DETERMINATION OF THE PARAMETERS AND PHASE SHIFTS OF LOW-ENERGY πN SCATTERING

V. F. PLESHAKOV

Mathematics Institute, Siberian Branch Academy of Sciences U.S.S.R.

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We analyze the role of Coulomb effects and of the contribution from inelastic processes in calculations of the parameters and phase shifts of low-energy πN scattering. We show, with a determination of the s-wave phase shift as an example, that a noticeable role is played by the π-mesic atom. The main deduction is that we cannot claim high accuracy for the results obtained with the aid of the theoretical reduction of the experimental data [2,3] without a consistent account of the Coulomb effect.

1. The calculation of the parameters and phase shifts of low-energy πN scattering was made essentially by a group of physicists in Dubna [1] and by British physicists [2,3]. The method of calculation can be traced back to the work of Chew et al. [4] whose notation will be used here. We consider here the elastic scattering

$$\pi^+ + p \rightarrow \pi^+ + p, \qquad \text{(I)}$$
$$\pi^- + p \rightarrow \pi^- + p \qquad \text{(II)}$$

in the interval of low kinetic energies $(\omega_L - \mu)$ of the incident meson in the laboratory system. For what follows we note that in the energy region under consideration the reaction (II) has two inelastic channels

$$\pi^- + p \rightarrow \pi^0 + n, \qquad \text{(III)}$$
$$\pi^- + p \rightarrow \gamma + n \qquad \text{(IV)}$$

—scattering with charge exchange and a process of the K-capture type. It was found [1-3] that an analysis of the elastic scattering (I) and (II) is best carried out with the aid of the dispersion relations (d.r.).

To this end, [4] in order to determine the parameters and the phase shifts, the real parts of the amplitudes for processes (I) and (II), which are located in the right sides of the d.r., are expressed in terms of the phase shift of purely nuclear scattering. The integrals of the total cross sections, contained in the right sides of the d.r., are approximated by using the experimental data and other premises [1-3]. This yields a system of relations for the determination of the quantities of interest to us.

For example, when we determine the s-wave phase shifts [2,3] in the region $0 < (\omega_L - \mu) < 45$ MeV, we obtain

$$\frac{\sin 2\delta_1 + 2\sin 2\delta_3}{2q}\frac{W}{M+1} = (a_1 + 2a_3) + C^{(+)}q_L^2,$$

$$\frac{\sin 2\delta_1 - \sin 2\delta_3}{2q}\frac{W}{M+1} = (a_1 - a_3)\,\omega_L + C^{(-)}q_L^2, \qquad (1)$$

where $C^{(+)} = -0.096 \pm 0.026$, $C^{(-)} = -0.094 \pm 0.013$ and $a_1 = 0.171 \pm 0.005$, $a_3 = -0.088 \pm 0.004$—in accordance with the data of Hamilton and Woolcock [3]

$$W = (q^2+1)^{1/2} + (q^2+M^2)^{1/2}$$

—total energy in the c.m.s., M—mass of the nucleon (in pion mass units).

Many authors [1-6] indicate that an analysis of this type is incomplete, particularly at very low energy. This incompleteness, in particular, consists in the fact that account is taken only of strong interactions satisfying charge invariance, and the Coulomb interaction is neglected. We note that the role of the Coulomb effects (a) were not explained with sufficient completeness and (b) were studied essentially in connection with other problems at a time when there were no good experimental data in the low-energy region. Now that such data are available, the Coulomb problem is of interest for determining the parameters and phase shifts, and also for the derivation of an interpolation formula for a precise reduction of the experimental data in the low-energy region. We shall show how the method of analysis varies in this case, and we shall make suitable estimates with (1) as an example.

2. A convenient method for the analysis of the Coulomb problem is the analysis of the so-called "nuclear scattering amplitude," defined as the difference between the total amplitude and the

amplitude of pure Rutherford scattering. It is obvious that, in addition to pure nuclear scattering, there is left in this case a non-additive contribution from the interference of the two types of interactions.

