THE THERMAL CONDUCTIVITY OF THE INTERMEDIATE STATE IN SUPERCONDUCTORS

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It is shown that, owing to over-the-barrier reflection of electron excitations at the boundary of the normal and superconducting phases, a temperature drop occurs when there is a flow of heat. The additional thermal resistance of a superconductor in the intermediate state is calculated. It is shown that it increases exponentially as the temperature is lowered and does not depend on the electron mean free path.

1. As was first discovered by Mendelssohn and Olsen,[1] the thermal resistance of superconductors in the intermediate state greatly exceeds the thermal resistance in the superconducting state. A systematic study of this effect over a wide range of temperatures has been made by Zavaritskiĭ.[2] It thus became clear that the anomalous thermal resistance of the intermediate state occurs both at low temperatures when phonons make the major contribution to the thermal conductivity, and at high temperatures when electrons transport the heat. A mechanism for the additional phonon scattering, which explains the phenomena in the phonon thermal conductivity range, was proposed by Abrikosov and Zavaritskiĭ[3] and by Laredo and Pippard.[4] To explain the effect at high temperatures it was proposed[5,6] that the electrons are reflected in some way from the boundary dividing the normal and superconducting phases. However, so far no one has clarified the nature of this reflection. The present work is devoted to this problem.

It will be shown below that at the boundary dividing the two phases an effect occurs of a type involving over-the-barrier reflection of quasiparticles, the probability of which is of order unity, although the ratio of the transition layer width to the wavelength of the quasi-particle is very large ($\sim 10^4$). For this reason a temperature drop arises at each of the boundaries when a thermal flux is present; this causes the additional thermal resistance in the intermediate state.

We consider a plane boundary between the normal and superconducting phases. Let the z axis be normal to the boundary with the superconducting phase in the region of positive z. The energy gap $\Delta(z)$ tends to zero as $z \to -\infty$, and as $z \to +\infty$ it tends to Δ_0—the equilibrium value at the given temperature. We introduce the functions

$$f(\mathbf{r}, t) = \langle \Phi_0 | \psi(\mathbf{r}, t) | \Phi_1 \rangle, \quad \varphi(\mathbf{r}, t) = \langle \Phi_0 | \psi^+(\mathbf{r}, t) | \Phi_1 \rangle, \tag{1}$$

where ψ and ψ^+—Heisenberg operators, Φ_0 and Φ_1—ground and excited states. Using the equations of motion[7]

$$(i\partial/\partial t + \nabla^2/2m + \mu)\psi - g(\psi^+\psi)\psi = 0,$$
$$(i\partial/\partial t - \nabla^2/2m - \mu)\psi^+ + g\psi^+(\psi^+\psi) = 0, \tag{2}$$

where μ is the chemical potential, g is the interaction constant, and m is the mass of an electron, we obtain (cf.[7])[1)]

$$i\partial f/\partial t = -(\nabla^2/2m + \mu) f + i\Delta(\mathbf{r})\varphi,$$
$$i\partial\varphi/\partial t = (\nabla^2/2m + \mu)\varphi - i\Delta(\mathbf{r}) f. \tag{3}$$

The quantities f and φ taken together obviously signify the wave function of the quasi-particle. We multiply the first equation of (3) by f*, the second by φ*, and add the results:

$$i\frac{\partial}{\partial t}(|f|^2 + |\varphi|^2) = \frac{1}{2m}(f\nabla^2 f^* - f^*\nabla^2 f + \varphi^*\nabla^2\varphi - \varphi\nabla^2\varphi^*).$$

This equation can be rewritten as

$$\partial\rho/\partial t + \operatorname{div}\mathbf{j} = 0, \tag{4}$$

$$\rho = |f|^2 + |\varphi|^2,$$
$$\mathbf{j} = \frac{i}{2m}\left\{(f\nabla f^* - f^*\nabla f) + (\varphi^*\nabla\varphi - \varphi\nabla\varphi^*)\right\}. \tag{5}$$

Equation (4) shows that ρ defines the probability density of the quasi-particle coordinates, and j is the probability flux density.

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1)In the equations we do not take into account a magnetic field, since usually the penetration depth of the field is much less than the width of the transition layer between the phases. In the normal phase the field is in general unimportant, since the Larmor radius in a field of the order of the critical field is greater than the thickness of the normal layers.

