SOVIET PHYSICS JETP VOLUME 19, NUMBER 6 DECEMBER, 1964

EFFECT OF COLLECTIVE EXCITATIONS ON THE ELECTRODYNAMICS OF SUPERCONDUCTORS

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Submitted to JETP editor December 30, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 2188-2201 (June, 1964)

The electromagnetic properties of superconductors are studied without assuming a weak interaction between the electrons. The surface impedance is found by taking collective excitations into account. The result obtained can be used to explain the absorption of infrared radiation in lead and mercury. The coefficient of reflection of collective excitations from a surface is found and is used in investigations of superconducting films whose thickness is greater than the penetration depth.

1. INTRODUCTION

IN the study of the electrodynamics of superconductors, [1-3] only that part of the interaction between particles is usually taken into account which leads to pairing and to a gap in the spectrum of one-particle excitations. The quasiparticles are assumed to be noninteracting. The effect of the remaining interaction on the electrodynamics of superconductors in a constant field leads to a renormalization of the constants which determine the penetration depth. It will be shown below that the number of free electrons, which enters into the London constant depends on the remaining interaction and on the form of the periodic potential, and is identical with the corresponding constant in the dielectric permittivity of metals in the infrared region. In the Pippard limiting case, the penetration depth is determined by the momentum on the Fermi boundary and does not depend on the interaction.

In the region of microwave radiation, new qualitative effects appear, because the remaining interaction leads to the generation of collective oscillations in the superconductor. These excitations are similar to zero sound in a Fermi liquid; however, their spectrum has a gap whose width is less than 2Δ (Δ is the gap in the spectrum of one-particle excitations). The absorption of the electromagnetic wave incident on the surface of the superconductor takes place without account of the collective excitations only at frequencies that are greater than 2Δ, when the energy of the quantum suffices for the formation of two one-particle excitations. If the collective excitations can propagate in the superconductor, then electromagnetic radiation can be absorbed at frequencies less than 2Δ. Here the energy of the electromagnetic waves transforms into the energy of the collective excitations. At a frequency greater than 2Δ, the usual one-particle mechanism of radiation absorption begins to take effect.

Therefore, there will be two maxima on the curve which shows the difference in the absorption coefficients for metals in the normal and superconducting states as a function of frequency. The first is for the frequency of the collective excitations, the second for $\omega = 2\Delta$. These maxima were observed in experiments [4] on the absorption of microwave radiation in lead and mercury. These metals have the highest ratio of transition temperature to the Debye temperature, which means a strong interaction between the electrons. It is therefore natural that the collective excitations which arise from the interaction are most noticeable in them. A comparison with experiment is given below.

An interesting effect should be observed in the passage of radiation through a superconducting film whose thickness is much greater than the penetration depth. Because of the reflection of the collective excitations from the second surface, interference arises and a component that varies periodically with the frequency and with the film thickness appears in the reflection coefficient. The transmission coefficient of radiation through such films is also computed.

The interaction between the electrons is not assumed to be weak. As in the theory of the Fermi liquid, it is effectively described by certain constants. The Fermi surface is assumed to be isotropic; one can hope that the qualitative results do not change upon consideration of the anisotropy. The temperature is assumed to be equal to zero and the effect of impurities is not taken into account.

The Wiener-Hopf method is applied in the Appendix for the solution of the problem of the diffuse reflection of the collective excitations from the surface.

2. THE POLARIZATION OPERATOR

For the solution of the electrodynamic problem, it is necessary to solve Maxwell's equations, in which the current density is expressed in terms of the vector potential

$$-4\pi c^{-1} j_\alpha = K_{\alpha\beta}(\mathbf{k}, \omega) A_\beta . \tag{1}$$

The polarization operator $K_{\alpha\beta}$ depends on the frequency and the wave vector $\mathbf{k}$. The general expression for $K_{\alpha\beta}$ in systems with pairing has been obtained earlier;[5] however, a relation is used there which follows from the Galilean invariance, and which is not satisfied for electrons in the field of the crystalline lattice. In order to find $K_{\alpha\beta}$ in the superconductor, we average the current operator

$$\hat{\mathbf{j}} = \frac{eh}{2im}(\psi^+\nabla\psi - \psi\nabla\psi^+) - \frac{e^2}{mc}\mathbf{A}\psi^+\psi \tag{2}$$

over the state of the system. In the approximation linear in the field, it is necessary, for averaging the second term, to use the Green's function without account of the field, and, in the first term, to substitute the change in the Green's function G′ in the absence of the external field

$$\mathbf{j} = \frac{e}{m}\int d\mathbf{r}\mathbf{p}G'(\mathbf{r}, \mathbf{r}') - \frac{e^2}{mc}\mathbf{A}\int d\mathbf{r}G(\mathbf{r}, \mathbf{r}), \tag{3}$$

where the momentum operator acts on the coordinate

$$\mathbf{p} = \frac{h}{2i}\int d\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}')\left(\frac{\partial}{\partial\mathbf{r}} - \frac{\partial}{\partial\mathbf{r}'}\right). \tag{4}$$

From the diagrammatic representation of G′ follows its connection with the vertex parts, and in the superconductor it is necessary to introduce not only the vertex T for the creation of a particle and hole, but also the vertex T for the creation of two particles or two holes:

$$G' = -ec^{-1}A_\alpha\{G(p_+)T_\alpha(p)G(p_-) - F(p_+)T_\alpha(-p)F(p_-)$$
$$+ [G(p_+)F(p_-) - F(p_+)G(p_-)]\widetilde{T}_\alpha(p)\},$$
$$p_\pm = \{\mathbf{p} \pm \mathbf{k}/2, \varepsilon \pm \omega/2\}. \tag{5}$$

The one-particle function G and the function F introduced by Gor'kov[6] have close to the Fermi surface the form

$$G = a\frac{\varepsilon + \varepsilon_p}{\varepsilon^2 - \varepsilon_p^2 - \Delta^2 + i\delta},$$
$$F = a\frac{\Delta}{\varepsilon^2 - \varepsilon_p^2 - \Delta^2 - i\delta}, \tag{6}$$

where p is the quasimomentum and a is the renormalization constant.