To determine the "nuclear amplitude" we make use the model of van Hove [7]. We assume that the radius of interaction of the nuclear forces is $r_0 \sim h/\mu c$. In order to carry out the πN-scattering analysis in the usual fashion, i.e., with the aid of purely nuclear phase shift $\delta_{2J,\,2I}$, it is necessary to assume the charge invariance hypothesis for the nuclear part of the interaction. We shall therefore neglect the Coulomb interaction for $r < r_0$, and the nuclear interaction for $r > r_0$. At the same time we neglect the mass difference inside the pion and nucleon isomultiplets. Violation of the isotopic invariance in this model is the consequence of an interference between two types of interaction on the boundary of a sphere of radius r_0. We note that in the model no assumption is made at all whether the potential of the nuclear interaction is real or complex. This fact will be used below to take into account the contribution from inelastic processess.

Using the usual procedure of nonrelativistic scattering theory and neglecting the spin flip of the nucleon in the Coulomb field, we can obtain for each of the reactions (I), (II), and (III) the scattering amplitudes, without and with change in the nucleon polarization:

$$f = \sum_{l=0}^{\infty} [(l+1) f_{l+} + l f_{l-}] P_l(\cos\theta),$$

$$g = \sum_{l=1}^{\infty} [f_{l+} - f_{l-}] e^{i\varphi} \sin\theta P_l'(\cos\theta), \tag{2}$$

where the ± signs correspond to the total angular momentum $J = l \pm 1/2$, and P_l' is the derivative of the Legendre polynomial with respect to $\cos\theta$.

The partial amplitudes $f_{l\pm}$ can be readily reconstituted from the s-wave amplitude written below, by affixing the indices $l\pm$ to all the phase shifts and the index l to the quantities A, B, C, and D, and by adding l to the argument of the Γ function. In the c.m.s. for each of the three reactions (I), (II), and (III), the s-wave "nuclear scattering amplitudes" are equal to

$$f(p^+ \to p^+) = \frac{1}{2iq} \frac{\Gamma(1 + i\alpha/q)}{\Gamma(1 - i\alpha/q)} \left[\frac{A\exp(2i\delta_3) + B}{C\exp(2i\delta_3) + D} - 1\right], \tag{3}$$

$$f(p^- \to p^-) = \frac{1}{2iq} \frac{\Gamma(1 - i\alpha/q)}{\Gamma(1 + i\alpha/q)}$$
$$\times \left[\frac{A\,(^2/_3\exp(2i\delta_1) + {}^1/_3\exp(2i\delta_3)) + B}{C\,(^2/_3\exp(2i\delta_1) + {}^1/_3\exp(2i\delta_3)) + D} - 1\right], \tag{4}$$

$$f(p^- \to n^0) = \frac{1}{2iq}\left[\frac{\Gamma(1 - i\alpha/q)}{\Gamma(1 + i\alpha/q)}\right]^{1/2}$$
$$\times \frac{\sqrt{2}\,[\exp(2i\delta_3) - \exp(2i\delta_1)]/3}{i\,[C\,(^2/_3\exp(2i\delta_1) + {}^1/_3\exp(2i\delta_3)) + D]/2\Delta_1}. \tag{5}$$

Here q—momentum of the incident pion in the c.m.s. (we used $\hbar = c = \mu = 1$), and $\alpha = e^2/\hbar c$ is the fine-structure constant. This result is obtained in the usual manner by joining the solutions of the Schrödinger equation for the regions $r < r_0$ and $r > r_0$. The final result (3), (4), and (5) contains only the phase shifts of the purely nuclear interaction. The following expressions are obtained for A, B, C, and D

$$A = (G_1' - iF_1')(G_0 + iF_0) - (G_1 - iF_1)(G_0' + iF_0'),$$
$$D = -A^*,$$
$$B = -(G_1' - iF_1')(G_0 - iF_0) + (G_1 - iF_1)(G_0' - iF_0'),$$
$$C = -B^*. \tag{6}$$

