INVARIANT EXPANSIONS OF RELATIVISTIC AMPLITUDES

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The problem of expanding a function into integrals over orthogonal systems of eigenfunctions of the Laplace operator on a hyperboloid is considered. Expressions for the direct and inverse expansions are given in four orthogonal coordinate systems in Lobachevskiĭ space. The connection of these expansions with expansions of functions on a cone is exhibited. The expansions which are obtained are in general not unitary and become unitary for certain values of the quantum numbers.

1. INTRODUCTION

WHEN properties of relativistically invariant functions are used, in particular properties of the scattering amplitude, the geometric properties of the manifold (velocity space) on which these functions are defined are usually not utilized.

It is known that the space of 4-velocities of the theory of relativity is described by a Lobachevskian geometry. The geometric properties of the space determine those properties of the scattering amplitude which are independent of the concrete form of the interaction and refer to its kinematic properties. Since the velocity space possesses a constant negative curvature (equal to $1/c$), the kinematic properties of the relativistic amplitude lead to a series of interesting problems.

One of these problems is the investigation of invariant expansions of functions.

In nonrelativistic theory the variables in the Schrödinger equation (or, what amounts to the same thing, in the Laplace operator) separate in any elliptic orthogonal coordinate system. Special cases of such coordinate systems are Cartesian, cylindrical, and spherical coordinates. In these three coordinate systems, functions can be expanded in Fourier integrals, Bessel functions, or spherical functions, respectively. In other coordinate systems the expansion leads to more complicated functions (for instance, when the problem of scattering by a Coulomb potential is considered in parabolic coordinates).

The relativistic amplitude can be expanded in a Fourier integral in four-dimensional space. In this case the relation between the components of the momentum four-vector (or the four-velocity) is taken into account by including a delta function. It is convenient, however, to make use only of independent components of the velocity and to go over from a Minkowski space to a Lobachevskiĭ space, realized as the upper sheet of the double-sheeted hyperboloid $u^2 = 1$.

In the present paper we consider the problem of expanding relativistic functions in series and integrals of eigenfunctions of the Laplace operator on Lobachevskiĭ space.

The problem consists in studying the Laplace operator on the double-sheeted hyperboloid (or the angular part of the D'Alembert operator), in various coordinate systems. Later we will consider a similar problem for a single-sheeted hyperboloid[1)]. In investigating these problems it is extremely useful to employ geometric constructs which make more intuitive the whole way of reasoning. An essential role in this exposition is played by the expansion on a cone, first obtained by Shapiro[1,2]. Using methods of integral geometry, Gel'fand and Graev[3,4] have constructed a general theory of such expansions. These methods allow one to replace the analysis of functions on a hyperboloid by an analysis of functions on a cone, where essentially the expansions reduce to ordinary Fourier transforms. At the same time a connection with the theory of representations of the Lorentz group is established[2)].

The systems of eigenfunctions which are constructed are generalizations of nonrelativistic systems: the spherical, cylindrical and Cartesian systems. However, one of the systems described here, which is called cylindrical throughout this

[1)]A single-sheeted hyperboloid corresponds to an unphysical region of the variables, which is also of interest.

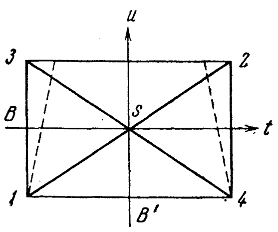
[2)]The expansion of relativistic functions has also been studied by Dolginov[5,6], but the whole system of functions was not constructed in his papers.

paper, has no analog in nonrelativistic mechanics. Its eigenfunctions are not expressible in terms of Legendre and Bessel functions, but can be expressed in terms of the general hypergeometric function.

A function defined on the cone ($k^2 = 0$) describes a particle of zero mass. Therefore we can say that the problem posed in the present paper consists in finding expansions of the amplitude for a particle of nonvanishing mass in terms of amplitudes for particles with $m = 0$.

2. THE SCATTERING AMPLITUDE

The scattering of particles, or a binary reaction, is described by a function of four variables: the velocities of the two particles in the initial and the final states. Out of the twelve scalar variables, ten can be chosen arbitrarily, owing to the invariance of the scattering amplitude with respect to transformations of the Lorentz group. A convenient way to describe the kinematics is the kinematic diagram which has been considered in preceding papers[7,8]. The velocities of the four particles are represented by four points on the upper sheet of a double-sheeted hyperboloid. Due to the invariance of the quadrangle formed in this way, it



can be placed in any convenient position (six parameters are used for this). We place the quadrangle so that its center is situated in the vertex of the hyperboloid, i.e., the point with coordinates (1, 0, 0, 0). We direct one of the coordinate axes along the normal to the hyperboloid and construct the two other axes as follows. The three 4-vectors[3)] $A_s = p_1 + p_2$, $A_t = p_1 - p_4$ and $A_u = p_1 - p_3$, and the 4-vector A_n, which is orthogonal to them, form an orthogonal coordinate system. The lines of intersection of these coordinate surfaces with our hyperboloid will be chosen as a coordinate system on the hyperboloid.

The velocity quadrangle is represented in the figure. The points 1 and 2 correspond to the initial velocities, the points 3 and 4 correspond to the final velocities. The point s represents the velocity of the center of inertia of the system. One of the coordinate axes bisects the scattering angle. As has been shown in[8] the points of intersection of the sides 14 and 23 are situated on a single-sheeted hyperboloid and correspond to the velocity of the center of inertia of the crossed reaction. In the equal mass case the point of intersection of sides 13 and 24 corresponds to the velocity of the center of inertia of the u-channel. In this case the coordinate axes coincide with the directions of the lines st and su, and the three points s, t, and u form a so-called autopolar triangle. In the case of unequal masses (the dotted lines in the figure) the connection between the coordinate axes and the u-system is less direct.

The directions of the coordinate axes A_s, A_t, A_u, and A_n vary with the energy and scattering angles. Specification of the velocity of one of the particles (e.g., the first) in such a coordinate system completely determines the other three velocities; this is due to the energy-momentum conservation laws (four other parameters of the Lorentz group).

Thus the argument of the scattering amplitude is defined by the position of one point on the hyperboloid, in the coordinate system defined by the vectors A. We agree to denote the angle between the direction of the velocity and one of the axes by $\theta/2$ (θ is the scattering angle $0 \leq \theta \leq 2\pi$). The coordinates of the point 1 can be chosen in various ways and this leads to different systems of eigenfunctions.