The equation for the vertex part can be written out[5] if one introduces the irreducible 4-poles U and V, which do not contain diagrams joined only by two lines in the channels particle—hole and particle—particle, respectively:

$$\mathbf{T}(p) = \frac{1}{m}\mathbf{p} + \int\frac{d^4p'}{(2\pi)^4 i}U(pp')\{G(p_+')\mathbf{T}(p')G(p_-')$$
$$- F(p_+')\mathbf{T}(-p')F(p_-')$$
$$- [G(p_+')F(p_-') - F(p_+')G(p_-')]\widetilde{\mathbf{T}}(p')\},$$
$$\widetilde{\mathbf{T}}(p) = \int\frac{d^4p'}{(2\pi)^4 i}V(pp')\{[G(p_+')G(-p_-')$$
$$+ F(p_+')F(p_-')]\widetilde{\mathbf{T}}(p') - G(p_+')\mathbf{T}(p')F(p_-')$$
$$+ F(p_+')\mathbf{T}(-p')G(-p_-')\}. \tag{7}$$

The 4-poles U and V close to the Fermi surface do not depend on $\mathbf{k}$, ω, Δ.

In those terms of Eq. (7) where there are two functions G, the integration is not only over the regions near the Fermi surface, but also far ones. As in the theory of the Fermi liquid,[7] these regions can be eliminated from the equations by redefining the irreducible 4-poles and the bare vertex:

$$\Gamma^k = U + U(GG)^k\Gamma^k, \quad \Gamma^\xi = V + V(GG + FF)^0\Gamma^\xi,$$
$$\mathbf{T}^k = \frac{\mathbf{p}}{m}[1 + (GG)^k\Gamma^k]. \tag{8}$$

The index k means $\omega = \Delta = 0$, $k \to 0$; the index 0 corresponds to $k = \omega = 0$.

Eliminating U and V from (7) and (8), we get a system in which the integration goes only over the regions close to the Fermi surface, where one must use the formula (6) for G and F, and one can integrate over ϵ and ϵ_p (the integrals were computed previously[8]). We have

$$T = T^k + f^k\{LT + M\widetilde{T}\}, \qquad \widetilde{T} = f^\xi\{N\widetilde{T} + OT\}. \tag{9}$$

Here

$$L = \frac{1}{a^2}\int\frac{d\varepsilon d\varepsilon_p}{2\pi i}[GG - (GG)^k - FF\hat{P}]$$
$$= \frac{\omega}{\omega - \mathbf{kv}}(1 - g(\beta)) + g(\beta)\frac{1 - \hat{P}}{2},$$

$$M = \frac{2}{a^2}\int\frac{d\varepsilon d\varepsilon_p}{2\pi i}GF = \frac{\omega + \mathbf{kv}}{2\Delta}g(\beta),$$

$$N=\frac{1}{a^2}\int\frac{d\varepsilon d\varepsilon_p}{2\pi i}[GG(-p_-)+FF-(GG-FF)^0]=\beta^2 g(\beta),$$

$$O=\frac{1}{a^2}\int\frac{d\varepsilon d\varepsilon_p}{2\pi i}(GF-FG(-p_-)\hat{P})$$

$$=-g(\beta)\left(\frac{\omega+\mathbf{kv}}{4\Delta}+\frac{\omega-\mathbf{kv}}{4\Delta}\hat{P}\right),$$

$$g(\beta)=\frac{\arcsin\beta}{\beta\sqrt{1-\beta^2}},\qquad \beta^2=\frac{\omega^2-(\mathbf{kv})^2}{4\Delta^2}. \tag{10}$$

The operator $\hat{P}$ means $\hat{P}T(p) = T(-p)$.

The dimensionless amplitudes f^k and f^ξ are expressed in terms of Γ^k and Γ^ξ:

$$f^k=a^2\rho\Gamma^k,\qquad f^\xi=a^2\rho\Gamma^\xi, \tag{11}$$

where ρ is the density of levels close to the Fermi surface. For an isotropic surface, ρ is expressed in terms of the limiting quasimomentum, and the velocity on the Fermi surface

$$\mathbf{v}=\partial\varepsilon_p/\partial\mathbf{p} \tag{12}$$

is given by the relation

$$\rho=2\int\frac{d\mathbf{p}}{(2\pi)^3}\delta(\varepsilon_p)=\frac{p_0^2}{\pi^2 v}. \tag{13}$$

Making use of Eqs. (1), (3), (5), and (9) we get the following expression for the polarization operator

$$K_{\alpha\beta}=\frac{4\pi e^2}{mc^2}\Big\{\int[p_\alpha(GG)^k T_\beta+\delta_{\alpha\beta}G(\mathbf{r},\mathbf{r})]\,d\mathbf{r}$$

$$+a\rho^2\langle p_\alpha(LT_\beta+M\tilde{T}_\beta)\rangle\Big\}, \tag{14}$$

where $\langle\ldots\rangle$ means averaging over the Fermi surface. The momenta and the energy over which integration is performed in the last term of Eq. (14) are close to their values on the Fermi surface, but in the first terms, regions far from the Fermi surface, including other zones, are important. Therefore these terms are written in the coordinate representation.

For the elimination of the distant region, we substitute T^β from Eq. (9) and use Eq. (8) for T^k:

$$K_{\alpha\beta}=\frac{4\pi e^2}{c^2}\Big\{\frac{1}{m}\int[p_\alpha(GG)^k T_\beta^k+\delta_{\alpha\beta}G(\mathbf{r},\mathbf{r})]\,d\mathbf{r}$$

$$+a^2\rho\langle T_\alpha^k(LT_\beta+M\tilde{T}_\beta)\rangle\Big\}. \tag{15}$$

The first two terms in this equality cancel out if we use the expression for T^k which follows from the condition for gauge invariance. This expression was obtained by Pitaevskiĭ.[9] His derivation must be changed, since it is not possible to use the momentum representation for electrons in the field of the lattice for regions far from the Fermi surface.

