NONLINEAR EFFECTS IN THE ELECTRODYNAMICS OF SUPERCONDUCTORS

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The dependence of a superconductor "energy gap" and the penetration depth of a constant magnetic field on the field strength is investigated on the basis of a microscopic theory. The investigation is carried out for arbitrary temperatures; the limiting cases of absolute zero and of temperatures close to T_c are examined in detail for Pippard and London superconductors.

A linear electrodynamics of superconductors within the framework of a microscopic theory was first constructed in the article by Bardeen, Cooper, and Schrieffer.[1] This enabled them to explain the Meissner effect and to determine one of the important characteristics of a superconductor—the penetration depth of a weak magnetic field. Later on, Gor'kov[2] succeeded in writing down the equations for the electrodynamics of superconductors in an arbitrary magnetic field in gauge invariant form. However, in view of the complexity of these equations, aside from the weak field case it has been possible to obtain concrete results for pure superconductors only in a comparatively small number of cases: the Ginzburg-Landau equations, the supercooling field, and others. Therefore, it is of definite interest to investigate nonlinear effects related to the influence of an external magnetic field on basic characteristics of a superconductor, such as the energy gap, the Meissner effect, and the penetration depth. These problems are indeed the subject of the present article. The field will be treated by perturbation theory; the corresponding criteria will be evident from the results obtained.

The problem is solved for an infinite superconducting half-space. It is obvious that the results obtained can also be directly applied to bulk superconductors whose dimensions exceed the penetration depth. The presence of a boundary poses the question of the nature of the reflection of electrons from a superconductor surface. In the present article, we shall assume the reflection to be specular. A separate article will be devoted to the case of diffuse reflection.

1. GENERAL EQUATIONS

According to Gor'kov,[2] a superconductor in an arbitrary (constant in time) magnetic field may be described by a system of gauge-invariant equations for the thermodynamic functions:

$$\left(i\omega + \frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}\right)^2 + \mu\right)\mathfrak{G}_\omega(\mathbf{r}, \mathbf{r}') + \Delta(\mathbf{r})\mathfrak{F}_\omega^+(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

$$\left(-i\omega + \frac{1}{2m}\left(\nabla + \frac{ie}{c}\mathbf{A}\right)^2 + \mu\right)\mathfrak{F}_\omega^+(\mathbf{r}, \mathbf{r}') - \Delta^*(\mathbf{r})\mathfrak{G}_\omega(\mathbf{r}, \mathbf{r}') = 0, \tag{1.1}$$

where $\omega \equiv \pi T(2n+1)$ (n is an integer). The quantity $\Delta^*(\mathbf{r})$ (the energy gap in the superconductor's spectrum in the absence of a field) is related to $\mathfrak{F}^+$ by the equation

$$\Delta^*(\mathbf{r}) = |\lambda| T \sum_\omega \mathfrak{F}_\omega^+(\mathbf{r}, \mathbf{r}'), \tag{1.2}$$

where λ is the "superconducting" interaction coupling constant. In what follows, we shall conventionally call $\Delta^*(\mathbf{r})$ the "energy gap."

It is possible to combine the system (1.1) together with its Hermitian conjugate into a single matrix equation

$$\begin{pmatrix} i\omega + \frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}\right)^2 + \mu & \Delta(\mathbf{r}) \\ -\Delta^*(\mathbf{r}) & -i\omega + \frac{1}{2m}\left(\nabla + \frac{ie}{c}\mathbf{A}\right)^2 + \mu \end{pmatrix} \times \begin{pmatrix} \mathfrak{G}_\omega & -\mathfrak{F}_\omega \\ \mathfrak{F}_\omega^+ & \mathfrak{G}_\omega^+ \end{pmatrix} = 1, \tag{1.3}$$

where

$$\mathfrak{F}_\omega(\mathbf{r}, \mathbf{r}') = \mathfrak{F}_{-\omega}^+(\mathbf{r}', \mathbf{r}), \qquad \mathfrak{G}_\omega^+(\mathbf{r}, \mathbf{r}') = \mathfrak{G}_{-\omega}(\mathbf{r}', \mathbf{r}).$$

Let us denote the solution of Eq. (1.3) by

$$\hat{\mathfrak{G}}_\omega(\mathbf{r}, \mathbf{r}') = \begin{pmatrix} \mathfrak{G}_\omega & -\mathfrak{F}_\omega \\ \mathfrak{F}_\omega^+ & \mathfrak{G}_\omega^+ \end{pmatrix}. \tag{1.4}$$