3. THE COORDINATE SYSTEM AND THE D'ALEMBERT OPERATOR

As has already been said, the Lobachevskian velocity space is realized in the form of the three-dimensional surface of the upper sheet of the double-sheeted hyperboloid $u^2 = 1$ in Minkowski space. We intersect this hyperboloid with a plane parallel to one of the coordinate planes. The intersection will again be a (two-dimensional) hyperboloid, if the plane does not pass through the 0-axis, otherwise the intersection will be a two-dimensional sphere, and we can construct on this sphere a spherical coordinate system. In the case of the hyperboloid, one can continue the sections in two different ways: either along circles or along hyperbolas. Thus we obtain two other hyperbolic coordinate systems, to which, as will be seen, correspond different eigenfunctions of the D'Alembert operator.

There exists one other coordinate system, connected with the fact that two mutually orthogonal planes can be constructed in Minkowski space,

3)The squares $A_s^2 = s$, $A_t^2 = t$, and $A_u^2 = u$ are the Mandelstam variables.

one with a euclidean and the other with a pseudo-euclidean metric. Introducing in the first plane polar coordinates and in the second plane hyperbolic coordinates, we obtain a fourth coordinate system for the Lobachevskiĭ space, i.e., spheres with their centers (in the imaginary Lobachevskiĭ space, on the single-sheeted hyperboloid). If the center of the sphere is placed on the light cone, the coordinate surface is the so called "orisphere", a surface for which the geometry is isomorphic to the geometry of the euclidean plane. This orispheric coordinate system will also be described in the present paper.

The physical significance of the enumerated coordinate systems is easily understood if one observes that the transformation from one reference frame to any other can be achieved by means of three transformations, at least one of which is a Lorentz transformation.

In the spherical coordinate system a point is characterized by two spatial rotations and one Lorentz rotation, whereas in the other coordinate systems two or three Lorentz transformations are needed. We note that for the description of binary reactions without spin, for which the amplitude is independent of one (space) angle, the purely hyperbolic system (three Lorentz rotations) is inconvenient[4)].

We will describe the coordinate systems in velocity space by specifying four homogeneous Cartesian coordinates of the velocity u_0, u by means of angles. Since in this case a three-dimensional point is specified by four numbers, these numbers represent projective coordinates of the point. Besides the projective coordinates we introduce also inhomogeneous coordinates, obtained by dividing the projective coordinates by u_0.

Since the ratios u_i/u_0, i = 1, 2, 3 vary within 0 and 1, it is convenient to denote them by tanh z_i, where now $0 \le z_i \le \infty$. Thus, we select as inhomogeneous coordinates the coordinates z_i, defined by the equations

$$\text{th } z_i = u_i//u_0 \quad (i = 1.\ 2.\ 3). \tag{3.1)*}$$

Finally, we use a convention in which spatial rotations are denoted by Greek letters and hyperbolic angles by Latin letters.

We go over now to a description of the coordinate systems.

The Spherical System S

The system S is defined by the equations

$$u_0 = \text{ch } a, \qquad u_2 = \text{sh } a \sin\theta \cos\varphi,$$
$$u_3 = \text{sh } a \cos\theta, \qquad u_1 = \text{sh } a \sin\theta \sin\varphi, \tag{3.2}$$

or

$$\text{th } z_3 = \text{th } a \cos\theta, \qquad \text{th } z_2 = \text{th } a \sin\theta \cos\varphi,$$
$$\text{th } z_1 = \text{th } a \sin\theta \sin\varphi. \tag{3.3}$$

The inhomogeneous coordinates coincide with three-dimensional spherical coordinates, in which tanh a plays the role of the radial variable. Therefore in the S system the equations resemble most closely the nonrelativistic ones. The D'Alembert operator in four-dimensional Minkowski space can be written in the form

$$\Box = \frac{1}{U^3}\frac{\partial}{\partial U} U^3 \frac{\partial}{\partial U} - \Delta_L, \tag{3.4}$$

where U is the length of the four-vector in Minkowski space and Δ_L is the Laplace operator on the hyperboloid[5)], for which we have to find the eigenfunctions.

In the coordinate system S

$$\Delta_L = \text{sh}^{-2} a \frac{\partial}{\partial a} \text{sh}^2 a \frac{\partial}{\partial a} + \text{sh}^{-2} a \left(\sin^{-1}\theta \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \sin^{-2}\theta \frac{\partial^2}{\partial\varphi^2}\right). \tag{3.5)*}$$

The Hyperbolic Coordinate System (the Lobachevskiĭ System L)

The hyperbolic coordinate system has been considered already by Lobachevskiĭ, and the corresponding coordinates are called Lobachevskian coordinates (the coordinates a, b and c):

$$u_0 = \text{ch } a \text{ ch } b \text{ ch } c, \qquad u_3 = \text{ch } a \text{ ch } b \text{ sh } c,$$
$$u_2 = \text{ch } a \text{ sh } b, \qquad u_1 = \text{sh } b; \tag{3.6}$$

$$\text{th } z_3 = \text{th } c, \qquad \text{th } z_2 = \text{th } b/\text{ch } c,$$
$$\text{th } z_1 = \text{th } a/\text{ch } b \text{ ch } c. \tag{3.7}$$

The homogeneous coordinates u_0, u are called Weierstrass coordinates. In this system all three angles are hyperbolic.

The Laplace operator is given by the expression

$$\Delta_L = \text{ch}^{-2} a \frac{\partial}{\partial a} \text{ch}^2 a \frac{\partial}{\partial a} + \text{ch}^{-2} a \left(\text{ch}^{-1} b \frac{\partial}{\partial b} \text{ch } b \frac{\partial}{\partial b} + \text{ch}^{-2} b \frac{\partial^2}{\partial c^2}\right). \tag{3.8}$$

[4)]The problem of finding the coordinate systems in which the variables in the Klein-Gordon equation separate has been considered in papers by Olevskiĭ[9] and Shum[10].

*th = tanh, sh = sinh, ch = cosh, cth = coth.

[5)]I. e., a second-order differential operator on the hyperboloid, commuting with the Lorentz transformations.

*Note that in this article $\sin^{-1} = 1/\sin$ and $\text{ch}^{-1}\, b = 1/\cosh b$ – Tr.