With the aid of (3), we can consider the reflection of quasi-particles from the boundaries of the two phases in the usual way. We note first of all that for energies of the order of the Fermi energy of the medium under consideration is completely homogeneous (with an accuracy Δ/ϵ_F). We shall therefore seek a solution of (3) in the form

$$f = e^{ip_0\mathbf{nr}-i\omega t}\eta(\mathbf{r}), \qquad \varphi = e^{ip_0\mathbf{nr}-i\omega t}\chi(\mathbf{r}), \tag{6}$$

where p_0 is the Fermi momentum, $\omega > 0$ is the energy of the quasi-particle, $\mathbf{n}$ is some unit vector and η and χ are functions that vary slowly compared with $e^{ip_0\mathbf{n}\cdot\mathbf{r}}$. Substituting (6) in (3) and neglecting higher derivatives of η and χ, we obtain

$$\left(iv\mathbf{n}\frac{\partial}{\partial\mathbf{r}} + \omega\right)\eta - i\Delta(\mathbf{r})\chi = 0,$$

$$\left(iv\mathbf{n}\frac{\partial}{\partial\mathbf{r}} - \omega\right)\chi - i\Delta(\mathbf{r})\eta = 0, \tag{7}$$

where $v = p_0/m$.

We find the asymptotic form when $z \to \pm\infty$ of the solutions of (7) describing the reflection of quasi-particles falling on the boundary of the normal phase. When $z \to -\infty$ we can put $\Delta(\mathbf{r}) = 0$. Then

$$\begin{pmatrix}\eta\\ \chi\end{pmatrix} = A\begin{pmatrix}1\\ 0\end{pmatrix}e^{i\mathbf{k}_1\mathbf{r}} + B\begin{pmatrix}0\\ 1\end{pmatrix}e^{i\mathbf{k}_2\mathbf{r}}, \tag{8}$$

where $\mathbf{k}_1 = \mathbf{n}\omega/v$, $\mathbf{k}_2 = -\mathbf{n}\omega/v$; A and B are arbitrary constants. The first term corresponds to a "particle" whose velocity (or j) lies along $\mathbf{n}$, and the second term to a "hole" whose velocity lies in the opposite direction to $\mathbf{n}$. If $n_z > 0$, then (8) describes a "particle" incident on the boundary and a reflected "hole"; if $n_z < 0$, it describes an incident "hole" and a reflected "particle."

We must put $\Delta(\mathbf{r}) = \Delta_0$ in (7) as $z \to +\infty$. The solution describing the transmitted wave ($j_z > 0$) has for $\omega > \Delta_0$ the form

$$\begin{pmatrix}\eta\\ \chi\end{pmatrix} = \frac{C}{\sqrt{2}}\begin{pmatrix}\sqrt{1 + v\mathbf{n}\mathbf{k}_3/\omega}\\ -i\sqrt{1 - v\mathbf{n}\mathbf{k}_3/\omega}\end{pmatrix}e^{i\mathbf{k}_3\mathbf{r}}, \tag{9}$$

where C is a constant,

$$\mathbf{k}_3\mathbf{n} = v^{-1}\sqrt{\omega^2 - \Delta_0^2} \quad \text{for } n_z > 0,$$

$$\mathbf{k}_3\mathbf{n} = -v^{-1}\sqrt{\omega^2 - \Delta_0^2} \quad \text{for } n_z < 0.$$

If $\omega < \Delta_0$, then the functions η and χ decay exponentially as $z \to +\infty$.

We note that, because the medium is completely homogeneous in the x and y directions, we have $k_{1x} = k_{2x} = k_{3x}$ and $k_{1y} = k_{2y} = k_{3y}$.

To find the transmission coefficient w, defined as the ratio of the fluxes j_z in the transmitted and incident waves, it is necessary to find the asymptotic form as $z \to -\infty$ of the solution to equation (7), that has the asymptotic form (9) as $z \to +\infty$. This problem can be easily solved, for example, if $\Delta(z) = 0$ for $z < 0$ and $\Delta(z) = \Delta_0$ for $z > 0$. In fact, formulae (8) and (9) are then valid for $z < 0$ and $z > 0$, respectively. The continuity condition for the functions η and χ at $z = 0$ gives

$$w = \begin{cases} 2\sqrt{\omega^2 - \Delta_0^2}/[\omega + \sqrt{\omega^2 - \Delta_0^2}] & \text{for } \omega > \Delta_0 \\ 0 & \text{for } \omega < \Delta_0. \end{cases} \tag{10}$$