It is possible to transform Eq. (1.4) into an integral equation with the aid of its solution in the absence of a field, $\hat{\mathfrak{G}}^{(0)}$:

$$\hat{\mathfrak{G}}_\omega(\mathbf{r}, \mathbf{r}') = \hat{\mathfrak{G}}_\omega^{(0)}(\mathbf{r}, \mathbf{r}') + \frac{ie}{m}\int \hat{\mathfrak{G}}_\omega^{(0)}(\mathbf{r}, \mathbf{r}_1)\,\sigma_z\left(\mathbf{A}(\mathbf{r}_1)\frac{\partial}{\partial \mathbf{r}_1}\right)\hat{\mathfrak{G}}_\omega(\mathbf{r}_1, \mathbf{r}')\,d\mathbf{r}_1 - \int \hat{\mathfrak{G}}_\omega^{(0)}(\mathbf{r}, \mathbf{r}_1)\,\Delta_1(\mathbf{r}_1)\,\hat{\mathfrak{G}}_\omega(\mathbf{r}_1, \mathbf{r}')\,d\mathbf{r}_1, \tag{1.5}$$

where Δ_1 denotes the correction to the energy gap, appearing due to the presence of a field

$$\Delta_1(\mathbf{r}) = \Delta(\mathbf{r}) - \Delta. \tag{1.6}$$

In Eq. (1.5), we have neglected terms of second order in the field, since the radius of the electron's helical motion, $\sim cp_0/eH$, is considerably greater than the penetration depth for the fields under consideration [$p_0 \gg eA/c \sim eH\delta/c$ (p_0 is the Fermi momentum)], and we used the gauge $\nabla \cdot \mathbf{A} = 0$.

In Eq. (1.5), the coordinates $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}'$ belong to the half-space occupied by the metal. It is easy to show, similarly to what was done by Abrikosov and Fal'kovskiĭ,[3] that the determination of $\hat{\mathfrak{G}}_\omega(\mathbf{r}, \mathbf{r}')$ in the case of specular reflection can be reduced to the solution of Eq. (1.5) for an unbounded region of space, provided $\mathbf{A}(\mathbf{r})$ and $\Delta(\mathbf{r})$ are continued, as even functions of z, for negative values of z:

$$A(x, y, z) = A(x, y, -z), \qquad \Delta(x, y, z) = \Delta(x, y, -z).$$

We have chosen the coordinate system so that the xy plane coincides with the interface, and the z axis is directed into the metal. Expanding the right side of Eq. (1.5) in powers of A, and then going over to the momentum representation, we arrive at the two types of vertex parts shown in Fig. 1:

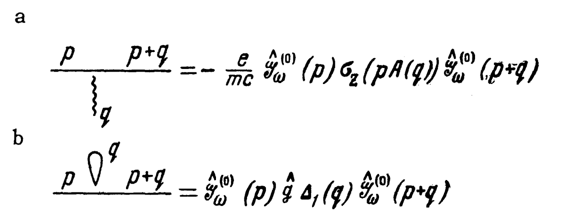

FIG. 1

($\hat{g}$ is the unit antisymmetric matrix, σ_z is a Pauli matrix). From the homogeneity of the problem in the xy plane, it is obvious that q has only a z-component. In the absence of a field, the solution $\hat{g}_\omega(p)$ is given by the expression

$$\hat{\mathfrak{G}}_\omega^{(0)}(\mathbf{p}) = -(\omega^2 + \xi(\mathbf{p})^2 + \Delta^2)^{-1}\begin{pmatrix} i\omega + \xi(\mathbf{p}) & -\Delta \\ \Delta & -i\omega + \xi(\mathbf{p})\end{pmatrix} \tag{1.7}$$

[$\xi(p) = (p^2/2m) - \mu \approx v(p - p_0)$, v is the velocity at the Fermi surface].

Finally, we write down expressions for the "gap" and the current density in terms of the Green's functions in the momentum representation:

$$\Delta(\mathbf{q}) = |\lambda|\,T\sum_\omega\int \mathfrak{F}_\omega(\mathbf{p}, \mathbf{p} - \mathbf{q})\frac{d\mathbf{p}}{(2\pi)^3}, \tag{1.8}$$

$$\mathbf{j}(\mathbf{q}) = \frac{2e}{m}T\sum_\omega\int \mathbf{p}\mathfrak{G}_\omega(\mathbf{p}, \mathbf{p} - \mathbf{q})\frac{d\mathbf{p}}{(2\pi)^3} - \frac{Ne^2}{mc^2}\mathbf{A}(\mathbf{q}), \tag{1.9}$$

where $N = p_0^3/3\pi^2$ is the number of electrons per unit volume.