where G_1 and F_1 are solutions of the Schrödinger equation in the Coulomb field [8], which are "singular" and "regular" at zero, respectively, and which have asymptotic values

$$\cos\left(qr - \frac{\pi l}{2} \pm \frac{\alpha}{q}\ln 2qr + \sigma_l\right),$$
$$\sin\left(qr - \frac{\pi l}{2} \pm \frac{\alpha}{q}\ln 2qr + \sigma_l\right),$$
$$\sigma_l = \arg\Gamma(l + 1 \pm i\alpha/q);$$

G_0 and F_0 are the corresponding solutions of the Schrödinger equation without the interaction. All the functions in (6) are taken at the point $r = r_0$, and the complex conjugation does not affect the functions G and F themselves. The functions A, B, C, and D for reactions (II) and (III) are obtained from the corresponding expressions for reaction (I) by making the substitution $\alpha \to -\alpha$, which corresponds to the sign of the interference. We note that A and D $\to -2i$ and B, C $\to 0$ as $\alpha \to 0$, and we then obtain the well known expressions for the amplitudes of purely nuclear πN scattering. In expression (5) the quantity $\Delta_1 = G_1F_1' - G_1'F_1$ is the Wronskian determinant taken at $r = r_0$. A prime denotes a derivative with respect to qr.

Thus, we should substitute the real parts of the amplitudes (3) and (4) in the left side of the d.r.

3. The amplitude $f(p^- \to p^-)$ has poles corresponding to the energy levels of the π-mesic atom, which are shifted by the nuclear interaction. All the poles are located on the imaginary axis in the upper half of the complex q plane if

the potential of the nuclear interaction is real. This can be proved directly, although by a cumbersome method. Assuming the nuclear level shift to be small, i.e., $q_n = i\alpha(1 + \Delta q_n)/n$ (here $|q| \sim \alpha$, n = 1, 2, ...) which is indeed the case, we obtain

$$\Delta q_n = \frac{2\alpha a}{n}\left[1 - 2\alpha r_0\left(1 - \frac{1}{n}\right)\right], \tag{7}$$

where $a = 2a_1/3 + a_3/3$—length for scattering by the proton, or $a = Za_p + (A - Z)a_n$ ($a_n = a_3$) for not too heavy nuclei. Formula (7) offers good corroboration of the result of Deser et al.[9], obtained by a cruder method ($\Delta q_1 = 2\alpha a$) and experimentally supported. Using the data of Hamilton and Woolcock for a_1 and a_3, we obtain for the level shift of the π-mesic atom ($\delta E = -2\alpha^3 a/n^3$) and hydrogen a value $\delta E = -9$ eV (attraction). For all other elements these data, as well as the data of Orear ($a_1 = 0.16$, $a_3 = -0.11$) lead to a positive value of δE (repulsion).

Since there are inelastic processes (III) and (IV), the potential of the nuclear interaction is complex. The width of the levels of the π-mesic atom can be estimated near the threshold with the aid of the relation[9,10] $\mathrm{Im}\, a = (\sigma_r q)_0/4\pi$, where σ_r—total cross section of the inelastic processes (III) and (IV), and the zero subscript denotes that the quantity is taken at q = 0. We obtain for the level width $\Gamma = -\mathrm{Im}\,\delta E$ near threshold

$$\Gamma = \frac{4}{9}\frac{\alpha^3}{n^3} q\,|a_3 - a_1|^2\left(1 + \frac{1}{P}\right), \tag{8}$$

where P—Panofsky ratio; at threshold P = 1.5. An estimate (for n = 1) shows that the width is $\Gamma = 0.4$ eV, i.e., small compared with the level shift. Obviously here, as in (10), for estimates near the threshold we must take $q^2/2\mu = \delta M$, where δM—relative mass difference of the ($\pi^- p$) and ($\pi^0 n$) systems.