Let the fictitious arbitrary static field

$$-eA_\alpha/c=\nabla_\alpha f(\mathbf{r}),\qquad f\sim e^{i\mathbf{kr}},$$

$$\omega=0,\qquad p_0\gg k\gg\Delta/v \tag{16}$$

act on the electron. Then it follows from Eq. (5) that the change in the Green's function is equal to

$$G'=(GG)^k T_\alpha^k ik_\alpha f. \tag{17}$$

On the other hand, introduction of the fictitious field (16) in the system is equivalent to change in the ψ operators: $\psi\to\psi\exp(if)$, $\psi^+\to\psi^+\exp(-if)$; with

$$G(\mathbf{r},\mathbf{r}')=G(\mathbf{r},\mathbf{r}')\exp\{i[f(\mathbf{r})-f(\mathbf{r}')]\}. \tag{18}$$

Expanding this expression in a series in f and in $\mathbf{k}(\mathbf{r}-\mathbf{r}')$ and comparing with (17) we get

$$(GG)^k T_\alpha^k=iG(\mathbf{r},\mathbf{r}')(r_\alpha-r_\alpha'). \tag{19}$$

Close to the Fermi surface, G has the form (6) in the quasimomentum representation, and we get from (19)

$$T_\alpha^k=-\frac{\partial G^{-1}}{\partial p_\alpha}\Big|_{\Delta=0}=\frac{v_\alpha}{a}, \tag{20}$$

where the velocity on the Fermi surface is determined by Eq. (12).

Substituting Eqs. (4) and (16) in the first term of Eq. (15), we can easily demonstrate that it cancels with the second term. In the remaining expression, integration is carried out over the region close to the Fermi surface and one can use Eq. (20) for T^k both in Eq. (15) and in Eq. (9). As a result, we obtain the basic equations for the polarization operator:

$$K_{\alpha\beta}(\mathbf{k},\omega)=4\pi e^2c^{-2}\rho\langle v_\alpha(LT_\beta+M\tilde{T}_\beta)\rangle,$$

$$\mathbf{T}=\mathbf{v}+f^k\{L\mathbf{T}+M\tilde{\mathbf{T}}\},\qquad \tilde{\mathbf{T}}=f^\xi\{N\tilde{\mathbf{T}}+O\mathbf{T}\}. \tag{21}$$

Here and below, the renormalized vertices $\mathbf{T}$ and $\tilde{\mathbf{T}}$ are introduced; they differ by a factor a from the corresponding quantities in the previous formulas.

3. LONDON AND PIPPARD LIMITS

Equation (21) becomes simplified in the limiting cases of small and large $\mathbf{k}$. In the London limit we have $\mathbf{k} = 0$, $L = 1$, $O = 0$; consequently, $\tilde{T} = 0$ and we get

$$K_{\alpha\beta}(0,\omega)=4\pi e^2c^{-2}\rho\langle v_\alpha T_\beta\rangle,\qquad \mathbf{T}=\mathbf{v}+f^k\mathbf{T}. \tag{22}$$

For a spheroidal Fermi surface

$$K_{\alpha\beta}=\delta_{\alpha\beta}K,\qquad K=\lambda_L^{-2}=4\pi/c^2\Lambda=4\pi N_0e^2/mc^2, \tag{23}$$

where Λ is the London parameter, λ_L is the London penetration depth, and N_0 is the so-called number of free electrons

$$N_0 = \frac{\rho v^2 m}{3(1-f_1{}^k)} = \frac{p_0{}^2 v m}{3\pi^2(1-f_1{}^k)}. \qquad (24)$$

This number is identical with the corresponding quantity entering into the dielectric constant $\epsilon = -4\pi N_0 e^2/m\omega^2$ in the infrared region, [10] and depends on the periodic field and on the interelectron interaction. Only in the absence of a periodic field, when the Galilean invariance leads to the following relation between the velocity and the momentum

$$mv = p_0(1-f_1{}^k) = p_0/(1+f_1{}^\omega),$$

does the number N_0 coincide with the electron density $p_0^3/3\pi^2$.

In the Pippard limiting case $kv \gg \Delta$ we have $L \sim M \sim N \sim O \sim \Delta/kv \ll 1$, $\mathbf{T} = \mathbf{v}$ in Eqs. (21), and we get

$$K_{\alpha\beta} = 4\pi e^2 c^{-2}\rho\, \langle v_\alpha L v_\beta\rangle. \qquad (25)$$

For an isotropic Fermi surface and a transverse vector potential, we get

$$K_{\alpha\beta} = K\delta_{\alpha\beta}, \qquad K = \frac{e^2 p_0{}^2}{\pi c^2 k}\int_{-\infty}^{\infty} L d\mathbf{kv}. \qquad (26)$$

This expression is identical with that obtained in the weak coupling model. [2,3] It does not depend either on the velocity on the Fermi surface or on the interaction, and is determined only by the momentum on the Fermi surface, which is expressed in terms of the number of electrons in the conduction band. The integral in Eq. (26) can be reduced to an elliptic integral. We set down, for various frequencies ω, the limiting values needed for what follows:

$$K(0) = \frac{e^2 p_0{}^2}{c^2}\frac{\pi\Delta}{k}, \quad K(2\Delta) = \frac{e^2 p_0{}^2}{c^2}\frac{2\Delta}{k},$$

$$K(\omega \gg \Delta) = \frac{e^2 p_0{}^2}{c^2}\frac{i\omega}{k}. \qquad (27)$$

The latter case corresponds to the normal state.