2. LINEAR ELECTRODYNAMICS

In this section we present, in a form convenient for us, the results [1] of the BCS theory necessary for the subsequent development, obtained to first order in A. To this approximation, the relation between the current and the vector potential is linear:

$$\mathbf{j}(\mathbf{q}) = -\frac{c}{4\pi}K(q)\mathbf{A}(q), \tag{2.1}$$

where the kernel K(q) is given by the following expression:

$$K(q) = \frac{6\pi^2 Ne^2}{mc^2}T\sum_\omega\int_0^1 \frac{(1 - \mu^2)\,d\mu}{\omega^2 + \Delta^2 + {}^1/_4 v^2q^2\mu^2}\frac{\Delta^2}{\sqrt{\omega^2 + \Delta^2}}. \tag{2.2}$$

The finite radius $\xi_0 \sim v/T_c$ of the correlation of the electrons in the superconductor causes a dependence of the kernel on the momentum q which means in coordinate space a nonlocal relation between the current and the vector potential (ξ_0 has the meaning of a nonlocality parameter).

The field is determined by Maxwell's equation

$$\Delta\mathbf{A}(\mathbf{r}) = -4\pi c^{-1}\mathbf{j}(\mathbf{r}) \tag{2.3}$$

with the boundary condition $|\nabla \times \mathbf{A}|_{z=0} = H$. One can represent the result in the form

$$|\mathbf{A}(q)| = 2HY(q), \tag{2.4}$$

$$Y(q) = 1/[q^2 + K(q)]. \tag{2.5}$$

The field penetration depth is defined by the expression

$$\delta = \frac{1}{H}\int_0^\infty H(z)\,dz = \frac{|\mathbf{A}(z = 0)|}{H} \tag{2.6}$$

or, using Eqs. (2.4) and (2.5), we find

$$\delta = \frac{2}{\pi}\int_0^\infty Y(q)\,dq. \tag{2.7}$$

Depending on the temperature and the parameters of the metal, the ratio between the penetration depth δ and the coherence length may, in general, be arbitrary. We present results for limiting cases.

A. The London case ($\delta \gg \xi_0$). The kernel turns out to be constant for the significant values of the momentum, $q \sim \delta^{-1} \ll \xi_0$:

$$K(q) = 4\pi N_s e^2 / mc^2 \equiv \delta^{-2}, \tag{2.8}$$

i.e., the electrodynamics has a local nature. In this formula, N_S denotes the number of "superconducting" electrons at a given temperature:

$$N_s(T) / N = \pi\Delta^2 T \sum_\omega (\omega^2 + \Delta^2)^{-3/2}. \tag{2.9}$$

It is convenient to express results concerning the behavior of superconductors in a magnetic field in terms of the constant κ of the Ginzburg-Landau (GL) theory.[4] According to Gor'kov,[2]

$$\varkappa = \frac{3\pi T_c mc}{e}\sqrt{\frac{2\pi m}{7\zeta(3)\, p_0^5}}, \tag{2.10}$$

and one can represent the result (2.8) in the form

$$\delta = \frac{\pi}{2\gamma}\sqrt{\frac{7\zeta(3)}{6}}\varkappa\xi_0\sqrt{\frac{N}{N_s}} \approx 1.04\varkappa\xi_0\sqrt{\frac{N}{N_s}}, \tag{2.11}$$

where, following BCS, we assume

$$\xi_0 = \frac{\gamma}{\pi^2}\frac{v}{T_c} \approx 0.18\frac{v}{T_c}. \tag{2.12}$$

At absolute zero $N_S = N$ and

$$\delta_0 = 1.04\varkappa\xi_0, \tag{2.12'}$$

and near the transition temperature $[N_S/N = 2\{1 - (T/T_c)\}]$

$$\delta = \frac{0.74\varkappa}{\sqrt{1 - T/T_c}}\xi_0. \tag{2.13}$$

Superconductors for which $\delta_0 \gg \xi_0$ at absolute zero will be called London superconductors. As is evident from Eq. (2.12), $\kappa \gg 1$ for London superconductors. It is clear that their electrodynamics has a local nature for the entire range of temperatures.