For small positive $\omega - \Delta_0$

$$w = 2\sqrt{2(\omega - \Delta_0)/\Delta_0}. \tag{11}$$

It is impossible to calculate w in the general case, since to do this it is necessary to find an explicit form for the function $\Delta(z)$, which cannot be done even close to the critical temperature T_c. It is, however, possible to write down a number of properties of the function $w(\omega, \mathbf{n})$ that are independent of the specific nature of the function $\Delta(z)$. Firstly, it is clear that when $\omega < \Delta_0$ we must have $w = 0$. Close to threshold the following formula is valid in the general case instead of (11):

$$w = f(n_z)\sqrt{(\omega - \Delta)/\Delta}, \tag{12}$$

where f is some function of order unity. (Here and subsequently we omit the subscript zero on Δ.)

In fact, from the definition of w and relations (8), (9), and (5), it follows that

$$w = \frac{\sqrt{\omega^2 - \Delta^2}}{\omega}\left|\frac{C}{A}\right|^2 \quad \text{for } n_z > 0,$$

$$w = \frac{\sqrt{\omega^2 - \Delta^2}}{\omega}\left|\frac{C}{B}\right|^2 \quad \text{for } n_z < 0.$$

As $\omega \to \Delta$ the ratio $|C/A|^2 (|C/B|^2)$ tends in general to some finite quantity independent of $\mathbf{n}$, whence a formula of type (12) is obtained.

If $\omega - \Delta \sim \Delta$, then $w \sim 1$. This follows from the fact that in this case the distance over which the functions η and χ vary in (7) has the same order of magnitude as the width of the transition layer, i.e., the distance in which the function $\Delta(z)$ changes. For $\omega \gg \Delta$ the wavelength of the functions η and χ becomes much smaller than the width of the transition layer, and then w is close to unity.

We note the following curious feature. Usually when particles are reflected, only the component of the velocity normal to the boundary changes sign. The projection of the velocity on the plane of the boundary remains unchanged. In our case all three components of the velocity change sign.

2. We proceed to calculate the thermal resistance of the intermediate state. We assume that the temperature $T \ll T_c$ but still $T \gg T_0$, where

T_0 is the temperature at which the phonon contribution to the heat exchange becomes important. In this case the quasi-particles whose energy ω differs little from Δ play the principal part. Therefore, expression (12) can be used for the transmission coefficient w, whence it is clear that in the important range of energy values $w \ll 1$. Thus, at low temperatures thermal exchange through the boundary between the normal and superconducting phases is made extremely difficult. The situation which occurs is analogous to the exchange of heat between a solid body and liquid helium, where the small coefficient of transmission of the quasi-particles through the boundary is due to the large ratio of the acoustic resistances. As is well known,[8,9] when a thermal flux exists between helium and a solid body a temperature step occurs. Just the same thing will also happen at the boundary between the normal and superconducting phases. To calculate the relation between the thermal flux Q and the temperature drop δT we note that, since w is small, the flux Q coincides in magnitude with the thermal flux which arises at the instant when the two media under consideration are brought together—each of the media being in a state of thermodynamic equilibrium, but their temperatures differing by δT.

We calculate the thermal flux W directed from the normal to the superconducting phase:

$$W = \int\limits_{v_z > 0} 2n_0(\omega)\, \omega v_z\, w \frac{d^3\mathbf{p}}{(2\pi)^3}, \tag{13}$$

where $\mathbf{v} = \partial\omega/\partial\mathbf{p}$, $\omega = |\xi|$, $\xi = v(p - p_0)$, and n_0 is the equilibrium distribution function.

Substituting in (13) the value of w from (12) and denoting the angle between the vector p and the z axis by θ, we bring (13) to the form

$$W = \left(\frac{p_0}{\pi}\right)^2 \int\limits_{\Delta}^{\infty} \omega n_0(\omega) \sqrt{\frac{\omega-\Delta}{\Delta}}\, d\xi \int\limits_0^1 \cos\theta f(\cos\theta) d\cos\theta. \tag{14}$$

Because $T \ll \Delta$ it can be assumed that $n_0(\omega) = e^{-\omega/T}$. The integral with respect to ξ in (14) is easily evaluated and we finally obtain

$$W = \tfrac{1}{2}\sqrt{\pi} f_0 (p_0/\pi)^2 T \sqrt{T\Delta}\, e^{-\Delta/T}, \tag{15}$$

where $f_0 = \int\limits_0^1 \cos\theta f(\cos\theta) d\cos\theta$ is a constant of order unity.