The Hyperbolic System H

This system differs from the preceding one in that one of its angles is spatial:

$$u_0 = \operatorname{ch} a \operatorname{ch} b, \qquad u_3 = \operatorname{ch} a \operatorname{sh} b \cos\varphi,$$
$$u_2 = \operatorname{ch} a \operatorname{sh} b \sin\varphi, \qquad u_1 = \operatorname{sh} a \tag{3.9}$$

and the inhomogeneous coordinates have the form

$$\operatorname{th} z_3 = \operatorname{th} b \cos\varphi, \qquad \operatorname{th} z_2 = \operatorname{th} b \sin\varphi,$$
$$\operatorname{th} z_1 = \operatorname{th} a / \operatorname{ch} b. \tag{3.10}$$

In terms of the inhomogeneous coordinates this coordinate system resembles a cylindrical coordinate system with the (plane) radial variable tanh b. The coordinate plane $z_1 = 0$ is orthogonal to the z_1 axis and the distance a is measured from that plane.

For the Laplace operator we obtain

$$\Delta_L = \operatorname{ch}^{-2} a \frac{\partial}{\partial a} \operatorname{ch}^2 a \frac{\partial}{\partial a}$$
$$+ \operatorname{ch}^{-2} a \left(\operatorname{sh}^{-1} b \frac{\partial}{\partial b} \operatorname{sh} b \frac{\partial}{\partial b} + \operatorname{sh}^{-2} b \frac{\partial^2}{\partial\varphi^2}\right). \tag{3.11}$$

The Cylindrical System C

The system C is very similar to H:

$$u_0 = \operatorname{ch} b \operatorname{ch} a, \qquad u_3 = \operatorname{sh} b \cos\varphi,$$
$$u_2 = \operatorname{sh} b \sin\varphi, \qquad u_1 = \operatorname{ch} b \operatorname{sh} a. \tag{3.12}$$

The similarity becomes apparent in terms of inhomogeneous coordinates

$$\operatorname{th} z_3 = \operatorname{th} b \cos\varphi, \qquad \operatorname{th} z_2 = \operatorname{th} b \sin\varphi,$$
$$\operatorname{th} z_1 = \operatorname{th} a. \tag{3.13}$$

We see that the difference consists in the third (the axial) coordinate. In the system C the distance between planes is measured along the axis of the cylinder, so that the coordinate surfaces are equidistant planes.

In the coordinate system the operator Δ_L has the form

$$\Delta_L = \operatorname{ch}^{-1} b \operatorname{sh}^{-1} b \frac{\partial}{\partial b} \operatorname{ch} b \operatorname{sh} b \frac{\partial}{\partial b}$$
$$+ \operatorname{ch}^{-2} b \frac{\partial^2}{\partial a^2} + \operatorname{sh}^{-2} b \frac{\partial^2}{\partial\varphi^2}. \tag{3.14}$$

The "Orispheric" Coordinate System O

Finally, we consider the "orispheric" coordinate system:

$$u_0 = \tfrac{1}{2}\left[e^{-a} + (r^2 + 1)e^a\right], \qquad u_3 = \tfrac{1}{2}\left[e^{-a} + (r^2 - 1)e^a\right],$$
$$u_2 = re^a \cos\varphi, \qquad u_1 = re^a \sin\varphi. \tag{3.15}$$

In this coordinate system the coordinate surface a = constant is an "orisphere" in the Lobachevskiĭ space.

In the "orispheric" coordinate system the Laplace operator has the following form:

$$\Delta_L = \frac{\partial^2}{\partial a^2} + 2\frac{\partial}{\partial a} + e^{-2a}\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\varphi^2}\right). \tag{3.16}$$

It is easy to see that the C system can be obtained from the O system by means of the following coordinate transformation

$$re^a = \operatorname{sh} b, \qquad e^a / \operatorname{ch} b = \operatorname{ch} a_C, \tag{3.17}$$

where a_C denotes the coordinate a in (3.12). After carrying out the transformation (3.17), the two systems differ only by a permutation of the axes.

Let us now consider the kinematic diagram in the coordinate systems that were introduced. Since we restrict ourselves to binary reactions, and accordingly to plane diagrams only, we will not consider the Lobachevskian system.

In the S system the coordinates will be: the angle θ (one half of the scattering angle in the c.m.s.) and the length of the segment s1, which determines the energy of the particle 1 in the c.m.s.

In the H system the point 1 is determined by the segment sB′, i.e., the velocity of the Breit system with respect to the c.m.s., and by the length of the segment B′1, which determines the energy of this particle in the Breit system.

In the C system the particle 1 is determined by the segment sB, i.e., the velocity of the "recoil" Breit system (the system in which the sum of the momenta of the particle 1 and the recoil particle vanishes). The S, H, and C systems thus reflect the symmetries of the s-, t-, and u-channels, respectively.

In the O system, the coordinate a is the distance from the "orisphere" with the center in the point k(1, 1, 0, 0), defined by the equation a = ln (ku). A scalar product of the type ku is often used as an independent variable in problems of quantum electrodynamics. Two other variables u_2 and u_1 determine together with a, an invariant rectangular three-dimensional cartesian system in Lobachevskiĭ space.

4. THE EIGENFUNCTIONS

In this section we obtain the eigenfunctions of the Laplace operator on the hyperboloid in the five systems S, L, H, C and O. We use the usual method of separation of the variables. It is known (cf. [11,12]) that the eigenvalues of the n-dimensional Laplace operator are $-l(l + n - 2)$. For n = 4 the eigenvalues is $-l(l + 2)$. When one goes over from the sphere to the hyperboloid, l is no longer an integer but can take on any complex value. Since in this case the sign of Δ is negative [cf. Eq. (3.4)] the eigenvalue equation will have the form

$$\Delta_L f = \sigma(\sigma + 2) f. \tag{4.1}$$

In order that $\sigma(\sigma + 2)$ be real it is necessary that either

$$\sigma = -1 + ip \quad (p - \text{real}), \tag{4.2}$$

or

$$\sigma - \text{real}. \tag{4.3}$$

It is known from the theory of group representations that these values correspond to unitary representations of the Lorentz group. Here Eq. (4.2) corresponds to the fundamental series of representations and (4.3) corresponds to the secondary series (cf.[13]).