B. The Pippard case ($\delta \ll \xi_0$). Values $q \gg \xi_0^{-1}$ give the main contribution to the integral in Eq. (2.7). For such values of q

$$K(q) = q_0^3 / q, \tag{2.14}$$

therefore,

$$Y(q) = q / (q^3 + q_0^3), \tag{2.15}$$

where q_0 is the Pippard characteristic momentum

$$q_0^3 = \frac{\pi p_0^2 e^2}{c^2}\Delta\,\mathrm{th}\frac{\Delta}{2T}, \tag{2.16*}$$

*th = tanh.

which is related to the penetration depth by the following equation:

$$\delta = 4 / 3\sqrt{3}q_0. \tag{2.17}$$

Using expression (2.10) for κ, we obtain

$$q_0^3 = \frac{18\gamma^2}{7\pi\zeta(3)}\xi_0^{-3}\varkappa^{-2}\frac{\Delta}{\Delta_0}\mathrm{th}\frac{\Delta}{2T} \approx 2.2\xi_0^{-3}\varkappa^{-2}\frac{\Delta}{\Delta_0}\mathrm{th}\frac{\Delta}{2T}, \tag{2.18}$$

$$\delta = \frac{4}{3\sqrt{3}}\left(\frac{7\pi\zeta(3)}{18\gamma^2}\right)^{1/3}\varkappa^{2/3}\xi_0\left(\frac{\Delta}{\Delta_0}\mathrm{th}\frac{\Delta}{2T}\right)^{-1/3}$$
$$\approx 0.59\varkappa^{2/3}\xi_0\left(\frac{\Delta}{\Delta_0}\mathrm{th}\frac{\Delta}{2T}\right)^{-1/3}. \tag{2.19}$$

At absolute zero $q_{00} \approx 1.3\,\xi_0^{-1}\kappa^{-2/3}$ and $\delta_0 \approx 0.59\,\kappa^{2/3}\xi_0$.

Superconductors, for which at T = 0 we have $\delta_0 \ll \xi_0$, i.e., $\kappa \ll 1$, will be called Pippard superconductors. Near the critical temperature $[1 - (T/T_c) \ll 1]$ we have

$$q_0^3 = \frac{72\gamma^3}{49\zeta^2(3)}\left(1 - \frac{T}{T_c}\right)\varkappa^{-2}\xi_0^{-3} \approx 5{,}7\xi_0^{-3}\frac{1 - T/T_c}{\varkappa^2}, \tag{2.20}$$

$$\delta = \frac{4}{3\sqrt{3}\gamma}\left(\frac{49\zeta^2(3)}{72}\right)^{1/3}\xi_0\left(\frac{1 - T/T_c}{\varkappa^2}\right)^{-1/3}$$
$$\approx 0.43\xi_0\left(\frac{1 - T/T_c}{\varkappa^2}\right)^{-1/3}. \tag{2.21}$$

The validity of the formulas given above for Pippard metals is limited by the condition $1 - (T/T_c) \gg \kappa^2$. With further increase of the temperature, up to critical, the penetration depth increases indefinitely, and for $1 - (T/T_c) \ll \kappa^2$ the formulas for the London case become applicable. The expressions for the penetration depth in the Pippard (2.21) and London (2.13) cases become equal at temperatures $1 - (T/T_c) \approx 27\,\kappa^2$, i.e., the limiting formulas (2.21) and (2.13) are joined together in a fairly broad temperature range. A numerical calculation shows that the values given by these limiting formulas differ in the region of the junction from the true values by roughly 10%.

3. DEPENDENCE OF THE "ENERGY GAP" $\Delta(\mathbf{r})$ ON THE FIELD

As is well-known,[5] the change $\Delta_1(\mathbf{r})$ in the gap can be made to vanish in the approximation linear in the field, by appropriate choice of the vector potential. In our case of a half-space occupied by a superconductor, it is sufficient for this purpose to choose the gauge $\nabla \cdot \mathbf{A} = 0$.

To second order in the field, the correction $\hat{g}_\omega^{(2)}(\mathbf{p}, \mathbf{p} - \mathbf{q})$ to the Green's function has graphically the form shown in Fig. 2.

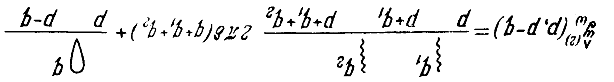

FIG. 2

Calculating the upper right-hand element of the matrix $\hat{\mathfrak{g}}_{\omega}^{(2)}$ and substituting it into Eq. (1.8), we find the equation for the correction to the "gap":