4. COLLECTIVE EXCITATIONS

In the limiting cases considered above, the interaction did not play an important role. The most noticeable effects arise from the fact that the interaction leads to the possibilities of excitation of collective excitations in the superconductor. The spectrum of these excitations is determined by the poles of the polarization operator. These poles arise for those frequencies $\omega(\mathbf{k})$ for which there is a solution of the homogeneous set of equations corresponding to the system (21).

Further, only the isotropic case will be considered. Here the amplitudes f^k and f^ξ depend on the angle $(\mathbf{n}\cdot\mathbf{n}')$ between the vectors $\mathbf{v} = v_0\mathbf{n}$ and $\mathbf{v}' = v_0\mathbf{n}'$ and can be represented in the form of a series in the spherical harmonics

$$f(\mathbf{nn}') = \sum_l f_l \frac{2l+1}{4\pi} P_l(\mathbf{nn}') = \sum_{lm} f_l Y_{lm}(\mathbf{n})\, Y_{lm}{}^*(\mathbf{n}'). \quad (28)$$

The frequencies of the excitations are easily found for $k = 0$. In this case, the coefficients L, M, N, and O in Eq. (21) do not depend on the angles and the integral equations reduce to algebraic equations. The condition for the solvability of the homogeneous equation has the form

$$(1-f_l{}^k L)(1-f_l{}^\xi N) - f_l{}^k f_l{}^\xi MO = 0. \qquad (29)$$

Using (10) we get, for even l,

$$1 = \left(f_l{}^\xi \frac{\omega^2}{4\Delta^2} - f_l{}^\omega\right) g\left(\frac{\omega}{2\Delta}\right), \qquad (30)$$

where f_l^ω is the amplitude introduced by Landau [7], which is expressed in terms of f_l^k by the formula

$$f_l{}^\omega = f_l{}^k / (1-f_l{}^k). \qquad (31)$$

In the weak interaction approximation, the amplitudes are expressed in terms of the harmonic potential

$$V_l = -\frac{4p_0}{v}\int_0^\infty V(r)\, J^2_{l+1/2}(p_0 r)\, r\, dr$$

in the following way:

$$f_l{}^\xi = V_0 V_l / (V_0 - V_l), \qquad f_l{}^\omega = V_l. \qquad (32)$$

The zeroth harmonic V_0 determines the ratio of the gap to the phonon frequency ω_c:

$$\Delta = 2\omega_c \exp(-1/V_0). \qquad (33)$$

Equation (30) with account of (32) was obtained in many researches. [8,11,12] Tsuneto [13] used for the determination of the excitation spectrum a random phase approximation in which f^ξ is determined by Eq. (32) while $f^\omega = 0$.

For small f^ξ and f^ω, Eq. (3) has a solution only for frequencies very close to 2Δ:

$$\omega_l^2(0) = 4\Delta^2\left[1 - \tfrac{1}{4}\pi^2 (f_l^\xi - f_l^\omega)^2\right]. \qquad (34)$$

Experimental data correspond to $\omega \approx \Delta$ for lead and $\omega = 1.5\Delta$ for mercury; for their explanation in the weak coupling approximation, it is assumed that V_2 is very close to V_0, which is strange. However, the interaction between the electrons in the metal is not small. The smallness of Δ in comparison with the Debye temperature means that the 4-pole which characterizes the interaction of a particle with a particle is comparatively small [in Eq. (33), $V_0 \approx 0.5 - 0.25$]. Therefore, one can think that the f_1^ξ determined by the other harmonics of this same quadrupole are comparatively small.

However, the amplitude of f^{ω}, which characterizes the interaction of a particle with a hole, is determined by completely different diagrams and hence is close to unity in order of magnitude.

In the determination of the excitation spectrum from the poles of the polarization operator, it was not taken into account that an electromagnetic field is present in the system. The collective excitations represent the simultaneous oscillations of the particles and the field and their frequencies are determined from the equation

$$[(\omega^2/c^2 - k^2)\delta_{\alpha\beta} + k_\alpha k_\beta - K_{\alpha\beta}(k,\omega)] \times A_\beta(\mathbf{k},\omega) = 0. \tag{35}$$

The most important effect of the electromagnetic field is on the spectrum of longitudinal oscillations. Their frequencies at $\mathbf{k} = 0$ are equal not to zero, as would follow from (30), but to the plasma frequency $\omega_0^2 = 4\pi e^2 N_0/m$. This is easily seen if we substitute the limiting expression for K from Eq. (23) in Eq. (35). However, for such large frequencies, the approximations of the theory of a Fermi liquid, which we used, are not applicable. For the low frequencies considered, the longitudinal field penetrates into the metal only to a depth of the order of the Debye radius.

Only the transverse field will be considered below; moreover, for frequencies $\omega \sim \Delta$, the wavelength of light is much larger than the penetration depth and one can neglect the first term in Eq. (35). As a result,

$$[k^2 + K(k,\omega)]A_\alpha = 0. \tag{36}$$

We limit ourselves to the first three terms in the expansion of the amplitudes in terms of the harmonics, and take it into account that f^{ξ} has only even harmonics for the spinless excitations considered. The system (21) for this case reduces to a set of algebraic equations. After long calculation, we get for the transverse field

$$K = \lambda_L^{-2}\left\{1 + \frac{1}{5}\left(\frac{kv}{\omega}\right)^2 (1 + f_1^{\omega}) \times \frac{L_2 - f^{\xi}_2(L_2N_2 - M_2O_2)}{1 - f_2^{\xi}N_2 - [f_2^k + {}^1/_5(kv/\omega)^2 f_1^{\omega}][L_2 - f_2^{\xi}(L_2N_2 - M_2O_2)]}\right\}. \tag{37}$$

Here

$$L_2 = \int d\mathbf{n}\, |Y_{21}(\mathbf{n})|^2 L(\omega, \mathbf{k}v\mathbf{n})$$

(and similarly for M_2, N_2, O_2) and λ_L is the London penetration depth (23).