It is convenient to carry out the substitution $\sigma = 1 + ip$, after which real values of p correspond to unitary representations (of the fundamental series) of the Lorentz group. In this case the eigenvalue is $\sigma(\sigma + 2) = -(p^2 + 1)$ and we can write (4.1) in the form

$$\Delta_L f_p = -(p^2 + 1) f_p. \tag{4.4}$$

The following calculations are aimed at finding the other quantum numbers. We note that in the case of a two-dimensional hyperboloid the eigenvalues are $\sigma(\sigma + 1)$; the reality condition yields $\sigma = 1/2 + iq$ (with q real), and the eigenvalue is $\sigma(\sigma + 1) = -(q^2 + 1/4)$.

We start with the S system. Separating the coordinates θ and φ by means of an ordinary spherical function (the factor $P_{lm}(\theta, \varphi)\, e^{im\varphi}$, $m = 0, \pm 1, \ldots$), we obtain the equation

$$\left[\frac{1}{\operatorname{sh}^2 a}\frac{d}{da}\operatorname{sh}^2 a\frac{d}{da} - \frac{l(l+1)}{\operatorname{sh}^2 a}\right] A(a) = -(p^2 + 1) A(a). \tag{4.5}$$

It is easy to check that the solution of this equation has the form

$$A(a) = (\operatorname{sh} a)^{-1/2} P_{-1/2+ip}^{-(l+1/2)}(\operatorname{ch} a). \tag{4.6}$$

Thus the (unnormalized) eigenfunctions in the S system are

$$\langle p, l, m \mid a, \theta, \varphi\rangle = (\operatorname{sh} a)^{-1/2} P_{-1/2+ip}^{-(l+1/2)}(\operatorname{ch} a) P_{lm}(\theta, \varphi)\, e^{im\varphi}. \tag{4.7}$$

The expansion with respect to the coordinate a is a generalization of the well-known Moller-Fock expansion[14].

In the Lobachevskiĭ system we can eliminate the coordinate c by means of the function e^{imc} (with m real; this is a Fourier expansion). Then we obtain for the determination of the eigenfunctions in the L system the two equations

$$\left(\frac{d^2}{da^2} + \frac{2}{\operatorname{cth} a}\frac{d}{da} - \frac{q^2 + 1/4}{\operatorname{ch}^2 a}\right) A(a) = -(p^2 + 1) A(a), \tag{4.8}$$

$$\left(\frac{d^2}{db^2} + \frac{1}{\operatorname{cth} b}\frac{d}{db} - \frac{m^2}{\operatorname{ch}^2 b}\right) B(b) = -\left(q^2 + \frac{1}{4}\right) B(b). \tag{4.9}$$

The solutions of these equations can be put in the form

$$A(a) = (\operatorname{ch} a)^{-1} P_{-1/2+iq}^{-ip}(\operatorname{th} a), \tag{4.10}$$

$$B(b) = (\operatorname{ch} b)^{-1/2} P_{-1/2+im}^{-iq}(\operatorname{th} b), \tag{4.11}$$

and the eigenfunctions are

$$\langle p, q, m \mid a, b, c\rangle = (\operatorname{ch} a \sqrt{\operatorname{ch} b})^{-1} P_{-1/2+iq}^{-ip}(\operatorname{th} a)\, P_{-1/2+im}^{-iq}(\operatorname{th} b)\, e^{imc}. \tag{4.12}$$

In the next system, H, the operator which is included in brackets in Eq. (3.11) has the eigenvalue $-(q^2 + 1/4)$ and the eigenfunctions

$$P_{-1/2+iq}^{im}(\operatorname{ch} b)\, e^{im\varphi}, \tag{4.13}$$

where m is an integer. Replacing the bracket by the eigenvalue, we obtain the equation

$$\left(\frac{d^2}{da^2} + \frac{2}{\operatorname{cth} a}\frac{d}{da} - \frac{q^2 + 1/4}{\operatorname{ch}^2 a}\right) A(a) = -(p^2 + 1) A(a). \tag{4.14}$$

The solutions of this equation are the functions

$$A(a) = (\operatorname{ch} a)^{-1} P_{-1/2+iq}^{ip}(\operatorname{th} a). \tag{4.15}$$

We thus obtain for the H system:

$$\langle p, q, m \mid a, b, \varphi\rangle = (\operatorname{ch} a)^{-1} P_{-1/2+iq}^{ip}(\operatorname{th} a)\, P_{-1/2+iq}^{im}(\operatorname{ch} b)\, e^{im\varphi}. \tag{4.16}$$

In the "orispheric" coordinate system we separate the variables r and φ by means of the Bessel function $J_m(\kappa r)\, e^{im\varphi}$. Equation (3.18) reduces to the form

$$\left[\frac{d^2}{db^2} - \frac{1}{b}\frac{d}{db} + \left(-\varkappa^2 + \frac{p^2 + 1}{b^2}\right)\right] B(b) = 0. \tag{4.17}$$

Its solution will be a Macdonald function (Bessel function of imaginary index and imaginary argument):

$$B(b) = \varkappa b\, K_{i\nu}(\varkappa b). \tag{4.18}$$

Thus in the O system the eigenfunctions will be

$$\langle \nu, \varkappa, m \mid b, r, \varphi\rangle = \varkappa b\, K_{i\nu}(\varkappa b)\, J_m(\varkappa r)\, e^{im\varphi}. \tag{4.19}$$

Finally, we consider the cylindrical system. Separating the variables φ and a in the operator (3.14) by means of the functions $e^{s\sin\varphi}$ and $e^{i\tau a}$, we obtain the equation

$$\frac{d^2B}{db^2} + (\operatorname{th} b + \operatorname{cth} b)\frac{dB}{db} + \left[p^2 + 1 - \frac{m^2}{\operatorname{sh}^2 b} - \frac{\tau^2}{\operatorname{ch}^2 b}\right] B = 0. \tag{4.20}$$

The solution of Eq. (4.20) is a product of a hyperbolic and a hypergeometric function:

$$B(b) = \mathrm{th}^m b\, \mathrm{ch}^{-1+ip} b\, F\left(\frac{m+1-ip+i\tau}{2},\right.$$

$$\left.\frac{m+1-ip-i\tau}{2}, m+1;\quad \mathrm{th}^2 b\right), \tag{4.21}$$

and the eigenfunctions in the C system are

$$\langle p, \tau, m \mid b, a, \varphi\rangle = e^{i(\tau a + m\varphi)} \frac{\mathrm{sh}^m b}{\mathrm{ch}^{m+1-ip} b}$$

$$\times F\left(\frac{m+1-ip+i\tau}{2}, \frac{m+1-ip-i\tau}{2}, m+1; \mathrm{th}^2 b\right). \tag{4.22}$$

This concludes the construction of sets of orthogonal functions in the five coordinate systems. It can be shown that the four systems S, H, O, and C exhaust all coordinate systems possessing axial symmetry and constructed out of spheres, "orispheres," and hyperboloids. (The L system does not possess this property and has been treated here only because it is very similar to the other ones; the remaining coordinate systems also include ellipsoidal coordinate surfaces.) It will be shown below that the restriction to orthogonal systems is not mandatory, and that one can carry out expansions also with respect to non-unitary representations (where the quantum number p is an arbitrary complex number).

