QUANTUM PROCESSES IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE AND IN A CONSTANT FIELD

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The effect of the field of a plane electromagnetic wave on quantum processes occurring in the absence of the field are considered. The main features of such a process are analyzed for the particular case of the $\pi \to \mu + \nu$ decay. A general formula is derived for the probability of $\pi \to \mu + \nu$ decay in the field of the wave. The formula is investigated for various limiting cases defined by the magnitude of the parameters ea/m and ea/ω (a is the wave potential amplitude). For ea/m, $ea/\omega \ll 1$ the formula yields the corresponding probabilities of perturbation theory, and for $ea/m \gg 1$—the probability $F(\chi)$, $\chi = e\sqrt{(F_{\mu\nu}P_\nu)^2/m^3}$ for the process in constant crossed field ($\mathbf{E}\cdot\mathbf{H} = 0$, $E = H$). If the conditions considered in [1] are fulfilled, $F(\chi)$ describes the process in an arbitrary constant field. A significant dependence of the probability on decay energy is demonstrated; for small values of χ the probability of $\pi \to \mu + \nu$ decay increases and the probability $\pi \to e + \nu$ decreases. With decreasing decay energy the sensitivity of the probability to the field strength increases. The exact expression for the probability removes the infrared divergence for photon absorption. In view of the importance of processes in crossed field, a straightforward method of calculation is presented, not connected with the limiting transition $ea/m \to \infty$.

1. INTRODUCTION

THIS paper treats the effect of the field of an electromagnetic wave and the effect of a constant field on processes which take place even in the absence of the field; an example, exhibiting the simplest type of kinematics, is the decay $\pi \to \mu + \nu$ (three particles, one of which has zero mass). In the real $\pi \to \mu + \nu$ decay the masses of the charged particles and their difference are of the order of magnitude of the mass of the π meson; hence it is clear from the results of the preceding paper [1] (hereafter cited as I) that in order to have any significant effect on this process the field must be of order m^3/ep_0, where m and p_0 are the mass and energy of the π meson; such fields will hardly be available in the near future. However there are many processes for which the mass difference between the decaying particle and its products is small (of the order of an electron mass). In this case the decay probability will depend on this small mass difference and much smaller fields will be capable of influencing the process.

Using the $\pi \to \mu + \nu$ decay as an example, we will investigate the following general questions: How does the decay probability depend on the decay energy? Does the decay probability decrease or increase in the presence of the field? etc. It is shown that the decay probability $F(\chi)$, where $\chi = e\sqrt{(F_{\mu\nu}P_\nu)^2/m^3}$, depends strongly on the decay energy or on $\Delta = 1 - (m'/m)^2$, since for small χ the value of $F(\chi)$ decreases with increasing χ if $\Delta \ll 1$ or if $1 - \Delta \ll 1$, and increases with increasing χ if Δ is not close either to zero or to unity. For large values of χ, $F(\chi)$ increases as $\chi^{2/3}$ independently of Δ. For small values of Δ, the probability $F(\chi)$ depends essentially on χ/Δ, so that its sensitivity to the field increases.

In Sec. 3 we consider the general expression for the probability of the process in the field of a wave, in various limiting cases depending on the parameters ea/m and ea/ω. For ea/m and $ea/\omega \ll 1$ the probability goes over into the corresponding formulas of perturbation theory; for $ea/m \gg 1$ one obtains an essentially nonlinear theory. For $ea/m \ll 1$ and $ea/\omega \gtrsim 1$ the probabilities for the processes involving absorption from the wave (or emission into the wave) of 0, 1, 2, ... photons differ significantly from the corresponding expressions of perturbation theory.

On the other hand they differ little from each other kinematically, and their sum, which gives the total probability, is approximately the same as the probability for the process in the absence of the field.

Whereas in perturbation theory the probability for a process involving the absorption of a low-energy photon is proportional to $e^2 n_\gamma t^3 = \frac{1}{2}(ea/\omega)^2$ and exhibits an infrared divergence, the general expressions obtained here for the probability of a process involving the absorption of 0, 1, 2, ... photons apply to arbitrary values of the parameters ea/m and ea/ω and exhibit regular behavior in the infrared region.

2. THE $\pi \rightarrow \mu + \nu$ DECAY

According to Gell-Mann and Feynman [2] the decay matrix element may be written in the form

$$M = \frac{G}{\sqrt{2}} f (\nabla_\alpha - ieA_\alpha)\, \varphi_\pi (\bar{\psi}_\mu \gamma_\alpha (1 + \gamma_5)\psi_\nu), \qquad (1)$$

where φ_π, ψ_μ, and ψ_ν are the wave functions of the π meson, the μ meson, and the neutrino, G is the weak interaction constant, and f is a phenomenological constant which takes account of the virtual strong interactions of the π meson [3]. In the field $A_\alpha = a_\alpha \cos(kx)$ of a plane electromagnetic wave the wave functions of the charged particles are given by the expressions (cf. I, (1))

$$\varphi_\pi = \frac{1}{\sqrt{2q_0}} \exp\left[i\frac{e(ap)}{(kp)} \sin(kx) - i\frac{e^2a^2}{8(kp)} \sin 2(kx) + i(qx)\right], \qquad (2)$$

$$\psi_\mu = \left[1 + e\frac{\hat{k}\hat{A}}{2(kp')}\right] u(p') \exp\left[i\frac{e(ap')}{(kp')}\sin(kx) - i\frac{e^2a^2}{8(kp')}\sin 2(kx) + i(q'x)\right], \qquad (3)$$

where q, q′ are the "quasi-momenta" of the π and μ mesons, and $\psi_\nu = u(l)\exp[i(lx)]$. It is assumed that all of the functions are normalized so that the average particle number density is equal to unity (cf. I, Sec. 2). Using these functions in (1), we obtain

$$M = -\frac{Gfm'}{2\sqrt{q_0}} \sum_s \bar{u}(p') \left[A_0 - \frac{e\hat{k}\hat{a}}{2(kp')}A_1\right](1+\gamma_5)\, u(l) \times (2\pi)^4 \delta(q + sk - q' - l). \qquad (4)$$