If the second harmonics of the interaction f_2^{ξ} and f_2^k are small, then the frequencies of the collective excitations are close to 2Δ, while the momenta are small ($kv \ll \Delta$). In this case, Eq. (37) becomes simplified:

$$K = \lambda_L^{-2}\left\{1 + \frac{1}{5}\left(\frac{kv}{\omega}\right)^2 \frac{1 + f_1^{\omega}}{f_2 - g_2^{-1}}\right\}, \quad f_2 = f_2^{\xi} - f_2^{\omega}, \tag{38}$$

$$g_2 = \int \frac{\pi\Delta\, |Y_{21}(\mathbf{n})|^2 d\mathbf{n}}{[4\Delta^2 - \omega^2 + (\mathbf{v}k\mathbf{n})^2]^{1/2}} = \frac{15\pi\Delta}{16kv}\left[(2 + 3a^2)\sqrt{1 + a^2} - (4 + 3a^2)\, a^2 \operatorname{arcsh}\frac{1}{a}\right],$$

$$a = \sqrt{4\Delta^2 - \omega^2}/kv. \tag{39*}$$

The limiting expressions for the function g have the following form

$$\frac{1}{g_2} = \frac{\sqrt{4\Delta^2 - \omega^2}}{\pi\Delta}\left(1 + \frac{3}{14}\frac{k^2v^2}{4\Delta^2 - \omega^2}\right)$$

for $k^2v^2 \ll 4\Delta^2 - \omega^2$,

$$\frac{1}{g_2} = \frac{8kv}{15\pi\Delta}\left(1 + \frac{4\Delta^2 - \omega^2}{k^2v^2}\ln\frac{k^2v^2}{4\Delta^2 - \omega^2}\right)$$

for $k^2v^2 \gg 4\Delta^2 - \omega^2$. (40)

5. SURFACE IMPEDANCE

Absorption of microwave radiation incident on the surface of the superconductor will take place only when the frequency of the radiation is higher than the frequencies of excitation in the superconductor. The reflection coefficient is expressed in terms of the real part of the surface impedance

$$D_{\text{ref}} = 1 - \frac{c}{\pi}\operatorname{Re} Z, \tag{41}$$

$$Z = \frac{4\pi}{c}\left(\frac{E}{H}\right)_{\text{sur}} = \frac{4\pi i\omega}{c^2}\left(A \Big/ \frac{\partial A}{\partial x}\right)_{\text{sur}}. \tag{42}$$

The surface impedance can be expressed in terms of the polarization operator. For specular reflection of electrons from the surface,

$$Z = \frac{8i\omega}{c^2}\int_0^\infty \frac{dk}{k^2 + K(k,\omega)}. \tag{43}$$

For diffuse reflection[14]

$$Z = \frac{4i\pi^2\omega}{c^2}\left\{\int_0^\infty \ln[1 + k^{-2}K(k,\omega)]\,dk\right\}^{-1}. \tag{44}$$

This formula can be regarded as a special case of Eq. (A.19) derived in the Appendix.

The polarization operator K has an imaginary part only for $\omega > 2\Delta$. However, the integrals that enter into Eqs. (43), and (44) have imaginary parts

*arcsh = $\sinh^{-1}$.

also at lower frequencies, if $k^2 + K(k, \omega)$ vanishes for some $k = k_1$, that is, a collective excitation can be propagated in the superconductor with frequency ω and momentum k_1. In the most important case of diffuse scattering, the imaginary part of the integral in Eq. (44) is equal to

$$I = \frac{1}{\pi}\,\mathrm{Im}\int \ln\left(1 + k^{-2}K(k, \omega)\right) dk = k_2 - k_1, \qquad (45)$$

here k_1 is the zero of the argument of the logarithm, and k_2 is the pole of this expression. Making use of Eqs. (38) and (39) for the determination of k_1 and k_2, we get

$$I = \frac{\pi\Delta}{v}\left(1 + f_1^\omega\right)\left(1 - \frac{\omega_0^2}{4\Delta^2}\right)\varphi\left(\frac{4\Delta^2 - \omega^2}{4\Delta^2 - \omega_0^2}\right), \qquad (46)$$

where ω_0 is the frequency of collective excitations for zero momentum, determined by Eq. (34). The function $\varphi(x)$ has in the limiting cases the form

$$\varphi(x) = \frac{1}{5}\left(\frac{15}{8}\right)^3\left(1 - \left(\frac{8}{15}\right)^2 x \ln\frac{1}{x}\right) \quad \text{for } x \to 0,$$
$$\varphi(x) = \frac{2}{5}\left(\frac{7}{3}\right)^{3/2}\sqrt{1 - x} \quad \text{for } x \to 1 \qquad (47)$$

and it can be written in the form of the interpolation formula

$$\varphi(x) = 1.3\sqrt{1 - x}\left(1 - 0.28 \ln\frac{0.7}{x}\right). \qquad (48)$$

In the calculation of the real part of the integral (44) for the Pippard metals, the principal contribution is made by the region $kv \gg \Delta$, where Eqs. (26), (27) can be used for K. For $\omega = 2\Delta$ we have

$$\frac{1}{\pi}\int \ln\left(1 + k^{-2}K\right) dk = \lambda_\omega^{-1}$$
$$= \frac{2}{\sqrt{3}}\left(\frac{e^2 p_0^2}{c^2}\, 2\Delta\right)^{1/3} = \left(\frac{2}{\pi}\right)^{1/3}\lambda^{-1}, \qquad (49)$$

where λ is the Pippard penetration depth. Taking it into account that the imaginary part of the integral is small in comparison with the real part, we get

$$\mathrm{Re}\, Z = 4\omega c^{-2}(\pi/2)^{2/3}\lambda^2 I. \qquad (50)$$

Using the last equation of (27), we get the impedance of the normal metal:

$$Z_n = \frac{4i\omega}{c^2}\frac{\sqrt{3}}{2}\left(i\omega\frac{e^2 p_0^2}{c^2}\right)^{-1/3} = \frac{i\omega}{\pi c}\left(\frac{i\omega}{\pi\Delta}\right)^{-1/3}\lambda. \qquad (51)$$