5. THE "ORISPHERE" METHOD AND EXPANSIONS ON THE CONE

We have constructed various eigenfunction systems for the Laplace operator on the double-sheeted hyperboloid. We must now normalize these functions, or what is equivalent, we must find inversion formulas for the expansions with respect to these functions. The derivation of all these inversion formulas is based on the method of "orispheres," developed by Gel'fand and Graev.

In order to explain this method, we consider the classical Fourier transform in n-dimensional space:

$$F(k) = \int f(x)\, e^{ikx}\, d^n x, \tag{5.1}$$

where (only in this and the following equation)

$$kx = k_1 x_1 + k_2 x_2 + \ldots + k_n x_n.$$

The expression (5.1) can be reduced to integrations over planes:

$$\Phi(k, p) = \int f(x)\, \delta(kx - p)\, d^n x \tag{5.2}$$

followed by the one-dimensional Fourier transformation

$$F(k) = \int \Phi(k, p)\, e^{ipk}\, dp. \tag{5.3}$$

The integral transformation (5.2) (the integration over planes) is called a Radon transform.

Gel'fand and Graev[3] have shown that invariant expansions of functions on any homogeneous manifold can be decomposed similarly. In this case the role of the planes is taken by the so-called "orispheres". For the upper sheet of the double-sheeted hyperboloid $u^2 = 1$, $u^2 = u_0^2 - u_1^2 - \ldots - u_n^2$ the "orispheres" are the sections of the hyperboloid by planes of equation $uk = 1$, where k is a point on the cone $k^2 = 0$ (these planes are parallel to the generators of the cone).

With each function f(u) defined on the hyperboloid we associate a function on the cone

$$h(k) = \int f(u)\, \delta(uk - 1) \frac{d^n u}{u_0}, \tag{5.4}$$

$$d^n u = du_1 \ldots du_n, \tag{5.5}$$

where $d^n u/u_0$ is the invariant measure on the hyperboloid. The mapping of f(u) on h(k) will be called the Gel'fand-Graev integral transform.

For $n = 2m + 1$ the inversion of this transformation has the form

$$f(u) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int \delta^{(n-1)}(uk - 1)\, h(k) \frac{d^n k}{k_0}, \tag{5.6}$$

and for $n = 2m$ it has the form[6)]

$$f(u) = \frac{(-1)^{n/2}\Gamma(n)}{(2\pi)^n} \int (ku - 1)^{-n} h(k) \frac{d^n k}{k_0}. \tag{5.7}$$

Here

$$\frac{d^n k}{k_0} = \frac{dk_1 \ldots dk_n}{k_0} \tag{5.8}$$

is the invariant measure on the cone.

It is convenient to consider functions on the cone because the concept of homogeneous function is meaningful on the cone. The expansion of a function in homogeneous components $\Phi(k, \sigma)$ has the form[7)]

$$h(k) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \Phi(k, \sigma)\, d\sigma, \tag{5.9}$$

$$\Phi(\sigma) = \int h(tk)\, t^{-\sigma - 1}\, dt. \tag{5.10}$$

It follows from Eqs. (5.4) and (5.10) that

$$\Phi(k, \sigma) = \int f(u)\, (uk)^{\sigma} \frac{d^n u}{u_0}. \tag{5.11}$$

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6)Here the divergent integral has to be understood in the sense of its regularized value, obtained by means of analytic continuation in the power exponent.

7)The value of δ is chosen so that all poles of the function $\Phi(k, \sigma)$ be situated outside the strip $0 \leq \mathrm{Re}\, \sigma \leq \delta$.

From Eqs. (5.6), (5.7) and (5.9) it follows that, for $n = 2m + 1$

$$f(u) = \frac{(-1)^{(n-1)/2}}{2i(2\pi)^n} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\sigma+n-1)}{\Gamma(\sigma)} \int_\Gamma \Phi(k,\sigma)(uk)^{-\sigma-n+1} d^{n-1}k\, d\sigma \tag{5.12}$$

and for $n = 2m$

$$f(u) = -\frac{(-1)^{n/2-1}}{2i(2\pi)^n} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\sigma+n-1)}{\Gamma(\sigma)}$$

$$\times \operatorname{ctg}\pi\sigma \int_\Gamma \Phi(k,\sigma)(uk)^{-\sigma-n+1} d^{n-1}k\, d\sigma. \tag{5.13}$$

The integration contour Γ is an arbitrary path on the cone, which intersects all the generators of the cone, and $d^{n-1}k$ is the measure on this cone, defined by the equality $d(tk) = t^{n-3}dt\, dk$. For $\sigma = -(n-1)/2 + ip$, where p is real, the integral transformation (5.11) and (5.12) is unitary.

On the double-sheeted hyperboloid the equations so obtained[8)] are the analogs of the Fourier expansion of functions defined in n-dimensional euclidean space. The analogy becomes even more obvious if one notes that $uk = e^{l(u,k)}$, where $l(u,k)$ is the distance from the point u to the orisphere $uk = 1$.

The further expansion of the function is now carried out on the contour Γ. To different choices of this contour will correspond different expansions. We restrict ourselves to the physically interesting case of $n = 3$. In this case the following choices of the contour are of interest:

a) Γ is the intersection of the cone by the plane $k_0 = 1$, i.e. the sphere $k_1^2 + k_2^2 + k_3^2 = 1$, $k_0 = 1$;

b) Γ is the section of the cone by the planes $k_3 = 1$ and $k_3 = -1$, i.e., the upper sheets of hyperboloids;

c) Γ is the section of the cone by the plane $k_0 + k_3 = 2$ (a paraboloid);

d) Γ is the section of the cone by the cylinder $k_0^2 - k_1^2 = 1$.