The square of the matrix element, summed over polarizations of the μ meson and the neutrino, is equal to

$$\sum \frac{|M|^2}{VT} = \frac{G^2f^2m^2m'^2}{4q_0q_0'l_0} \sum_s \left\{\Delta A_0^2 + \left(\frac{ea}{m}\right)^2 \frac{(kl)}{(kp')}(A_1^2 - A_0A_2)\right\} \times (2\pi)^4 \delta(q + sk - q' - l). \qquad (5)$$

Here m and m′ are the mass of the π and μ mesons, and $\Delta = 1 - (m'/m)^2$. Integrating this expression over the momenta l and q′ we obtain the total decay probability in the field of the plane electromagnetic wave:

$$W = \frac{G^2f^2m^2m'^2}{16\pi^2 q_0} \sum_{s>s_0} \int_0^{2\pi} d\varphi \int_0^{\frac{m^2\Delta - 2s(kq)}{m_*'^2}} \frac{du}{(1+u)^2} \times \left\{\Delta A_0^2 + \left(\frac{ea}{m}\right)^2 \frac{(kl)}{(kp')}(A_1^2 - A_0A_2)\right\}, \qquad (6)$$

where $u = (kl)/(kq')$, φ is the angle between the planes $(\mathbf{k}, \mathbf{q}')$ and $(\mathbf{k}, \mathbf{a})$ in the center of mass system, and $s_0 = m^2\Delta/2(kq)$.

Since this general expression for the decay probability contains the complicated functions $A_0^2(s, \alpha, \beta)$ and $A_1^2(s, \alpha, \beta) - A_0(s, \alpha, \beta)A_2(s, \alpha, \beta)$, which cannot be integrated analytically in general, it is natural to consider various special cases. In I we considered the limiting cases determined by the value of the invariant parameter $x = ea/m$. For the processes considered in I, which do not take place without a field, the arguments α and β of the functions $A_n(s, \alpha, \beta)$ were very much smaller than unity if $x \ll 1$ ($\alpha \sim xs$, $\beta \sim x^2 s$), and the probability went over into the perturbation-theory result. For processes which occur even in the absence of a field, the parameters α and β are in general not small when x is small. Hence the case $x \ll 1$ must be considered separately and will be treated in Sec. 3. We now consider the case of large x.

We therefore let $x \gg 1$. Then since the process $\pi \rightarrow \mu + \nu$ belongs to the group of processes obeying the conservation law $q + sk = q' + l$, the discussion given at the beginning of Sec. 4 of I is applicable. Using I (25), I (20) and I (21) and the asymptotic expressions for the functions A_0^2, and $A_1^2 - A_0A_2$, which were given in I, we obtain for the decay probability W in the slowly varying field of a plane wave

$$W(\chi) = \frac{2}{\pi}\int_0^{\pi/2} F(\chi \sin\psi)\, d\psi, \qquad (7)$$

where $F(\chi)$ is the decay probability in a constant crossed field given by

$$F(\chi) = \frac{G^2f^2m^2m'^2c}{\pi^2} \int_0^\infty dv \int_0^\infty du \frac{\operatorname{sh} v}{\operatorname{ch}^3 v} \operatorname{ch} u \left(\frac{\operatorname{sh}^2 v}{2\chi}\right)^{1/3} \left\{\Delta\Phi^2(y) + \operatorname{sh}^2 v(\operatorname{ch}^2 u - \Delta \operatorname{cth}^2 v)\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\right\}. \qquad (8)^*$$

Here

$$y = (\operatorname{sh}^2 v/2\chi)^{2/3}(\operatorname{ch}^2 u - \Delta \operatorname{cth}^2 v), \qquad \chi = e\sqrt{(F_{\mu\nu}q_\nu)^2}/m^3.$$

*sh = sinh, ch = cosh, cth = coth.

Although they are well defined for an oscillating field, the concepts of the average particle number density (assumed normalized to unity) and the mean kinetic energy (q_0) lose their meaning when considered separately in the case of a constant crossed field. However their ratio (which is what enters the expression for the probability) is the invariant quantity c, which, both for the alternating and for the constant crossed field, equals the ratio of the true particle number density to their kinetic energy or the ratio of the particle number density in the rest system to the mass of the particles (cf. I, Sec. 2). Therefore the factor $1/q_0$ in (8) is replaced by the invariant c. We note also that the lifetime of the π meson in the rest system is $cm/F(\chi)$.

The decay probability $F(\chi)$ has the same form as the probabilities for the processes which do not occur in the absence of a field. There is however one new feature. The argument of the Airy function which occurs in $F(\chi)$ may now take on both positive and negative values. This property is characteristic of the probabilities for all processes which occur even in the absence of a field. The use of the Airy function $\Phi(y)$, which is analytic for all real y, leads to a single form for the probabilities of the various processes. This unified expression would not obtain if in place of $\Phi(y)$ we use the more usual Bessel function of order $1/3$, since the latter is not analytic for $y = 0$.

All of the considerations set forth in Sec. 7 of I are applicable to the present processes. Hence when condition (43) of I is satisfied, (8) gives the decay probability in an arbitrary constant field $F_{\mu\nu}$.

We now consider the asymptotic behavior of $F(\chi)$ both for small and for large values of χ. A calculation, given in the appendix, indicates that for small χ

$$F(\chi) = \frac{G^2 f^2 m^2 m'^2 \Delta^2 c}{8\pi} \times \left\{1 + C(\Delta)\chi^{5/3} + \frac{6\Delta^2 - 16\Delta + 6}{3\Delta^2}\chi^2 + \ldots\right\}. \tag{9}$$