It follows from Eqs. (46)—(51) that the ratio of the absorption coefficients of metals in the superconducting state to those in the normal state at ω close to 2Δ is equal to

$$\frac{\mathrm{Re}\, Z}{\mathrm{Re}\, Z_n} = 2\left(\frac{\pi}{2}\right)^{1/3}\lambda I = \frac{\pi\Delta}{v}\lambda\left(1 + f_1^\omega\right)\left(1 - \frac{\omega_0^2}{4\Delta^2}\right)$$
$$\times\, 3.2\sqrt{1 - x}\left[1 - 0.28x \ln\frac{0.7}{x}\right],$$
$$x = (4\Delta^2 - \omega^2)/(4\Delta^2 - \omega_0^2). \qquad (52)$$

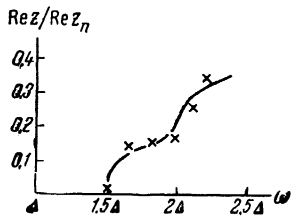
In the derivation of this formula, use was made of the condition that ω_0 is close to 2Δ. This is comparatively well satisfied for mercury. For lead, $\omega_0 \approx \Delta$; therefore, Eq. (52) is changed; however, in the general case, the vanishing will take place at a rate proportional to $(\omega - \omega_0)^{1/2}$ and a logarithmic singularity occurs in the derivative for ω close to 2Δ. Calculation of the specular reflection by Eq. (43) leads to the additional factor $2(2\pi)^{2/3}\lambda_L^2/\lambda^2$ in Eq. (52). Similar calculations of Tsuneto[13] are valid only for ω close to ω_0, since he assumed that $(kv)^2 \ll 4\Delta^2 - \omega^2$. For frequencies close to 2Δ, the absorption is determined by one-particle approximations[2,3] and, close to $\omega = 2\Delta$,

$$\mathrm{Re}\, Z / \mathrm{Re}\, Z_n = (\omega - 2\Delta)/3\Delta. \qquad (53)$$

However, the region very close to 2Δ is determined by small k in the integral (44) even for Pippard metals. Computing the contribution of the region $(\omega^2 - 4\Delta^2)^{1/2} \ll kv \ll \Delta$ in Eq. (44) with logarithmic accuracy, we get

$$\frac{\mathrm{Re}\,[Z(\omega) - Z(2\Delta)]}{\mathrm{Re}\, Z_n} = \left(1 + f_1^\omega\right)\frac{\pi\Delta}{v}\lambda\,\frac{3}{8}\left(\frac{\pi}{2}\right)^{1/3}\left(\frac{\omega^2}{4\Delta^2} - 1\right)$$
$$\times \ln\left[\frac{\Delta\sqrt{\omega^2 - \omega_0^2}}{\omega^2 - 4\Delta^2}\right]. \qquad (54)$$

The logarithmic factor can be shown to be of the order $v/\pi\Delta\lambda$, and the contribution from (54) will be equal to the contribution from (53). A plot of the function represented by Eqs. (52)—(54) is shown in the drawing. The values corresponding



to mercury[4] are taken as parameters: $\omega_0 = 1.5\Delta$, $v/\pi\Delta\lambda = \xi_0/\lambda = 7$. The corrected experimental values are denoted by crosses. Equation (52) gives an excessive value for lead. This can be explained by the fact that Eq. (52) is applicable only for ω_0 close to 2Δ, and it is necessary to use the general formulas (37) and (44). However, too detailed agreement should not be expected for the isotropic model under consideration.

6. PASSAGE THROUGH THICK FILMS

Films whose thickness is greater than the penetration depth can be regarded as semi-infinite bodies if there are no collective excitations. In the frequency range from ω_0 to 2Δ, the electromagnetic field is carried by the collective excitations even through thick films. The transmission coefficient through such films is not equal to zero, and interference phenomena should be observed in the reflection coefficient.

In the case of practical interest of diffuse reflection from the surface, the equation for the vector potential has the form [1)]

$$\frac{\partial^2 A(x)}{\partial x^2} = \int_0^a K(x-y) A(y)\, dy, \tag{55}$$

a is the thickness of the film; the Fourier component of the kernel of the equation is determined by Eqs. (37) and (38).

In contrast with the semi-infinite case, Eq. (55) cannot be solved in the general case. However, for a thick plate, the solution close to each boundary can be sought independently and thus the problem can be reduced to the semi-infinite case. Usually, a solution which falls off at infinity is sought in the semi-infinite case; if there is a weakly damped excitation, then an outgoing wave also exists at large distances in this solution. The waves inside the plate go in both directions and one must find the solution of the semi-infinite problem for the case in which it has the form

$$A(x) = Ce^{ikx} + De^{-ikx} \tag{56}$$

far from the edge.

This problem is solved in the Appendix, and Eqs. (A.13) and (A.19) give an expression for the logarithmic derivative of the vector potential on the surface in terms of the ratio C/D.

No electromagnetic wave incident from the vacuum is present at the rear surface of the plate; therefore, E = H, i.e., $A^{-1}\partial A/\partial x = i\omega/c$. Taking it into account that the penetration depth is small in comparison with the electromagnetic wavelength in the vacuum and with the wavelength of the collective excitations, we get for the reflection coefficient of the collective excitations (A.20)

$$\gamma = C/D = (k_1 - k_2)/(k_1 + k_2). \tag{57}$$

The roles of the incoming and outgoing waves change in relation to the edge irradiated. Making the substitution in Eq. (56), we get

$$C' = De^{-ik_1 a}, \qquad D' = Ce^{ik_1 a}. \tag{58}$$

Again making use of Eqs. (A.13) and (A.19), we get an expression for the surface impedance of the film:

$$Z^{-1} = \frac{ic^2}{4\pi\omega}\frac{\partial \ln A}{\partial x} = -\frac{ck_1}{\omega}\frac{2\gamma^2}{e^{2ik_1 a} - \gamma^2} + Z^{-1}_\infty, \tag{59}$$

where Z_∞ is the surface impedance of the semi-infinite specimen.