These types of sections correspond to the different coordinate systems S, H, O and C considered above. Case a) corresponds to the S system, case b) to the H system, case c) to the O system, and case d) to the C system.

6. DERIVATION OF THE INVERSION FORMULAS

We now derive the inversion formulas belonging to the eigenfunction systems of the Laplace operator which were derived in Sec. 4.

[8)]For $n = 3$ these equations (5.11) and (5.12) have been obtained before Gel'fand and Graev by I. Shapiro[1].

The Inversion Formula in the S System

We start from the case when the section Γ is the sphere $k_0 = 1$. In this case the function $\Phi(k,\sigma)$ is defined on the sphere and therefore can be expanded in a series of spherical harmonics

$$\Phi(k,\sigma) = \sum_{l,m} a_{lm}(\sigma) Y_{lm}(k), \tag{6.1}$$

where k is a point on the sphere. We substitute the expansion (6.1) in Eq. (5.12). It follows that

$$f(u) = \frac{i}{2(2\pi)^3} \sum_{l,m} \int_{\delta-i\infty}^{\delta+i\infty} \sigma(\sigma+1) a_{lm}(\sigma)$$

$$\times \int_\Gamma (uk)^{-\sigma-2} Y_{lm}(k)\, d^2k\, d\sigma, \tag{6.2}$$

where d^2k is the usual euclidean measure on the sphere Γ.

We have to compute the integral

$$I = \int_\Gamma (uk)^{-\sigma-2} Y_{lm}(k)\, d^2k. \tag{6.3}$$

Let $u = (u_0, u')$, where $u_0 = \cosh a$. We choose the polar axis along u' and go over to spherical coordinates; after simple calculations we obtain

$$I = \frac{(-1)^l (2\pi)^{3/2} \Gamma(-\sigma-1)}{\Gamma(-\sigma-l-1)(\operatorname{sh} a)^{1/2}} P_{-\sigma-3/2}^{-(l+1/2)}(\operatorname{ch} a) Y_{lm}(\theta,\varphi). \tag{6.4}$$

Substituting the value of I in Eq. (6.2), we obtain

$$f(u) = \sum_{l,m} \frac{(-1)^{l+1}}{2i(2\pi)^{3/2}} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(1-\sigma)}{\Gamma(-\sigma-l-1)} \frac{a_{lm}(\sigma)}{(\operatorname{sh} a)^{1/2}}$$

$$\times P_{-\sigma-3/2}^{-(l+1/2)}(\operatorname{ch} a) Y_{lm}(\theta,\varphi)\, d\sigma. \tag{6.5}$$

We now compute the coefficients a_{lm}. From Eqs. (6.1) it follows that

$$a_{lm} = \int_\Gamma \Phi(k,\sigma) Y_{lm}^*(k)\, d^2k. \tag{6.6}$$

Substituting in this equation the expression (5.11) for $\Phi(k,\sigma)$, we obtain

$$a_{lm}(\sigma) = \int f(u) \int_\Gamma (uk)^\sigma Y_{lm}^*(k)\, d^2k \frac{d^3u}{u_0}. \tag{6.7}$$

Repeating the calculation of the integral (6.4), we find

$$a_{lm}(\sigma) = \frac{(-1)^l (2\pi)^{3/2} \Gamma(\sigma+1)}{2\Gamma(\sigma-l+1)} \int f(u) (\operatorname{sh} a)^{-1/2} P_{\sigma+1/2}^{-(l+1/2)}$$

$$\times (\operatorname{ch} a) Y_{lm}^*(\theta,\varphi) \frac{d^3u}{u_0}. \tag{6.8}$$

In terms of the coordinates a, θ, and φ we have

$$u_0^{-1} d^3u = \operatorname{sh}^2 a \sin\theta\, da\, d\theta\, d\varphi. \tag{6.9}$$

Eqs. (6.5) and (6.8) give the expansion of the function f(u) in terms of the eigenfunctions of the Laplace operator.

The derived results become simpler if $\sigma = -1 + ip$, i.e., if we consider only the unitary case. In this case they become

$$f(u) = \sum_{l,m} \frac{(-1)^l}{(2\pi)^{3/2}} \int_0^\infty \frac{\Gamma(ip)}{\Gamma(ip-l)} \frac{a_{lm}(p)}{(\text{sh}\, a)^{1/2}} \times P^{-(l+1/2)}_{-1/2+ip}(\text{ch}\, a)\, Y_{lm}(\theta, \varphi)\, p^2\, dp, \quad (6.10)$$

$$a_{lm}(p) = \frac{(-1)^l (2\pi)^{3/2} \Gamma(-ip)}{2\Gamma(-ip-l)} \int f(u) (\text{sh}\, a)^{-1/2} \times P^{-(l+1/2)}_{-1/2+ip}(\text{ch}\, a)\, Y^*_{lm}(\theta, \varphi) \frac{d^3u}{u_0}. \quad (6.11)$$

The Lobachevskian system is inconvenient for the investigation of two-particle scattering, since it does not possess an azimuthal angle. Therefore we will not derive inversion formulas for this system.

The Expansion in the H System

The expansion in the H system is carried out in a completely analogous manner. We select as contour Γ the section of the cone by planes $u_3 = \pm 1$. This section decomposes into the upper sheets of two two-dimensional hyperboloids Γ_+ and Γ_-. We denote by $\Phi_+(k, \sigma)$ and $\Phi_-(k, \sigma)$ the values of $\Phi(k, \sigma)$ on these hyperboloids and apply to the functions $\Phi_+(k, \sigma)$ and $\Phi_-(k, \sigma)$ the expansion on the two-dimensional hyperboloid (cf. Appendix). Substituting the result in Eq. (5.12) we find

$$f(u) = -\frac{1}{8} \frac{1}{(2\pi)^4} \sum_{m=-\infty}^{\infty} \int_{\delta-i\infty}^{\delta+i\infty} \sigma(\sigma+1) \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\Gamma(1-\tau)}{\Gamma(m-\tau)} \tau \,\text{ctg}\, \pi\tau \times a_m(\tau, \sigma) \int_{(\Gamma_+ + \Gamma_-)} (uk)^{-\sigma-2} \mathfrak{D}^m_{-\tau-1}(k)\, d^2k\, d\tau\, d\sigma, \quad (6.12)$$

$$\mathfrak{D}^m_{-\tau-1}(k) = P^m_{-\tau-1}(\text{ch}\, b)\, e^{im\varphi} \equiv \mathfrak{D}^m_{-\tau-1}(b, \varphi).$$