The first term of this expansion is the probability for decay in the absence of the field (cf. for example [3]). The coefficient $C(\Delta)$ in the second term is given by the expression

$$C(\Delta) = \left(\frac{3}{2}\right)^{2/3} \frac{\Gamma(2/3)}{6\pi\Delta} \int_0^\Delta dt \frac{\alpha'\alpha''' - 3\alpha''^2}{\sqrt{-\alpha\alpha'^4}},$$

$$\alpha^3 = \frac{(t-\Delta)^3}{t(1-t)^2}. \tag{10}$$

The dependence of this coefficient on Δ is shown in Fig. 1; in particular for $\Delta \to 0$

$$C(\Delta) = -\left(\frac{2}{3}\right)^{1/3} \frac{\Gamma(2/3)}{\pi} \int_0^1 dx \frac{x^{5/6}\sqrt{1-x}\,(5+x)}{(1+2x)^4} \Delta^{-5/3};$$

for $\Delta = 0.427$, $C(\Delta) = 0.385$; for $\Delta = 1$, $C(\Delta) = (-5/24)(3/2)^{2/3}\Gamma(2/3)$. Thus $C(\Delta)$ depends strongly on Δ (i.e., on the decay energy) and changes sign twice as Δ varies from zero to one. Thus for values of Δ close to zero or unity the decay probability decreases when the field is turned on; for intermediate values of Δ the probability increases when the field is turned on (cf. Fig. 2). It is also clear from (9) that for $\Delta \ll 1$ the expansion must be in powers of the parameter χ/Δ, and hence it is valid under the condition that $\chi/\Delta \ll 1$. This also means that when $\Delta \ll 1$ the decay probability in the presence of a field differs essentially from the probability for free decay when $\chi/\Delta \sim 1$, i.e., in fields of order $m^3\Delta/ep_0$.

For large χ the integral in (8) is determined essentially by values $v \sim 1$. Using this fact, it is not difficult to show that

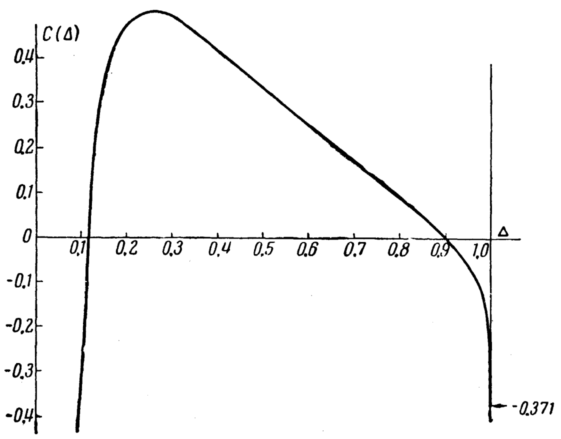

FIG. 1

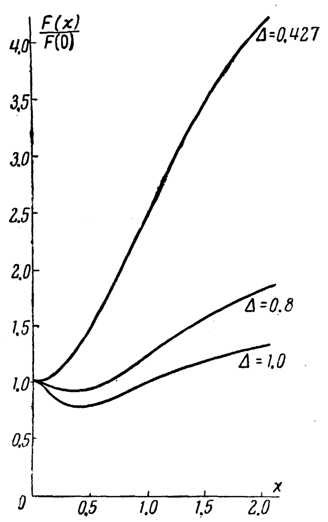

FIG. 2

$$F(\chi) = \frac{G^2 f^2 m^2 m'^2 c \Gamma(2/3)}{36\pi} (3\chi)^{2/3}, \qquad \chi \gg 1. \qquad (11)$$

Figure 2 shows the function $F(\chi)$ calculated from (8) for two values of the parameter Δ, equal to 0.427 and 1, corresponding to the real decay processes $\pi \to \mu + \nu$ and $\pi \to e + \nu$.

3. LIMITING CASES OF THE EXACT FORMULA AND THE INFRARED DIVERGENCE ACCOMPANYING ABSORPTION OF LONG WAVELENGTH PHOTONS

We now treat qualitatively several limiting cases which can be obtained from the general expressions for the probabilities of processes in the field of a plane monochromatic wave. We have in mind those limiting cases which depend on the properties of the electromagnetic wave, i.e., its frequency ω and its amplitude $B = \omega a$ (or on two other related parameters, for example the dimensionless parameters $x/ea/m$ and ea/ω).[1)]

A. Processes which take place in the absence of a field ($\Delta \neq 0$). In this case it is not difficult to show that to an order of magnitude

$$\alpha \sim \frac{ea}{\omega} \frac{\Delta}{2\sqrt{(1+x^2/2)(1-\Delta+x^2/2)}} + \frac{sx}{\sqrt{1-\Delta+x^2/2}},$$

$$\beta \sim \frac{x\alpha}{\sqrt{1-\Delta+x^2/2}}. \qquad (12)$$

Then for $ea/\omega \ll 1$, $x \ll 1$ we obtain $\beta \ll \alpha \ll 1$, which leads immediately to perturbation theory. In fact, in this case $A_0^2 \approx 1$, $A_1^2 - A_0A_2 \approx -1/2$ for $s = 0$, $A_0^2 \approx \alpha^2/4$, $A_1^2 - A_0A_2 \approx 1/4$ for $s = 1$, etc. Thus for example in the case of a decay, the term with $s = 0$ in (6) gives the probability for the ordinary decay, the term with $s = 1$ gives the probability for decay with the absorption of one photon, etc, calculated according to perturbation theory.

For $x \gg 1$ and for arbitrary ea/ω we obtain an essentially nonlinear theory which is considered in more detail in Sec. 2. There remains the case $ea/\omega \gtrsim 1$, $x \ll 1$.

We recall that the total probability for the process has the following structure

$$W = \sum_s W_s = \sum_s \int \frac{d^3q'd^3l}{q_0' l_0} [f_0 A_0^2 + f_1 (A_1^2 - A_0A_2)]\, \delta(q + sk - q' - l),$$

where W_s is the probability of a process involving the absorption from or emission into the wave of s photons. The functions A_0^2 and $A_1^2 - A_0A_2$ may be considered different from zero only for $|s| \lesssim \alpha$, since for $|s| > \alpha$ they decrease exponentially with increasing s [in this regard the functions $A_n(s, \alpha, \beta)$ behave like the Bessel functions $J_s(\alpha)$]. Therefore the probability for the process is determined essentially by the sum

$$\sum_{s=-s_{\text{eff}}}^{s_{\text{eff}}} W_s, \qquad s_{\text{eff}} \sim \frac{ea}{\omega}.$$

