of zero order in $\hbar$ [2)]).

First of all, as a result of averaging over $t \gg \omega_1^{-1}$ Eqs. (5) yield

$$\dot{\bar{\xi}}_{\parallel} = 0,\qquad \dot{\bar{\xi}}_{\perp} = 0.\tag{7}$$

On averaging (4) [in doing this it is necessary to utilize relations (5), (6) and (7)] we obtain

$$\dot{\bar{\xi}}_{n} = g\frac{e\hbar}{2mc}\bar{\xi}_{\parallel}^2(\overline{p_{\perp}^2/p^3})\nabla_{\tau_0}H,\tag{8}$$

where the bar over a letter indicates quantities averaged over the period ω_1^{-1}.

Integration of (8) yields

$$\bar{\xi}_{n} = \mu\bar{\xi}_{\parallel}^2\int_0^t dt'\,(\overline{p_{\perp}^2/p^3})(\tau_0\nabla)H + \xi_n^{(0)}\tag{9}$$

2)The neglected terms lead to a small correction $\sim\hbar^2$ to the correlation between spin and momentum.

(μ is the magnetic moment of the electron). Averaging (9) over the beam under the condition that initially the beam of electrons was unpolarized, and all the electrons of the beam had the same momentum, leads to the expression

$$(\bar{\xi}_{n}) = \frac{1}{3}\mu\int_0^t (\overline{p_{\perp}^2/p^3})(\tau_0\nabla)H\,dt';\tag{10}$$

here $(\bar{\xi}_n)$ is the longitudinal polarization of the electron beam.

Thus, we see that an electron beam in passing through an inhomogeneous magnetic field with a gradient along the lines of force acquires a certain longitudinal polarization. The time during which the quantity (10) becomes of the order of magnitude of unity is itself of the order of magnitude of the quantity $T = 3\bar{p}^3/\bar{p}_{\perp}^2\,|\nabla H| \sim 10^{-5}$ sec, and during this time the electrons must traverse a distance $L = c\beta T \sim 10^2$ cm.

We also note that the polarization properties of the beam are not altered when the electrons are accelerated in systems without a magnetic field. In our case, in virtue of the well-known relation $\sigma = \gamma_5\hat{n}$ [4] which is valid at high energies, electrons accelerated to high energies remain spiral.

[1] V. M. Hughes, Sources of Polarized Electrons, Proc. of the Conference on Photon Interactions in the BeV Energy Range, Cambridge, 1963.

[2] Progress in Elementary Particles and Cosmic Ray Physics, Amsterdam, 5, 67 (1960).

[3] N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asymptoticheskie metody v teorii nelineĭnykh kolebaniĭ (Asymptotic Methods in the Theory of Nonlinear Oscillations), Moscow, 1958, p. 315.

[4] Bose, Gamba, and Sudarshan, Phys. Rev. 113, 1161 (1959).

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