$$\Big[|\lambda|^{-1} - T\sum_{\omega}\int \frac{d\mathbf{p}}{(2\pi)^3}\,(\mathfrak{G}_{\omega}(\mathbf{p})\,\mathfrak{G}_{\omega}^{+}(\mathbf{p}+\mathbf{q}) - \mathfrak{F}_{\omega}(\mathbf{p})\,\mathfrak{F}_{\omega}(\mathbf{p}+\mathbf{q}))\Big]\,\Delta_1(q) = \left(\frac{e}{mc}\right)^2 T\sum_{\omega}\int\frac{d\mathbf{p}}{(2\pi)^3}$$
$$\times\iint_{-\infty}^{+\infty}(\mathfrak{F}_{\omega}\mathfrak{F}_{\omega}\mathfrak{F}_{\omega} + \mathfrak{G}_{\omega}\mathfrak{G}_{\omega}\mathfrak{F}_{\omega} + \mathfrak{F}_{\omega}\mathfrak{G}_{\omega}\mathfrak{G}_{\omega} - \mathfrak{G}_{\omega}\mathfrak{F}_{\omega}\mathfrak{G}_{\omega}^{+})$$
$$\times(\mathbf{p}\mathbf{A}(q_1))\,(\mathbf{p}\mathbf{A}(q_2))\,\delta(q+q_1+q_2)\,\frac{dq_1\,dq_2}{2\pi}. \quad (3.1)$$

Here, for brevity, the index (0) on the Green's functions is omitted. The order of the arguments on the right side of this formula is the same as in Fig. 2.

In the integrals over $\mathbf{p}$ in Eq. (3.1), let us go over to spherical coordinates ξ, θ, and φ, with the z axis chosen as the polar axis ($d\mathbf{p} = mp_0 \times d\xi\, d\mu\, d\varphi$, $\mu = \cos\theta$). The integral standing to the left is logarithmically divergent for large values of ξ; the divergence is eliminated, as usual, by a cutoff at the Debye frequency ω_D. With account of the relationship

$$|\lambda|^{-1} = \frac{mp_0}{2\pi^2}\ln\frac{2\omega_D}{\Delta_0}, \quad (3.2)$$

after simple integrations over ξ and φ, one can represent Eq. (3.1) in the form[1)]

$$L(q)\,\Delta_1(q) = \iint_{-\infty}^{+\infty} L_2(q, q_1, q_2)\,|\mathbf{A}(q_1)|\,|\mathbf{A}(q_2)|\,\delta(q+q_1+q_2)\,\frac{dq_1\,dq_2}{2\pi}, \quad (3.3)$$

where

$$L(q) = \pi T\sum_{\omega}\frac{1}{\tilde{\omega}}\int_0^1\frac{\Delta^2 + {}^1/_4(vq\mu)^2}{\tilde{\omega}^2 + {}^1/_4(vq\mu)^2}\,d\mu = \pi T\sum_{\omega}\frac{1}{\tilde{\omega}}\left[1 - 2\frac{\omega^2}{\tilde{\omega}vq}\operatorname{arctg}\frac{vq}{2\tilde{\omega}}\right], \quad (3.4)^*$$

$$L_2(q, q_1, q_2) = \frac{\Delta}{2}\left(\frac{ev}{c}\right)^2\pi T\sum_{\omega}\frac{1}{\tilde{\omega}}\int_{-1}^{+1}(1-\mu^2)\,d\mu$$
$$\times\left[\left(\frac{1}{(2i\tilde{\omega} - vq\mu)(2i\tilde{\omega} + vq_2\mu)} + q_1 \rightleftarrows q_2\right)\left(1 - \frac{\Delta^2}{\tilde{\omega}^2}\right) - \frac{\Delta^2}{\tilde{\omega}^2}\frac{1}{(2i\tilde{\omega} + vq_1\mu)(2i\tilde{\omega} - vq_2\mu)}\right] \quad \text{for } q + q_1 + q_2 = 0,$$
$$\tilde{\omega} \equiv \sqrt{\omega^2 + \Delta^2}. \quad (3.5)$$

Further calculations for arbitrary ratios of δ to ξ_0 have not been successfully carried out; therefore, we confine ourselves to limiting cases.

A. London case ($\delta \gg \xi_0$). Since, as is evident from (3.5), L_2 varies considerably during a change of the momenta by an amount $\sim \xi_0^{-1}$, and the potential A has a sharp peak at much smaller values of $q \sim \delta^{-1} \ll \xi_0^{-1}$, one can set all momenta in L_2 equal to zero in formula (3.3). Using expressions (2.4), (2.5), and (2.8) for A(q) and integrating over μ and q_1, q_2, we obtain after simple transformations

$$\frac{\Delta_1(q)}{\Delta} = -2\frac{N}{N_s}\frac{S_3(T) - (\Delta/\Delta_0)^2 S_5(T)}{L(q)}\frac{\delta}{4 + (q\delta)^2}\left(\frac{H}{H_{c0}}\right)^2. \quad (3.6)$$