Furthermore, since in the present case we have $s_{\text{eff}}\omega \sim ea \ll m$ the kinematics of the processes described by the individual terms in this sum are very similar to the kinematics of free decay. Therefore in the expression for W we may take out from the summation sign all factors except A_0^2 and $A_1^2 - A_0A_2$, since these factors are essentially independent of s as long as $|s| \lesssim s_{\text{eff}}$. We then obtain

$$W \approx \sum_{s=-s_{\text{eff}}}^{s_{\text{eff}}} W_s \approx \int \frac{d^3q'd^3l}{q_0' l_0} \delta(q - q' - l) \times \left[f_0 \sum_{s=-s_{\text{eff}}}^{s_{\text{eff}}} A_0^2 + f_1 \sum_{s=-s_{\text{eff}}}^{s_{\text{eff}}} (A_1^2 - A_0A_2)\right].$$

Thus, neglecting exponentially small terms, we replace s_{eff} by ∞, and since

$$\sum_{s=-\infty}^{\infty} A_0^2 = 1, \qquad \sum_{s=-\infty}^{\infty} (A_1^2 - A_0A_2) = 0$$

[cf. (A5) and (A6) in I], we have

$$W \approx \int \frac{d^3q'd^3l}{q_0' l_0} \delta(q - q' - l) f_0,$$

i.e., the total probability of the process is essentially the same as the probability for free decay.

Thus for $ea/\omega \gtrsim 1$, $x \ll 1$ the probabilities for processes involving absorption from the wave (or absorption into the wave) of 0, 1, 2 ... photons differ only slightly as far as kinematics are concerned but differ significantly from the corresponding expressions of perturbation theory; on the other hand, the total probability is approximately the same as the probability for the process in the absence of the field.

B. Processes which do not occur in the absence

[1)]For simplicity we limit ourselves to processes obeying the conservation law $q + sk = q' + l$, where $l^2 = 0$ and we will characterize these processes by the parameter $\Delta = 1 - (m'/m)^2$, where m and m' are the masses of the charged particles. Then for processes occurring in the absence of a field $\Delta \neq 0$, and for processes which do not occur in the absence of a field $\Delta = 0$. Moreover in this section ω designates the frequency in the system in which $q = 0$, $q_0 = m*$, so that ea/ω is invariant and equal to $-eam*/(kq)$.

of the field ($\Delta = 0$). In this case we have, in order of magnitude,

$$\alpha \sim \frac{sx}{\sqrt{1 + x^2/2}}, \quad \beta \sim \frac{sx^2}{1 + x^2/2}. \tag{13}$$

Therefore for $x \ll 1$ we obtain $\beta \ll \alpha \ll 1$, i.e., perturbation theory. For $x \gg 1$ we have the nonlinear theory discussed in I.

We now return to the processes which occur in the absence of a field and consider them from the point of view of the infrared divergence of quantum electrodynamics. As is well known [4], in dealing with processes which involve the emission of photons with energy $\omega \to 0$, the expansion parameter of perturbation theory is not e^2, but $e^2 \ln(E/\omega)$, where E is an energy of the order of magnitude of the electron energy. This parameter arises because the squared matrix element of a process involving emission of a photon with energy $\omega \to 0$ is proportional to $e^2/\omega^3 V$, and its integral over the number of final states $4\pi\omega^2 d\omega V/(2\pi)^3$ of the emitted photon gives the result $e^2 \ln(E/\omega)$. For processes involving the absorption of a low energy photon the squared matrix element is proportional to $e^2/\omega^3 V$ as before, but there is no integration over final states. Hence in this case the expansion parameter of the perturbation theory is $e^2/\omega V \equiv e^2 n_\gamma \lambda\!\!\!^{-3}$, where n_γ is the number density of incident photons. If one uses the relation $a^2 \sim n_\gamma/\omega$ between the amplitude a of the 4-potential of the plane monochromatic wave and the number density of photons n_γ, then the parameter $e^2 n_\gamma \lambda\!\!\!^{-3}$ may also be written $(ea/\omega)^2$. Thus the condition that perturbation theory be applicable to the absorption of long wavelength photons is

$$e^2 n_\gamma \lambda\!\!\!^{-3} \ll 1 \quad \text{or} \quad ea/\omega \ll 1. \tag{14}$$

Thus for $\omega/m \ll 1$ and $ea/\omega \ll 1$ the probability of a process involving the absorption of a single photon may be written down using perturbation theory and is proportional to $e^2 n_\gamma \lambda\!\!\!^{-3} = (1/2)(ea/\omega)^2$. As the frequency decreases this probability diverges, which is physically absurd (this is the infrared divergence for the absorption of long wavelength photons). However even for $ea/\omega \sim 1$ the perturbation theory formula loses its validity. In this case it is convenient to use the theory proposed above treating the electromagnetic wave as an external field. Then, for the probability of the decay $\pi \to \mu + \nu$ for example we obtain (6), in which the s-th term, W_s, for $s \geq 0$ ($s < 0$) gives the probability for decay accompanied by absorption from the wave (or emission into the wave) of s photons. No term in this formula exhibits an infrared divergence but instead, as expected from physical considerations, each approaches zero as $\omega \to 0$, since the functions $A_n(s, \alpha, \beta) \to 0$ for α or $\beta \to \infty$.

4. PROCESSES IN A CONSTANT CROSSED FIELD

In I and in Sec. 2 of this paper, using general expressions for the probabilities of processes in a field of a plane wave and with the help of a rather complicated limiting transition ($ea/m \to \infty$), we obtained the probabilities for processes in a crossed field. As was shown in I, the probabilities for processes in a constant field play a very important role since, when condition (43) of I is fulfilled, they give the probabilities of the same processes in an arbitrary constant field. It will be useful to explain the direct method for calculating probabilities in a crossed field.

A constant crossed field may be considered as a special case of the field of a plane wave (cf. I, Sec. 2) if one takes a potential of the type $A_\mu = a_\mu \cdot (kx)$ as the 4-potential $A_\mu(\varphi)$, $\varphi = (kx)$. Then, using (1) of I, we obtain the solution of the Dirac equation in a crossed field:

$$\psi_{pr}(x) = \left[1 + e\frac{\hat{k}\hat{a}\varphi}{2(kp)}\right] u(pr) \times \exp\left[i\frac{e(ap)\varphi^2}{2(kp)} - i\frac{e^2a^2\varphi^3}{6(kp)} + i(px)\right]. \tag{15}$$

All of the general relations (2)-(4) of I are valid for this solution. The spinor $u(p)$ is normalized by the relation $u^+u = 1$.