If the temperatures of the normal and superconducting phases are equal, the flux W is compensated by a flux of the same magnitude directed from the superconducting phase to the normal. If there is a difference of temperature δT, the resulting thermal flux is $(\partial W/\partial T)\delta T$. Performing the differentiation and remembering that $\Delta \gg T$, we obtain

$$Q = \frac{\sqrt{\pi}}{2} f_0 \left(\frac{p_0}{\pi}\right)^2 \Delta \sqrt{\frac{\Delta}{T}}\, e^{-\Delta/T} \delta T. \tag{16}$$

The temperature drop δT determined by this equation occurs at each of the boundaries between the phases.

We calculate the additional thermal resistance due to this mechanism for a cylindrical specimen situated in a magnetic field perpendicular to the cylinder axis. In this case a structure of alternating normal and superconducting regions arises in the specimen. For the period of the structure a we have[10,2]

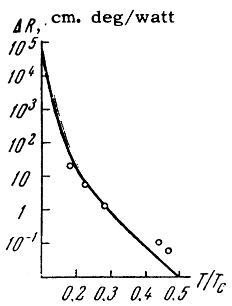
$$a = \sqrt{\alpha d/\varphi(\eta)}, \tag{17}$$

where d is the diameter of the specimen, $\alpha = 8\pi\sigma/H_c^2$, σ is the surface tension at the boundary between the phases, H_c is the critical magnetic field, $\eta = 2H/H_c - 1$, and $\varphi(\eta)$ is the function tabulated in [10].

On the basis of (16) and (17), the additional thermal resistance ΔR can easily be calculated if it is taken into account that at each period a two temperature drops of δT occur. The result is

$$\Delta R = f_0^{-1} (2\pi/p_0)^2 \sqrt{\varphi(\eta)\, T/\pi\alpha d\Delta^3}\, e^{\Delta/T}. \tag{18}$$

A graph of the function $\Delta R(T/T_c)$ for $\eta = 0.5$ is given in the figure for a specimen of tin with diameter 0.175 mm. In the calculations we put $f_0 = 1$, $\hbar/p_0 = 10^{-8}$, $\alpha = 3 \times 10^{-5}$ (see [11]). The experimental results of Zavaritskiĭ[2] are shown in the same figure. At low temperatures the agreement between the theory given and the experiment is completely satisfactory.



We note that the expression we obtain for ΔR does not contain the mean free path of electrons, in constrast, for example, to the result of Strassler and Wyder.[6] Our result agrees with the data of Zavaritskiĭ,[2] where it was shown that ΔR is almost independent of the impurity concentration, whilst the quantity $\Delta R/R_S$ (R_S is the thermal resistance of the superconducting phase) changes greatly when the concentration is changed.

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[1] K. Mendelssohn and J. L. Olsen, Proc. Phys. Soc. (London) **A63**, 2 (1950); Phys. Rev. **80**, 859 (1950).

[2] N. V. Zavaritskiĭ, JETP **38**, 1673 (1960), Soviet Phys. JETP **11**, 1207 (1960).

[3] A. A. Abrikosov and N. V. Zavaritskiĭ, Supplement to D. Shoenberg, Superconductivity (Russ. Transl., IIL, 1955).

[4] S. J. Laredo and A. B. Pippard, Proc. Cambridge Phil. Soc. **51**, 368 (1955).

[5] J. K. Hulm, Phys. Rev. **90**, 1116 (1953).

[6] S. Strassler and P. Wyder, Phys. Rev. Letters **10**, 225 (1963).

[7] L. P. Gor'kov, JETP **34**, 735 (1958), Soviet Phys. JETP **7**, 505 (1958).

[8] P. L. Kapitza, JETP **11**, 1 (1941).

[9] I. M. Khalatnikov, JETP **22**, 687 (1952).

[10] E. M. Lifshitz and Yu. V. Sharvin, DAN SSSR **79**, 783 (1951).

[11] Yu. V. Sharvin, JETP **33**, 1341 (1957), Soviet Phys. JETP **6**, 1031 (1958).

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