The reflection coefficient is determined by the real part of A; taking it into account that it is small in comparison with the imaginary part, and assuming $\gamma \ll 1$, we get

$$\operatorname{Re} Z = \operatorname{Re} Z_\infty (1 - \gamma \cos 2k_1 a). \tag{60}$$

The momentum k_1 of the collective excitations and the pole k_2 of the polarization operator are determined from Eqs. (38) and (39). As a result, we get for the limiting value of the coefficient

$$\gamma(\omega_0) = \frac{7\pi}{30}(1 + f_1^\omega)\sqrt{1 - \frac{\omega_0^2}{4\Delta^2}}. \tag{61}$$

These expressions are obtained under the assumption $\gamma \ll 1$. For lead and mercury, $\gamma \approx 0.5$; therefore, the oscillations of the reflection coefficient determined by Eqs. (41) and (60) will be clearly noticeable. In Eq. (60), the dependence of k_1 on the frequency close to ω_0 is determined by the equation

$$k_1(\omega) = v^{-1}\sqrt{{}^7/_3(\omega^2 - \omega_0^2)}. \tag{62}$$

Similarly, one can find the connection between the value of the field on one or the other surface:

$$A_2 = i\lambda(k_1 - k_2)e^{ik_1 a}A_1. \tag{63}$$

As a result, we get for the transmission coefficient

$$D_{\text{пр}} = (\omega\lambda/c)^2(\operatorname{Re} Z/\operatorname{Re} Z_n)^2. \tag{64}$$

This quantity is very small because of the smallness of the penetration depth λ.

For experimental proof of the fact that the additional absorption in lead and mercury is brought about by the collective oscillations, it would be useful to observe interference in the reflection coefficient.

The author expresses his gratitude to V. M. Galitskiĭ for valuable advice.

[1)]The problem of the applicability of this equation to the description of diffuse reflection requires clarification. It has little effect on the results (A.5), but can increase the coefficient of transmission through the film.

APPENDIX

We apply the Wiener-Hopf method to find the solution of the equation

$$\frac{\partial^2 A}{\partial x^2} = \int_0^\infty K(x - x')\, A(x')\, dx', \qquad (A.1)$$

which has the following form for large x:

$$A(x) = Ce^{ik_1x} + De^{-ik_1x}, \qquad (A.2)$$

where k_1 has a small positive imaginary part.

Usually, the case considered corresponds to D = 0, i.e., only the outgoing wave is present. Assuming that Eq. (A.1) defines the function A(x) also for negative x, while $A(-\infty) = 0$, we introduce the functions

$$A_+(x) = \begin{cases} A(x), & x > 0 \\ 0, & x < 0 \end{cases},$$

$$A_-(x) = \begin{cases} 0, & x > 0 \\ A(x), & x < 0 \end{cases}. \qquad (A.3)$$

The Fourier components of these functions

$$A_\pm(k) = \int_{-\infty}^{\infty} A_\pm(x)\, e^{-ikx}\, dx \qquad (A.4)$$

do not have any singularities: A_+ is in the half-plane $\mathrm{Im}\, k \le -c$, and A_- in the half plane $\mathrm{Im}\, k \ge -c$, where $c > \mathrm{Im}\, k_1$.

The inverse transformation has the form

$$A_\pm(x) = \int_{-ic-\infty}^{-ic+\infty} e^{ikx} A_\pm(k) \frac{dk}{2\pi}. \qquad (A.5)$$

Integration is carried out below the real axis, since A(x) can increase for large x.

Equation (A.1) in the k representation has the form

$$-k^2(A_+(k) + A_-(k)) = K(k)A_+(k) + a + ibk. \qquad (A.6)$$

The constants a and b are equal to the jumps in the function A(x) and its derivative at x = 0. To find the functions A_+ and A_-, we rewrite Eq. (A.6) in such a form that the left side of the equation contains a function which is analytic in the upper half plane and the right side contains one analytic in the lower half plane. For this purpose, we write

$$1 + K(k)\,/\,k^2 \equiv L_+(k)\,/\,L_-(k), \qquad (A.7)$$

$$L_\pm(k) = \exp\left[\frac{1}{2\pi i}\int_{-ic-\infty}^{-ic+\infty} \ln\left(1 + \frac{K(q)}{q^2}\right)\frac{dq}{q - k \pm i\delta}\right], \qquad (A.8)$$

where $L_+(k)$ is analytic for $\mathrm{Im}\, k \le -c$, while $L_-(k)$ is analytic for $\mathrm{Im}\, k \ge -c$. Taking (A.7) into account, Eq. (A.6) can be represented in the form

$$A_-L_- - \frac{a + ibk}{k^2}L_- - \frac{\alpha}{k^2} - \frac{\beta}{k} = A_+L_+ - \frac{\alpha}{k^2} - \frac{\beta}{k}. \qquad (A.9)$$

The constants α and β are so chosen that there are no poles at the point k = 0 in the left side of the equation. On the left side of the equation there is a function which is analytic in the upper half plane, and on the right side there is a function which coincides with the previous function on the line $\mathrm{Im}\, k = -c$, and which is analytic in the lower half-plane. Therefore, this function is analytic in the entire plane, and, consequently, is equal to zero. Thus,

$$A_+(k) = \left(\frac{\alpha}{k^2} + \frac{\beta}{k}\right)\frac{1}{L_+}. \qquad (A.10)$$