Using the function (15) the matrix element for an electron transition from the state ψ_p to the state $\psi_{p'}$, with the emission of a photon of momentum k' and polarization e' can be written

$$M = e\int(\bar{\psi}_{p'}\hat{e}'\psi_p)\frac{e^{-i(k'x)}}{\sqrt{2k_0'}}d^4x = e\int_{-\infty}^{\infty}\frac{ds}{\sqrt{2k_0'}}\bar{u}(p')\left\{\hat{e}'A + e\left(\frac{\hat{a}\hat{k}\hat{e}'}{2(kp')} + \frac{\hat{e}'\hat{k}\hat{a}}{2(kp)}\right)i\frac{\partial A}{\partial s} + \frac{e^2a^2(ke')\hat{k}}{2(kp)(kp')}\frac{\partial^2 A}{\partial s^2}\right\}u(p) \times (2\pi)^4\delta(p + sk - p' - k'), \tag{16}$$

where $A = A(s, \alpha, \beta)$ is the Fourier transform of the quantity $\exp(i\alpha\varphi^2/2 - i4\beta\varphi^3/3)$:

$$\exp\left\{i\left(\frac{\alpha\varphi^2}{2} - \frac{4\beta\varphi^3}{3}\right)\right\} = \int_{-\infty}^{\infty} ds e^{is\varphi} A(s, \alpha, \beta),$$

$$A(s, \alpha, \beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\varphi \exp\left\{i\left(\frac{\alpha\varphi^2}{2} - \frac{4\beta\varphi^3}{3} - s\varphi\right)\right\}, \tag{17}$$

and where the derivatives $i^n\partial^n A/\partial s^n$ are the Fourier trnasforms of the quantities $\varphi^n \exp(i\alpha\varphi^2/2 - i4\beta\varphi^3/3)$.

The matrix element (16) is analogous to the matrix element for the emission from an electron in the field of a plane monochromatic wave $A_\mu = a_\mu \cos(kx)$ and differs from the latter in replacing the discrete index s by a continuous index using the formula $(s - 2\beta)_{\text{disc}} \to s_{\text{cont}}$, so that the sum over s is replaced by an integral, the momenta q and q′ are replaced by a momenta p and p′, and the functions A_0, A_1, A_2 are replaced by the functions A, $i\partial A/\partial s$, $-\partial^2 A/\partial s^2$.

It follows from (17) that $A(s, \alpha, \beta)$ can be expressed in terms of the Airy function $\Phi(y)$:

$$A(s, \alpha, \beta) = \frac{1}{\sqrt{\pi}} (4\beta)^{-1/3} \exp\left\{- is\frac{\alpha}{8\beta} + i\frac{8\beta}{3}\left(\frac{\alpha}{8\beta}\right)^3\right\}\Phi(y),$$
$$y = (4\beta)^{2/3}\left[\frac{s}{4\beta} - \left(\frac{\alpha}{8\beta}\right)^2\right] \tag{18}$$

(as shown in I, $\beta > 0$). Using (18) it is not difficult to find a relationship between the function A and its first two derivatives with respect to s:

$$sA - i\alpha\, \partial A/\partial s - 4\beta\, \partial^2 A/\partial s^2 = 0, \tag{19}$$

which is analogous to (A3) of I.

In calculating $|M|^2$ one obtains the double integral

$$\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds'\, \delta(sk + p - p' - k')\,\delta$$
$$\times (s'k + p - p' - k') \ldots,$$

in which the product of delta functions may be written as follows

$$\delta(sk + p - p' - k')\,\delta(s'k + p - p' - k') = \frac{\delta(s - s')}{\delta(0)}$$
$$\times [\delta(sk + p - p' - k')]^2$$
$$= \frac{\delta(s - s')}{\delta(0)} \frac{VT}{(2\pi)^4}\,\delta(sk + p - p' - k'). \tag{20}$$

We then have

$$\sum_{r, r'} \frac{|M|^2}{VT} = \frac{(2\pi)^4 e^2}{\delta(0)} \int_{-\infty}^{\infty} \frac{ds}{2p_0 p_0' k_0'}\,\delta(sk + p - p' - k')$$
$$\times \Big\{[2(pe'')^2 - (pp') - m^2]\,|A|^2 + [\alpha(kk')$$
$$- 4e(ae'')(pe'')]\,\mathrm{Im}\, AA'^*$$
$$+ e^2\left[\frac{a^2(kk')^2}{2(kp)(kp')} + 2(ae'')^2\right]|A'|^2\Big\}, \tag{21}$$

where $e'' = e' - k'(ke')/(kk')$ and $A' = \partial A/\partial s$. If we sum (21) over the polarization directions e′ and make use of (19), then $\Sigma\,|M|^2/VT$ can be expressed via just two combinations of the functions A, A′, and A″; viz: A^2 and $|A'|^2 + \mathrm{Re}AA''^*$, which in view of (18) are given by

$$|A|^2 = \frac{\sigma}{\pi x^2 y}\Phi^2(y), \quad |A'|^2 + \mathrm{Re}AA''^*$$
$$= \frac{\sigma^2}{\pi x^4 y}\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right], \tag{22}$$

where

$$y = (4\beta)^{2/3}\left[\frac{s}{4\beta} - \left(\frac{\alpha}{8\beta}\right)^2\right], \qquad \sigma = x^2\left[\frac{s}{4\beta} - \left(\frac{\alpha}{8\beta}\right)^2\right],$$
$$x = \frac{ea}{m}.$$

Then we have

$$\sum \frac{|M|^2}{VT} = \frac{4\pi e^2 m^2}{\delta(0)\, x^2} \int_{-\infty}^{\infty} ds\, \frac{\sigma}{p_0 p_0' k_0' y} \times \Big\{- \Phi^2(y) + \sigma$$
$$\times \left(1 + \frac{(kk')^2}{2(kp)(kp')}\right)\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\Big\}$$
$$\times\, \delta(sk + p - p' - k'). \tag{23}$$