The functions $S_n(T)$ are determined in the Appendix. We use here also the expression for the critical field at T = 0,

$$H_{c0}^2 = 2mp_0\Delta_0^2/\pi. \quad (3.7)$$

With the aid of Eq. (3.4), it is easy to obtain an expansion of L(q) for small values of q:

$$L(q) = \frac{N_s}{N}\left(1 + \frac{1}{12}\left(\frac{qv}{\Delta}\right)^2\right), \quad q\xi_0 \ll 1. \quad (3.8)$$

In this formula the second term inside the parentheses, according to the condition $q\xi_0 \ll 1$, is small in comparison with unity, except for temperatures near critical. In the latter case

$$L(q) = \frac{1 - T/T_c}{\varkappa^2}(2\varkappa^2 + (q\delta_L)^2), \quad |T - T_c| \ll T_c, \quad (3.9)$$

where δ_L is the London penetration depth near T_c [see (2.13)]:

$$\delta_L = \frac{\varkappa}{\sqrt{6}}\frac{v}{\Delta} \approx 0.74\frac{\varkappa}{\sqrt{1 - T/T_c}}\,\xi_0. \quad (3.10)$$

Hence, near T_c we find

$$\frac{\Delta_1(q)}{\Delta} = -\frac{2\varkappa^2\delta}{(4 + (q\delta)^2)(2\varkappa^2 + (q\delta)^2)}\left(\frac{H}{H_c}\right)^2 \quad (3.11)$$

or, in the coordinate representation,

$$\Delta(z) = \Delta\left(1 - \frac{\varkappa}{2\sqrt{2}(2 - \varkappa^2)}\left(\exp\left\{-\frac{\varkappa\sqrt{2}}{\delta}z\right\} - \frac{\varkappa}{\sqrt{2}}\exp\left\{-\frac{2z}{\delta}\right\}\right)\left(\frac{H}{H_c}\right)^2\right), \quad (3.12)$$

which, of course, agrees with the result of Ginzburg and Landau.[4] Therefore, for Pippard met-

[1)]Expression (3.3) for q = 0 was previously obtained by Nambu and Tuan[6] at absolute zero; later Tsuzuki[7] generalized their result to arbitrary temperatures.

*arctg = $\tan^{-1}$.

als ($\kappa \ll 1$) it is evident that Δ varies over distances $\sim \delta/\kappa$, which considerably exceed the penetration depth. This fact was first noticed in [4] (see also [8]).

The correction to the "gap" decreases as the temperature is lowered. For temperatures $T \ll T_c$, expression (3.6) vanishes exponentially $[\sim \exp(-\Delta_0/T)]$ and the next term $\sim (\xi_0/\delta)^2$ of the expansion becomes significant. This term gegins to play a noticeable role at temperatures $T \sim T_c/\ln\kappa \ll T_c$. It is easy to show that at absolute zero it is equal to

$$\frac{\Delta_1(q)}{\Delta} = -\frac{1}{15}\left(\frac{v}{\Delta_0\delta_0}\right)^2 \frac{\delta_0}{(q\delta_0)^2+4}\left(\frac{H}{H_{c0}}\right)^2 \approx 0.6\,\varkappa^{-2}\frac{\delta_0}{(q\delta_0)^2+4}\left(\frac{H}{H_{c0}}\right)^2 \quad (3.13)$$

or in the coordinate representation

$$\Delta(z) = \Delta(1 - 0.15\,\varkappa^{-2}e^{-2z/\delta_0}(H/H_{c0})^2), \qquad \varkappa \gg 1. \quad (3.14)$$

B. Pippard case ($\delta \ll \xi_0$). In this case large values of the momenta[2)], $q_1, q_2 \gg \xi_0^{-1}$, are important in the integral (3.5). The expression for L_2 can be considerably simplified for such values of the momenta. Since small values of $\mu \sim (q\xi_0)^{-1} \ll 1$ give the major contribution upon integration over μ in (3.5), one can neglect the term μ^2 in the integrand in comparison with unity, and extend the limits of integration to infinity. One can represent the result in the form

$$L_2(q, q_1, q_2) = \frac{\pi}{8v}\left(\frac{ev}{c}\right)^2\frac{\Delta}{\Delta_0}\left[\left(S_2 - \frac{1}{2}\left(\frac{\Delta}{\Delta_0}\right)^2 S_4\right)\frac{|q_1|+|q_2|-|q|}{q_1q_2} - \frac{1}{2}\left(\frac{\Delta}{\Delta_0}\right)^2 S_4\frac{1}{q}\left(\frac{|q_1|}{q_1}+\frac{|q_2|}{q_2}\right)\right]. \quad (3.15)$$