This time we have to compute the integrals

$$I_\pm = \int_{\Gamma_\pm} (uk)^{-\sigma-2} \mathfrak{D}^m_{-\tau-1}(k)\, d^2k. \quad (6.13)$$

These are done in analogy to the integral (6.4) and yield the result

$$I_\pm = \frac{\Gamma(\sigma+\tau+2)\Gamma(\sigma-\tau+1)}{\Gamma(\sigma+2)} \times (\text{ch}\, a)^{-1} P^{-\sigma-1}_{-\tau-1}(\mp \text{th}\, a)\, \mathfrak{D}^m_{-\tau-1}(b, \varphi). \quad (6.14)$$

Therefore

$$f(u) = -\frac{1}{8(2\pi)^4 \text{ch}\, a} \times \sum_m \int_{\delta-i\infty}^{\delta+i\infty} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\Gamma(\sigma+\tau+2)\Gamma(1-\tau)\Gamma(\sigma-\tau+1)}{\Gamma(\sigma)\Gamma(m-\tau)} \tau \,\text{ctg}\, \pi\tau \times \mathfrak{D}^m_{-\tau-1}(b, \varphi) \{a^+_m(\tau, \sigma) P^{-\sigma-1}_{-\tau-1}(-\text{th}\, a) + a^-_m(\tau, \sigma) P^{-\sigma-1}_{-\tau-1}(\text{th}\, a)\}\, d\tau\, d\sigma. \quad (6.15)$$

The computation of a^+_m and a^-_m yields

$$a^\pm_m(\tau, \sigma) = \frac{\Gamma(\tau)\Gamma(-\sigma-\tau-1)\Gamma(\tau-\sigma)}{\Gamma(\tau-m+1)\Gamma(-\sigma)} \times \int f(u) P^{\sigma+1}_\tau(\mp \text{th}\, a)\, \mathfrak{D}^{-m}_\tau(b, \varphi)\, d^3u, \quad (6.16)$$

$$d^3u = \text{sh}^2 a\, \text{sh}\, b\, da\, db\, d\varphi.$$

As noted in Sec. 5, the unitary case corresponds to the values $\sigma = -1 + ip$, $\tau = -1/2 + iq$. For these values, (6.15) and (6.16) take the form

$$f(u) = \frac{1}{2} \frac{(\text{ch}\, a)^{-1}}{(2\pi)^4} \times \sum_m \int_0^\infty p^2 \int_0^\infty \frac{\Gamma(1/2+ip+iq)\Gamma(1/2+ip-iq)\Gamma(1/2-iq)}{\Gamma(1+ip)\Gamma(m+1/2-iq)} q \,\text{cth}\, \pi q \times \mathfrak{D}^m_{-1/2+iq}(b, \varphi) \{a^+_m P^{-ip}_{-1/2+iq}(-\text{th}\, a) + a^-_m P^{-ip}_{-1/2+iq}(\text{th}\, a)\}\, dp\, dq, \quad (6.17)$$

$$a^\pm_m(p, q) = \frac{\Gamma(-1/2+iq)\Gamma(1/2-ip-iq)\Gamma(1/2-ip+iq)}{\Gamma(1/2+iq-m)\Gamma(1-ip)} \times \int f(u) P^{ip}_{-1/2+iq}(\mp \text{th}\, a)\, \mathfrak{D}^{-m}_{-1/2+iq}(b, \varphi)\, d^3u. \quad (6.18)$$

The Expansion in the O System

In the "orispheric" coordinate system the calculations are performed in the same way as above. We select as contour Γ the section of the cone by the plane $k_0 - k_3 = 2$ and parametrize this section as follows:

$$k_0 = 1 + \rho^2, \quad k_2 = 2\rho \cos \alpha, \quad k_3 = -1 + \rho^2,$$
$$k_1 = 2\rho \sin \alpha.$$

we assume

$$\Phi(k, \sigma) = \int_0^{2\pi} \int_0^\infty \Psi(\varkappa, \theta, \sigma)\, e^{i\varkappa\rho \cos(\theta-\alpha)} \varkappa\, d\varkappa\, d\theta. \quad (6.19)$$

(Hankel transform) and substitute into Eq. (5.12). After simple transformations we obtain

$$f(u) = -\frac{1}{2i(2\pi)^3} \int_{\delta-i\infty}^{\delta+i\infty} \sigma(\sigma+1) e^{-a(\sigma+2)} \times \int_0^{2\pi} \int_0^\infty \Psi(\varkappa, \theta, \sigma)\, e^{i\varkappa r \cos(\theta-\varphi)} \times \int_0^{2\pi} \int_0^\infty e^{i\varkappa\rho \cos(\theta-\alpha)} (e^{-2a} + \rho^2)^{-\sigma-2} \varkappa\rho\, d\rho\, d\varkappa\, d\alpha\, d\theta\, d\sigma. \quad (6.20)$$

Further, we have

$$I=\int_0^{2\pi}\int_0^{\infty}e^{i\varkappa\rho\cos(\theta-\alpha)}(e^{-2a}+\rho^2)^{-\sigma-2}\rho\,d\rho\,d\alpha$$

$$=\frac{2\pi}{\Gamma(\sigma+2)}\left(\frac{\varkappa}{2}e^a\right)^{\sigma+1}K_{-\sigma-1}(e^{-a}\varkappa), \quad (6.21)$$

where K is a Macdonald cylinder function. Therefore (with $e^{-a} = b$), we find

$$f(u)=-\frac{b}{i(2\pi)^2}\int_{\delta-i\infty}^{\delta+i\infty}\frac{1}{\Gamma(\sigma)}$$

$$\times\int_0^{2\pi}\int_0^{\infty}\Psi(\varkappa,\theta,\sigma)\left(\frac{\varkappa}{2}\right)^{\sigma+2}e^{i\varkappa\rho\cos(\theta-\varphi)}K_{-\sigma-1}(b\varkappa)\,d\varkappa\,d\theta\,d\sigma. \quad (6.22)$$

The coefficients $\psi(\kappa, \theta, \sigma)$ are computed according to the Fourier formula:

$$\Psi(\varkappa,\theta,\sigma)=\frac{1}{(2\pi)^2}\int\Phi(k,\sigma)\,e^{-i\varkappa\rho\cos(\theta-\alpha)}\,d^2k,$$

where $d^2k = 4\rho\,d\rho\,d\alpha$. Substituting (5.11) into this expression we obtain

$$\Psi(\varkappa,\theta,\sigma)=\frac{2}{\pi\Gamma(-\sigma)}\left(\frac{2}{\varkappa}\right)^{\sigma+1}$$

$$\times\int f(u)\,e^{-i\varkappa r\cos(\theta-\varphi)}K_{\sigma+1}(b\varkappa)\,b^{-2}\,d^3u, \quad (6.23)$$

$$d^3u = r\,dr\,d\varphi\,db.$$

In the unitary case $\sigma = -1 + i\rho$.