An important feature of the squared matrix element (23) is the infinite factor $\delta(0)$ in the denominator. This factor, or more precisely $2\pi\delta(0)$, represents the "volume" of the phase φ, since

$$\delta(s - s') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(s - s')\varphi}\, d\varphi.$$

Photon emission by an electron. In order to obtain the probability F for the emission of a photon by an electron, (23) must be integrated over final states $d^3p' d^3k'/(2\pi)^6$ of the electron and the photon, the result divided by two (because of the average over the electron polarizations). After integrating over **k′** and s, the factor $k_0'/(kk'')$ replaces the 4-dimensional delta function and there remains an integral over **p′**, in which it is convenient to transform from the variable p_3' to the variable $\gamma' = p_0' - p_3'$. Finally the probability takes a form which is common to processes obeying the conservation law $p + sk = p' + l$, independent of the mass of the charged particles, the momenta p, p′, and the mass of the neutral particle with angular momentum l; this expression is

$$F = -\frac{1}{(2\pi)^2\,\delta(0)\, p_0} \int_{-\infty}^{\infty} dp_1' \int_{-\infty}^{\infty} dp_2' \int_0^{\gamma = p_0 - p_3} \frac{d\gamma'}{\gamma'(kk')} w, \tag{24}$$

where w is an invariant function (in the particular case of photon emission by an electron $p^2 = p'^2 = -m^2$, $l^2 = k'^2 = 0$.

The integrand in (23) depends on the independent variables s, α, β, (kp), and (kp′). In what follows however it is convenient to make the substitution $x \equiv ea/m$ and to choose the following new variables:

$$\chi = -\frac{(kp)}{m^2}x, \quad \chi' = -\frac{(kp')}{m^2}x, \quad \varkappa = -\frac{(kl)}{m^2}x,$$
$$\sigma = x^2\left[\frac{s}{4\beta} - \left(\frac{\alpha}{8\beta}\right)^2\right], \qquad \psi = \frac{\alpha}{8\beta} \tag{25}$$

[cf. (20) in I]. Only four of the five variables in (25) are independent, since $\varkappa = \chi - \chi'$ because of the conservation law $p + sk = p' + l$. The form of these variables in the "special" coordinate system (i.e., the system in which the axes 1,2,3 are parallel to the vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{E}\times\mathbf{H}$) and also their invariant expression in terms of the field amplitude $F_{\mu\nu}$ are given by (21) of I if q and q′ are replaced in the latter by p, p′ and cos ψ is replaced by ψ.

If we use (21) of I with q, q′, and cos ψ replaced by p, p′, and ψ, and if we transform from the variables p_1', p_2', γ' to the variables ψ, τ, χ' respectively, it is not difficult to see that the integrand in (24) is independent of the variable ψ and depends only on the variables τ and χ'. Therefore

$$F=\frac{1}{(2\pi)^2\,\delta(0)\,p_0}\int_{-\infty}^{\infty}d\psi\int_0^{\chi}d\chi'\int_0^{\infty}d\tau\,\frac{x^2\varkappa}{\chi^2\chi'}\,w, \tag{26}$$

and the integral becomes proportional to the infinite "volume" of the variable ψ. This infinite "volume" of the variable ψ cancels the infinite "volume" $2\pi\delta(0)$ of the phase φ, occurring in the denominator in (26), since these "volumes" are equal. In fact if one uses the integral expression (17) for the function A to find the phase making the primary contribution to A, one finds the following expression for the phase:

$$\varphi_{1,2}=\frac{\alpha}{8\beta}\pm\sqrt{\left(\frac{\alpha}{8\beta}\right)^2-\frac{s}{4\beta}}=\psi\pm i\,\frac{\sqrt{\sigma}}{x}, \tag{27}$$

from which it is clear that $\int_{-\infty}^{\infty}d\psi=2\pi\delta(0)$. Consequently the probability for emission by an electron in a crossed field can be obtained in final form which, with the substitutions $\tau=\sinh u$, $\chi/\chi'=\cosh^2 v$, can be written in the form

$$F(\chi)=\frac{e^2m^2}{\pi^2p_0}\int_0^{\infty}dv\int_0^{\infty}du\,\frac{\mathrm{sh}\,v}{\mathrm{ch}^3\,v}\sqrt{y}\,\Big\{-2\Phi^2(y)$$
$$+\,\mathrm{ch}^2\,u\,\Big\{\left(\mathrm{ch}^2\,v+\mathrm{ch}^{-2}\,v\right)\left(\Phi^2+\frac{1}{y}\,\Phi'^2\right)\Big\}. \tag{28}$$

Here

$$y=\left(\frac{\mathrm{sh}^2\,v}{2\chi}\right)^{2/3}\mathrm{ch}^2\,u,\qquad \chi=e\sqrt{(F_{\mu\nu}p_\nu)^2}/m^3.$$

This expression coincides with expression (27) in I obtained by a limiting transition from the probability for emission by an electron in the field of a plane monochromatic wave.

Pair production by a photon. The probability of this process may be obtained from (23) if in the latter expression we make the replacements $k'\to -l$ and $p\to -p$, change the sign of the whole expression, integrate over final states $d^3p\,d^3p'/(2\pi)^6$ of the electron and the positron, and finally divide the result by two (because of the average over the polarizations of the incident photon). After integrating over p and s there remains an integral over $\mathbf{p}'$ which, after replacing the variable p_3' by $\gamma'=p_0'-p_3'$ takes the form

$$F=-\frac{1}{(2\pi)^2\,\delta(0)\,l_0}\int_{-\infty}^{\infty}dp_1'\int_{-\infty}^{\infty}dp_2'\int_0^{\lambda=l_0-l_3}\frac{d\gamma'}{\gamma'(kp)}\,w, \tag{29}$$

which is valid for the probability of the more general process obeying the conservation law $l+sk=p+p'$. Changing from the variables p_1', p_2', and γ' to the variables ψ, τ, and χ' with the help of (21) in I (in which q, q′ cos ψ must be replaced by p, p′ψ and also $1\to -1$, $p\to -p$), we find that the integrand does not depend on ψ and further that