Substituting (2.4), (2.15), and (3.15) into (3.3), and changing to dimensionless variables q_1/q_0 and q_2/q_0 for the integration, we find

$$\frac{\Delta_1(q)}{\Delta} = -\frac{1}{\pi q_0}\left[\frac{\Delta_0}{\Delta\,\mathrm{th}(\Delta/2T)}\right] \times \frac{(2S_2-(\Delta/\Delta_0)^2S_4)F_1(q/q_0)-(\Delta/\Delta_0)^2S_4F_2(q/q_0)}{L(q)} \times \left(\frac{H}{H_{c0}}\right)^2; \quad (3.16)$$

$$F_1(y) = \int_0^\infty \frac{x\,dx}{(x^3+1)((x+y)^3+1)}, \quad (3.17)$$

$$F_2(y) = \frac{1}{2y}\int_0^y \frac{x(y-x)\,dx}{(x^3+1)((y-x)^3+1)} \quad (3.18)$$

for $y > 0$. For negative values of y, we have $F_{1,2}(y) = F_{1,2}(-y)$. These integrals are elementary. In view of the cumbersome nature of the final formulas, we shall not present them, and we only consider the limiting cases:

$$F_1(y) = \frac{2\pi}{3\sqrt{3}}\begin{cases}1/3\,(1-y), & 0<y\ll 1\\ y^{-3}, & y\gg 1\end{cases}, \quad (3.19)$$

$$F_2(y) = \begin{cases}1/12\,y^2, & 0<y\ll 1\\ 2\pi/3\sqrt{3}y^3, & y\gg 1\end{cases}. \quad (3.20)$$

As far as $L(q)$ is concerned, in the region of large momenta $q \gg \xi_0^{-1}$, we have for arbitrary temperatures

$$L(q) \approx \ln q\xi_0 \quad (3.21)$$

correct to terms of order $\ln q\xi_0$ (with logarithmic accuracy).

At low temperatures, $T \ll T_c$, we obtain

$$\frac{\Delta_1(q)}{\Delta} = -\frac{1}{\pi}\frac{F_1(q/q_0)-F_2(q/q_0)}{L(q)}\frac{1}{q_0}\left(\frac{H}{H_{c0}}\right)^2, \quad (3.22)$$

and near T_c in the Pippard region ($\kappa^2 \ll 1 - (T/T_c) \ll 1$)

$$\frac{\Delta_1(q)}{\Delta} = -\frac{2}{\pi}\frac{F_1(q/q_0)}{L(q)}\frac{1}{q_0}\left(1-\frac{T}{T_c}\right)\left(\frac{H}{H_c}\right)^2. \quad (3.23)$$

As already noted, a Pippard metal goes over into the London region at temperatures $1-(T/T_c) \sim \kappa^2$. Under this condition $\delta \sim \xi_0$ and $q_0 \sim \xi_0^{-1}$ in the region of momenta $q \lesssim \kappa/\delta$, i.e., at distances $z \gtrsim \delta/\kappa$, and the formulas for the Pippard (3.23) and London (3.11) cases are close in order of magnitude. At smaller distances, the dependence of the "gap" turns out to be different. In the coordinate representation

$$\Delta(z) = \int_{-\infty}^{+\infty}\Delta(q)\,e^{iqz}\frac{dq}{2\pi}. \quad (3.24)$$

Far from T_c (for this, as we shall see below, the condition $1-(T/T_c) \gg \kappa^{4/5}$ must be satisfied) we obtain with logarithmic accuracy [see (3.21)]

$$\frac{\Delta_1(z)}{\Delta} = -\frac{\Delta_0}{\Delta\,\mathrm{th}(\Delta/2T)} \times \frac{(2S_2-(\Delta/\Delta_0)^2S_4)\Phi_1(zq_0)-(\Delta/\Delta_0)^2S_4\Phi_2(zq_0)}{\ln q_0\xi_0}\left(\frac{H}{H_{c0}}\right)^2 \quad (3.25)$$

for $z \ll \xi_0$. We have introduced here the functions

$$\Phi_{1,2}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}F_{1,2}(y)\,e^{iyz}\,dy. \quad (3.26)$$