4. The integrals of the cross sections in the d.r. are preceded by additional pole terms corresponding to the π-mesic atom. The contribution to the real parts of the amplitudes of reactions (I) and (II) due to the π-mesic atom is equal to

$$\Delta\left(\frac{\mathrm{Re}\, f_{\mp}(q_L)}{r_0}\right) = \pm\alpha a\,\frac{q_L{}^2}{\omega_L - 1}\sum_{n\geqslant 0}\frac{(-1)^n}{n!\,(n+1)!}. \tag{9}$$

We confine ourselves to the following remarks:

A. If we neglect the change in the polarization of the nucleon in the Coulomb field, then the crossing symmetry, which ensures the usual form of the d.r., remains the same as before, although isotopic invariance is violated. To prove this it is necessary to determine in suitable manner[3] the amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ in terms of the real amplitudes, and to use the C- and T-invariance.

B. Usually the subtractions in the d.r. are made at the point $\omega_L = \mu$, in which the amplitude now has an essential singularity. The correctness of the procedure which leads to the result (9) can be demonstrated, however, by cutting off the Coulomb field at large $r = R_0$. Then the number of levels in such a "well" will be $N_0 < -\sqrt{2\alpha R_0} - 1$, and the correction itself depends on R_0: $\Delta q_n = 2\alpha a/n - (n+1)^2/4n\alpha R_0$. By calculating the contribution and then letting $R_0 \to \infty$ we obtain again the result (9).

5. An account of the pole terms (9) in the right sides of (1) causes the constants $C^{(\pm)}$ to depend on the energy. To contribution to the absolute values of the constants $C^{(\pm)}$ in (1), due to the mesic atom, changes from 40% at $(\omega_L - \mu) = 1$ MeV to 4% at 10 MeV. In accordance with the foregoing reasoning, the right side of (1) will contain phase shifts that include a non-additive contribution from the interference. The correction for the interference can be obtained from our formulas by expanding the functions A, B, C, and D in terms of the parameter $\alpha/q \gg 1$, which corresponds to $(\omega_L - \mu) \gg 200$ keV, i.e., to energies now in practical use (see also the paper by Van Hove[7]). The contribution from the interference is particularly large when $(\omega_L - \mu) < 15$ MeV. When $(\omega_L - \mu) = 15$ MeV it amounts to 20% of the absolute value of the phase, and at higher energies it decreases rapidly. Finally, it follows from the sum rule of Goldberger et al. (see, e.g.,[3]) modified with allowance for (9), that the contribution due to the mesic atom to $|a_1 - a_3|$ amounts to 11%.

Account can also be taken of an "inelastic component," which contributes to the elastic scattering amplitudes (I) and (II). To this end we note that, owing to the inelastic processes (III) and (IV), the total elastic scattering cross section can be interpolated in the low-energy region by the formula

$$\sigma_{\mathrm{el}} = \sigma_0/[1 + (\sigma_r q)_0 q + \beta q^2], \tag{10}$$

which, compared with the usual formula, contains a term linear in the momentum[10]. With the aid of the d.r. it is possible to determine the contribution from this term to the real part of the amplitudes. Calculation has shown that this contribution is very small, although it is opposite in sign to the contribution from the mesic atom and can play a role only near threshold. A characteristic feature is that for $(\omega_L - \mu) > 20$ MeV the contributions from the inelastic processes and from the mesic atom cancel each other with high degree of accuracy.

Our estimates show that it cannot be stated conclusively that the parameters and phase shifts have by now been determined with as high a degree of accuracy as indicated by Hamilton and Woolcock [3]. For final conclusions it is necessary to construct a precision formula describing πN scattering in the low-energy region. From the point of view of precision reduction of the experimental data, all the indicated effects must be consistently accounted for, particularly for $(\omega_L - \mu) < 20$ MeV.

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