$$F=\frac{1}{(2\pi)^2\,\delta(0)\,l_0}\int_{-\infty}^{\infty}d\psi\int_0^{\varkappa}d\chi'\int_{-\infty}^{\infty}d\tau\,\frac{x^2}{\chi\chi'}\,w$$
$$=\frac{1}{2\pi l_0}\int_0^{\varkappa}d\chi'\int_{-\infty}^{\infty}d\tau\,\frac{x^2}{\chi\chi'}\,w, \tag{30}$$

since

$$\int_{-\infty}^{\infty}d\psi=2\pi\delta(0).$$

In the case of pair formation the function w is symmetric in χ, χ' since $\int_0^{\kappa}d\chi'\ldots=2\int_0^{\kappa/2}d\chi'\ldots$. Putting $\chi'=\kappa(1+\tanh v)/2$ and $\tau=\sinh u$, we obtain the probability for pair formation by an unpolarized photon of angular momentum l in the form

$$F(\varkappa)=\frac{e^2m^2}{\pi^2l_0}\int_0^{\infty}dv\int_0^{\infty}du\,\frac{\sqrt{y}}{\mathrm{ch}^2 v}\,\Big\{\Phi^2(y)$$
$$+\,\mathrm{ch}^2u\,(2\,\mathrm{ch}^2\,v-1)\left[\Phi^2(y)+\frac{1}{y}\,\Phi'^2(y)\right]\Big\}, \tag{31}$$

where

$$y=(2\mathrm{ch}^2v/\varkappa)^{2/3}\,\mathrm{ch}^2u,\qquad \varkappa=e\sqrt{(F_{\mu\nu}l_\nu)^2}/m^3.$$

This expression coincides with the corresponding result obtained in I by a limiting transition.

Single photon annihilation. The probability of this process may also be obtained from (23) if we make the changes $p'\to -p'$, $k\to -k$, and $a\to -a$, change the sign of the whole expression, integrate over the final states $d^3k'/(2\pi)^3$ of the emitted photon, and divide the result by four (because of the averages over the polarizations of the electron and positron). After integrating over $\mathbf{k}'$ and s we obtain the annihilation probability per unit volume per unit time:

$$F=\frac{e^2}{2\delta(0)\,p_0p_0'x}\,\frac{\sigma}{\varkappa y}$$
$$\times\Big\{\Phi^2(y)+\sigma\,\frac{\chi^2+\chi'^2}{2\chi\chi'}\left[\Phi^2(y)+\frac{1}{y}\,\Phi'^2(y)\right]\Big\}, \tag{32}$$

This is zero because of the infinite volume $2\pi\delta(0)$ of the phase φ. However if in place of F we introduce the annihilation probability per unit volume for the time T, the latter probability will be finite. In fact by definition $\varphi = (kx)$, so that in the special system $\varphi = k_0(z - t)$, from which it follows that the "volume" of the phase φ is related to the total time T by the relation $2\pi\delta(0) = k_0 T$ is z is fixed. Hence the annihilation probability per unit volume for the time T is given by the expression

$$FT = \frac{\pi e m}{p_0 p_0' B} \frac{\sigma}{\varkappa y}\left\{\Phi^2(y) + \sigma\frac{\chi^2+\chi'^2}{2\chi\chi'}\left[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\right]\right\}, \quad (33)$$

where B is the field strength. The same expression is obtained for the annihilation probability per unit time in an infinite cylindrical volume with unit cross-section and with axis along the direction $\mathbf{E}\times\mathbf{H}$.

The decay $\pi \to \mu + \nu$. Calculation of the decay probability in a crossed field is carried out according to the same procedure as the calculation for the probability of emission from an electron [cf. (15)-(22) and also (1)-(6)], as a result of which we obtain in place of (6)

$$F = \frac{G^2 f^2 m^2 m'^2}{16\pi^3 p_0 x^2 \delta(0)} \int_{-\infty}^{\infty} ds \int \frac{\sigma}{y}\Big[\Delta\Phi^2(y) + \sigma\frac{(kl)}{(kp')}\Big(\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\Big)\Big]\delta(p + sk - p' - l)\frac{d^3l\,d^3p'}{l_0 p_0'}. \quad (34)$$

The integration over l and s replaces the delta function by the factor $-l_0/(kl)$, and the change from the variable p_3' to the variable $\gamma' = p_0' - p_3'$ leads to an expression of the type (24). Furthermore using the same change from the variables p_1', p_2', γ' to the variables ψ, τ, χ' and canceling the infinite integral $\int_{-\infty}^{\infty} d\psi$ by the factor $2\pi\delta(0)$, as in the case of emission from an electron, we obtain the final expression for the decay probability in a crossed field

$$F(\chi) = \frac{G^2 f^2 m^2 m'^2}{\pi^2 p_0}\int_0^\infty dv \int_0^\infty du \frac{\operatorname{sh} v}{\operatorname{ch}^3 v}\operatorname{ch} u\left(\frac{\operatorname{sh}^2 v}{2\chi}\right)^{1/3}\Big\{\Delta\Phi^2(y) + \operatorname{sh}^2 v(\operatorname{ch}^2 u - \Delta\operatorname{cth}^2 v)\Big[\Phi^2(y) + \frac{1}{y}\Phi'^2(y)\Big]\Big\}, \quad (35)$$

if one puts $\tau = \sinh u$, $\chi/\chi' = \cosh^2 v$. Here

$$y = \left(\frac{\operatorname{sh}^2 v}{2\chi}\right)^{2/3}(\operatorname{ch}^2 u - \Delta\operatorname{cth}^2 v), \quad \chi = e\sqrt{(F_{\mu\nu}p_\nu)^2}/m^3.$$

In conclusion we thank V. L. Ginzburg for stimulating discussions of this work. We are grateful also to L. V. Pariĭskaya for carrying out the numerical computations.

APPENDIX

THE ASYMPTOTIC EXPANSION OF $F(\chi)$ FOR SMALL χ.