The constants α and β can be expressed in terms of the amplitudes C and D of the incoming and outgoing waves. For this purpose, we separate the zeros and poles in L_+ close to the real axis:

$$1 + \frac{K(k)}{k^2} = \frac{k^2 - k_1^2}{k^2 - k_2^2}\left(1 + \frac{k_2^2}{k_1^2\lambda_L^2k^2}\right)Q(k),$$

$$L_+ = \frac{(k^2 - k_1^2)(k - ik_2/k_1\lambda_L)}{k^2(k - k_2)}Q_+(k). \qquad (A.11)$$

The function Q(k) and the function $Q_+(k)$ obtained from it by Eq. (A.8) have singularities only in the complex plane at distance of the order of Δ/v from the real axis; $Q(0) = Q(\infty) = 1$. For London metals, $Q(k) \equiv 1$. Substituting (A.11) in (A.10), we get

$$A_+(k) = \frac{(\alpha + \beta k)(k - k_2)}{2k_1(k - ik_2/k_1\lambda_L)\,Q_+(k)} \times\left(\frac{1}{k - k_1} - \frac{1}{k + k_1}\right). \qquad (A.12)$$

The distribution of the field in the superconductor can be found from Eq. (A.5); it is determined close to the line $\mathrm{Im}\, k = -c$ for large x by the singularities of the function $A_+(k)$, i.e., by the poles at the points $k = \pm k_1$. The residues at these poles are equal to the amplitudes C and D of the outgoing and incoming waves. Taking k_1 to be small in comparison with Δ/v, so that $Q_+(k_1) = Q_+(-k_1)$, we get

$$\frac{C}{D} = \frac{\alpha + \beta k_1}{\alpha - \beta k_1}\,\frac{k_1 - k_2}{k_1 + k_2}. \qquad (A.13)$$

The condition $\alpha = \beta k_1$ corresponds to the absence of an incoming wave.

The behavior of the field close to the boundary x = 0 is determined by the form of $A_+(k)$. From Eq. (A.8) we get

$$L_+ \to 1 + \frac{1}{2\pi ik}\int_{-ic-\infty}^{-ic+\infty} \ln\left(1 + \frac{K(q)}{q^2}\right)dq. \qquad (A.14)$$

We move the contour in this integral to the real axis, bypassing the singularity at the point $q = -k_1$, and we denote the resultant integral by

$$\lambda_\omega = 2\pi \left\{ \int_{-\infty}^{\infty} \ln \left(1 + \frac{k(q)}{q^2}\right) dq \right\}^{-1}. \qquad (A.15)$$

Substituting (A.14) and (A.15) in (A.10), we get

$$A_+(k) \to \frac{\beta}{k} + \frac{1}{k^2}\left(\alpha - \beta k_1 + \beta \frac{i}{\lambda_\omega}\right). \qquad (A.16)$$

From Eq. (A.5) we get, with the aid of (A.3),

$$\int_{-ic-\infty}^{-ic+\infty} A_+(k)\frac{dk}{2\pi} = \frac{A_+(+0) + A_+(-0)}{2} = \frac{1}{2}A_+(+0). \qquad (A.17)$$

Taking it into account that $A_+(k)$ does not have singularities in the lower half plane and displacing the contour of integration to the large lower half circle, we get

$$A(0) = A_+(+0) = -i\beta. \qquad (A.18)$$

The derivative $\partial A/\partial x$ is expressed in similar fashion in terms of the second term of the expansion (A.16). As a result,

$$\left.\frac{\partial \ln A}{\partial x}\right|_{x=0} = i\left(\frac{\alpha}{\beta} - k_1\right) - \frac{1}{\lambda_\omega}. \qquad (A.19)$$

Eliminating the ratio α/β from (A.19) and (A.13), we get the desired connection between the logarithmic derivative on the surface and the amplitudes of the incoming and outgoing waves of the collective excitations.

When there is no incoming wave, $D = 0$, $\alpha = \beta k_1$, we get for a penetration depth equal to the inverse logarithmic derivative the well known result of Reuter and Sondheimer.[14] In the other limiting case, where no electromagnetic radiation is incident on the surface and the logarithmic derivative is equal to the reciprocal of the wavelength of light $\omega/c \approx 0$, we get for the reflection coefficient of the collective excitations

$$\frac{C}{D} = \frac{k_1 - k_2}{k_1 + k_2}(1 + 2i\lambda_\omega k_1), \qquad (A.20)$$

where k_1 and k_2 are the zero and pole of the expression $k^2 + K(k)$.

[1] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] Abrikosov, Gor'kov, and Khalatnikov, JETP **35**, 265 (1958), Soviet Phys. JETP **8**, 182 (1959).

[3] D. C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958).

[4] M. Tinkham and D. M. Ginsberg, Phys. Rev. **118**, 990 (1960).

[5] A. I. Larkin and A. B. Migdal, JETP **44**, 1703 (1963), Soviet Phys. JETP **17**, 1146 (1963).

[6] L. P. Gor'kov, JETP **34**, 735 (1958), Soviet Phys. JETP **7**, 505 (1958).

[7] L. D. Landau, JETP **35**, 97 (1958), Soviet Phys. JETP **8**, 70 (1959).

[8] Vaks, Galitskiĭ, and Larkin, JETP **41**, 1655 (1961), Soviet Phys. JETP **14**, 1177 (1962).

[9] L. P. Pitaevskiĭ, JETP **37**, 1794 (1959), Soviet Phys. JETP **10**, 1267 (1960).

[10] L. P. Pitaevskiĭ, JETP **34**, 942 (1958), Soviet Phys. JETP **7**, 652 (1958).

[11] Bogolyubov, Tolmachev, and Shirkov, Novyĭ metod v teorii sverkhprovodimosti (New Method in the Theory of Superconductivity) (Fizmatgiz, 1960).

[12] A. Bardasis and J. R. Schrieffer, Phys. Rev. **121**, 1050 (1961).

[13] T. Tsuneto, Phys. Rev. **118**, 1029 (1960).

[14] B. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).

Translated by R. T. Beyer
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