The Expansion in the Cylindrical Coordinate C System

Let us finally consider the cylindrical coordinate system. It corresponds to a section of the cone by the cylinder $k_0^2 - k_1^2 = 1$. We chose on Γ the parameters c and α:

$$k_0=\operatorname{ch}c,\qquad k_3=\cos\alpha,$$

$$k_1=\operatorname{sh}c,\qquad k_2=\sin\alpha. \quad (6.24)$$

We perform a Fourier transformation of the function $\Phi(k, \sigma)$ with respect to the parameters c and α:

$$\Phi(k,\sigma)=\sum_m\int a_m(\tau,\sigma)\,e^{i(m\alpha+\tau c)}d\tau. \quad (6.25)$$

Substituting this expansion in Eq. (5.12) and assuming

$$u_0=\operatorname{ch}b\operatorname{ch}a,\qquad u_1=\operatorname{ch}b\operatorname{sh}a,$$

$$u_3=\operatorname{sh}b\cos\varphi,\qquad u_2=\operatorname{sh}b\sin\varphi, \quad (6.26)$$

we obtain

$$f(u)=-\frac{1}{2i(2\pi)^3}$$

$$\times\sum_{m=-\infty}^{\infty}\int_{-\infty}^{+\infty}e^{i(m\varphi+\tau a)}\int_{\delta-i\infty}^{\delta+i\infty}\sigma(\sigma+1)\,a_m(\tau,\sigma)$$

$$\times\int_0^{2\pi}\int_0^{\infty}(\operatorname{ch}b\operatorname{ch}c-\operatorname{sh}b\cos\alpha)^{-\sigma-2}e^{i(m\alpha+\tau c)}\,dc\,d\alpha\,d\sigma\,d\tau. \quad (6.27)$$

We have to compute the integral

$$I=\int_0^{2\pi}\int_0^{\infty}(\operatorname{ch}b\operatorname{ch}c-\operatorname{sh}b\cos\alpha)^{-\sigma-2}e^{i(m\alpha+\tau c)}\,dc\,d\sigma. \quad (6.28)$$

Taking out the factor cosh b cosh c in front of the bracket and expanding $(1 - \tanh b \cos\alpha/\cosh c)^{-\sigma-2}$ by means of Newton's binomial formula, we obtain after term-by-term integration

$$I=2^{\sigma+2}\pi\frac{\Gamma(A)\Gamma(B)}{\Gamma(m+1)\Gamma(\sigma+2)}F(A,B,m+1;\operatorname{th}^2b)\frac{\operatorname{sh}^m b}{\operatorname{ch}^{m+\sigma+2}b},$$

$$A=\frac{m+\sigma+i\tau}{2},\qquad B=\frac{m+\sigma-i\tau}{2}. \quad (6.29)$$

Therefore

$$f(u)=-\frac{1}{16\pi^2 i}\sum_m\frac{\operatorname{th}^m a\,e^{im\varphi}}{\Gamma(m+1)}\int_{-\infty}^{+\infty}e^{i\tau a}$$

$$\times\int_{\delta-i\infty}^{\delta+i\infty}\frac{a_m(\tau,\sigma)}{\Gamma(\sigma)}\Gamma(A')\Gamma(B)\left(\frac{2}{\operatorname{ch}a}\right)^{\sigma+2}$$

$$\times F(A,B,m+1;\operatorname{th}^2b)\,d\sigma\,d\tau. \quad (6.30)$$

The coefficients $a_m(\tau, \sigma)$ have the expression

$$a_m(\tau,\sigma)=\frac{1}{4\pi^2}\int_0^{\infty}\int_{-\infty}^{+\infty}\Phi(k,\sigma)\,e^{-i(m\varphi+\tau a)}\,d^2k$$

$$d^2k=d\varphi\,da. \quad (6.31)$$

We substitute for $\Phi(k, \sigma)$ the expression (5.11). After simple manipulations we find

$$a_m(\tau,\sigma)=\frac{\Gamma(A')\Gamma(B')}{4\pi\Gamma(m+1)\Gamma(-\sigma)}\int\operatorname{th}^m a\left(\frac{2}{\operatorname{ch}a}\right)^{-\sigma}$$

$$\times F(A',B',m+1;\operatorname{th}^2b)\,e^{-i(m\varphi+\tau a)}d^3u, \quad (6.32)$$

where

$$d^3u=\operatorname{sh}^2a\,da\,db\,d\varphi,$$

$$A'=\frac{m-\sigma+i\tau}{2},$$

$$B'=\frac{m-\sigma-i\tau}{2}.$$

The unitary case corresponds to $\sigma = -1 + i\varphi$. These formulas exhaust the problem of expanding the scattering amplitude in the four coordinate systems we have described.

APPENDIX

For the expansion in the H system we have made use of expansions of functions on a two-dimensional hyperboloid. Introducing the coordinates

$$u_0=\operatorname{ch}b,\qquad u_2=\operatorname{sh}b\cos\varphi,\qquad u_1=\operatorname{sh}b\sin\varphi,$$

we have

$$f(u) = -\frac{1}{8\pi i}\sum_{m=-\infty}^{\infty}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} a_m(\sigma)\frac{\Gamma(1-\sigma)}{\Gamma(m-\sigma)}$$

$$\times\, \mathfrak{D}^{m}_{-\sigma-1}(b, \varphi)\, \sigma \operatorname{ctg} \pi\sigma\, d\sigma,$$

$$a_m(\sigma) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-m+1)}\int f(u)\, \mathfrak{D}^{-m}_{\sigma}(b, \varphi)\, d^2u,$$

$$d^2u = \operatorname{sh} b\, db\, d\varphi.$$

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