In Eq. (8) we change from the variables u, v to the variables y, $\alpha = \sinh^{4/3} v(1 - \Delta\coth^2 v)$, respectively. Then $F(\chi)$ takes the form

$$F(\chi) = \frac{G^2 f^2 m^2 m'^2 c}{2\pi^2}\Big\{\Delta(2\chi)^{1/3}\int_{-\infty}^{\infty} dy\,\Phi^2(y) f((2\chi)^{2/3}y) + \chi\int_{-\infty}^{\infty} dy\,[\Phi^2(y)]'' g((2\chi)^{2/3}y)\Big\}, \quad (A1)$$

where

$$f(z) = \int_{-\infty}^{z} d\alpha\, A(\alpha)(z-\alpha)^{-1/2}, \quad g(z) = \int_{-\infty}^{z} d\alpha B(\alpha)(z-\alpha)^{-1/2},$$

and $A(\alpha) = (\tfrac{1}{2})d\tanh^2 v/d\alpha$, and $B(\alpha) = (\tfrac{1}{2})\sinh^{2/3} v\, d\tanh^2 v/d\alpha$ are implicit functions of α. It is not difficult to see that $A(\alpha) \sim \alpha^{-4}$ as $\alpha \to -\infty$, $\sim \alpha^{-5/2}$ as $\alpha \to \infty$, and is finite for $\alpha = 0$; similarly $B(\alpha) \sim \alpha^{-5}$ as $\alpha \to -\infty$, as α^{-2} for $\alpha \to \infty$ and is finite for $\alpha = 0$.

We consider first the leading term in (A1). We represent the integral over y as a sum of an integral J_1 between the limits $-\infty$ and 0 and an integral J_2 between the limts 0 and ∞. Since $\Phi^2(y)$ decreases exponentially as $y \to \infty$, the function $f(z)$ in the integral J_2 may be expanded in a series around $z = 0$. We then obtain a series in powers of χ for J_2:

$$J_2 \equiv \int_0^\infty dy\,\Phi^2(y) f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\int_0^\infty dy\, y^n \Phi^2(y)(2\chi)^{2n/3},$$

$$f^{(n)}(0) = \int_{-\infty}^{0}\frac{A^{(n)}(\alpha)\,d\alpha}{\sqrt{-\alpha}}. \quad (A2)$$

The integrals $\int_0^\infty dy\, y^n\Phi^2(y)$ can, after integration by parts and application of equation $\Phi'' = y\Phi$, be expressed in terms of $\Phi(0)$ and $\Phi'(0)$; they can also be found in [5] if one uses the representation $\Phi(y) = \sqrt{y/3\pi}K_{1/3}((2/3)y^{3/2})$. Such a procedure can not be used for the integral J_1, because of the slow fall-off of $\Phi^2(y)$ as $y \to -\infty$. However if one subtracts from Φ^2 the first term of its asymptotic expansion as $y \to -\infty$: $S_0(y) = (\tfrac{1}{2})\sqrt{-y}$ [6], the remaining function $\Phi^2 - S_0$ falls off at $-\infty$ more rapidly and we may make use of the partial expansion of $f(z)$ in powers of z. In other words we can write

$$J_1 \equiv \int_{-\infty}^{0} dy \Phi^2(y) f(z) = \int_{-\infty}^{0} dy S_0(y) f(z) + f(0) \int_{-\infty}^{0} dy\,[\Phi^2(y) - S_0(y)] + f'(0) \int_{-\infty}^{0} dy\,[\Phi^2(y) - S_0(y)]\,z + \int_{-\infty}^{0} dy\,(\Phi^2 - S_0)\,[f(z) - f(0) - f'(0)z]. \quad \text{(A3)}$$

The final "remainder" term can again be subjected to a similar procedure with the difference that now the function $\Phi^2 - S_0$ plays the role of Φ^2, and the role of $f(z)$ is played by $f(z) - f(0) - f'(0)z$, etc. The series obtained in this way for J_1 consists of terms in which χ is already clearly separated out and "special" terms

$$\int_{-\infty}^{0} S_0(y) f(z)\,dy, \quad \int_{-\infty}^{0} S_1(y)\,[f(z) - f(0) - f'(0)z]\,dy, \quad \int_{-\infty}^{0} S_2(y)\left[f(z) - f(0) - f'(0)z - \frac{1}{2!} f''(0) z^2\right], \ldots \quad \text{(A4)}$$

which require further analysis (here $S_i(y)$ are the successive terms in the asymptotic expansion of the function $\Phi^2(y)$, cf. [6]). The first of these "special" terms can be evaluated exactly and is equal to $(1/4)\pi\Delta(2\chi)^{-1/3}$. The remaining "special" terms may be calculated only asymptotically because of the presence of sinusoidal terms in $S_i(y)$, $i \geq 1$. Carrying this evaluation out accurate to terms which fall off faster than $\chi^{5/3}$, we obtain

$$J_1 = \frac{\pi\Delta}{4}(2\chi)^{-1/3} - f(0)\Phi'^2(0) + \frac{1}{3} f'(0)\Phi(0)\Phi'(0)(2\chi)^{2/3} - \frac{1}{2} f''(0)\left[\frac{1}{5}\Phi^2(0) + \frac{1}{12}\left(\frac{3}{4}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\right](2\chi)^{4/3} + \frac{\pi(3\Delta - 2)}{24}(2\chi)^{5/3} + \ldots \quad \text{(A5)}$$

Summing the series (A2) and (A5) we obtain an asymptotic series for the first integral in (A1):

$$\int_{-\infty}^{\infty} dy\,\Phi^2(y) f(z) = \frac{\pi\Delta}{4}(2\chi)^{-1/3} - \frac{1}{24} f''(0) \times \left(\frac{3}{4}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)(2\chi)^{4/3} + \frac{\pi(3\Delta - 2)}{24}(2\chi)^{5/3} + \ldots \quad \text{(A6)}$$

A double integration by parts transforms the second integral in (A1) into the first integral, with the function

$$g''(z) = \int_{-\infty}^{z} d\alpha\, B''(\alpha)(z - \alpha)^{-1/2}.$$

playing the role of $f(z)$.

It is sufficient to calculate the first term in the expansion

$$\int_{-\infty}^{\infty} dy\,[\Phi^2(y)]''\,g(z) = (2\chi)^{4/3}\int_{-\infty}^{\infty} dy\,\Phi^2(y)\,g''(z) = \frac{\pi(1 - 2\Delta)}{4} 2\chi + \ldots \quad \text{(A7)}$$

Using (A6) and (A7) in (A1) we obtain Eq. (9).

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