Using the limiting values of these functions

$$\Phi_1(z) = \begin{cases}4\pi/27 - 1/9\,\pi^{-1}\psi'(1/3) \approx 0.108, & z=0\\ 2\sqrt{3}/27z^2, & z\gg 1\end{cases}, \quad (3.27)$$

$$\Phi_2(z) = \begin{cases}5\pi/216 \approx 0.073, & z=0\\ 4/\pi z^6, & z\gg 1\end{cases}, \quad (3.28)$$

2)The contribution of smaller values of the momenta, $q_1, q_2 \lesssim \xi_0^{-1}$, turns out to be $(\delta/\xi_0)^2$ times smaller.

one can write the following expressions for the correction to the "gap"

$$\frac{\Delta_1(0)}{\Delta} = -\frac{0.216S_2 - (\Delta/\Delta_0)^2 \cdot 0.181S_4}{\ln q_0\xi_0} \frac{\Delta_0}{\Delta\,\mathrm{th}\,(\Delta/2T)} \left(\frac{H}{H_{c0}}\right)^2, \quad (3.29)$$

$$\frac{\Delta_1(z)}{\Delta} = -\frac{\Delta_0}{\Delta\,\mathrm{th}\,(\Delta/2T)} \frac{2\sqrt{3}}{27} \frac{2S_2 - \left(\frac{\Delta}{\Delta_0}\right)^2 S_4}{\ln q_0\xi_0} \frac{1}{(zq_0)^2} \times \left(\frac{H}{H_{c0}}\right)^2, \quad (3.30)$$

($q_0^{-1} \ll z \ll \xi_0$), i.e., for distances smaller than the nonlocality parameter ξ_0, the correction to the "gap" decreases according to a quadratic law. For distances exceeding ξ_0, the decrease is basically exponential in nature. For $1 - (T/T_c) \sim \kappa^{4/5}$, according to (3.9) the contribution of small q $\sim \kappa/\delta_L$ to the integral (3.24) becomes significant, and in the limit $\kappa^2 \ll 1 - (T/T_c) \ll \kappa^{4/5}$ it becomes the main contribution:

$$\frac{\Delta_1(z)}{\Delta} = -0.069\,\kappa^{2/3} \left(1 - \frac{T}{T_c}\right)^{1/6} \exp\left\{-\frac{\sqrt{2}\kappa}{\delta_L} z\right\} \left(\frac{H}{H_c}\right)^2. \quad (3.31)$$

From a comparison of formulas (3.29) and (3.31) one can see that their values become the same order of magnitude for $1 - (T/T_c) \sim \kappa^{4/5}$. Expression (3.31) agrees with the corresponding expression for the London region (see Eq. (3.12) for $\kappa \ll 1$) for $1 - (T/T_c) \sim \kappa^2$.

4. DEPENDENCE OF THE PENETRATION DEPTH ON THE FIELD

The third order (in powers of A) correction to the Green's function is shown in Fig. 3.

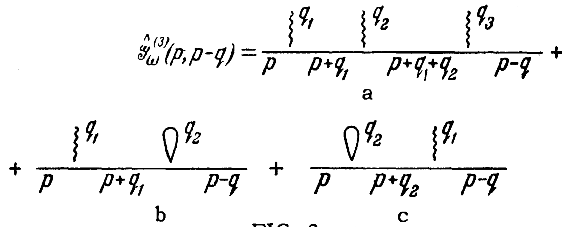

FIG. 3

Here a loop denotes the second order correction $\Delta_1(q)$ to the gap, which was determined in the preceding Section. With regard to the third-order correction, it vanishes like the first order correction because it is impossible to construct a scalar of the appropriate order in A from the vectors **q** and **A**(q) (**q** · **A**(q) = 0 in the chosen gauge).

We write the expression for the current (Fig. 3) in the following form:

$$\frac{4\pi}{c}\mathbf{j}_1(q) = \iiint_{-\infty}^{+\infty} K_3(q, q_1, q_2, q_3) \times |\mathbf{A}(q_1)|\,|\mathbf{A}(q_2)|\,|\mathbf{A}(q_3)|\,\delta(q + q_1 + q_2 + q_3) \times \frac{dq_1 dq_2 dq_3}{(2\pi)^2} + \iint_{-\infty}^{+\infty} K_2(q, q_1, q_2)\,|\mathbf{A}(q_1)|\,\Delta(q_2) \times \delta(q + q_1 + q_2)\frac{dq_1 dq_2}{2\pi}. \quad (4.1)$$

The kernel K_3 describes the contribution of diagram a, the kernel K_2 describes the contribution to the current from diagrams b and c, which are related to the change in the "gap" (see Fig. 3). By definition

$$K_3(q_1, q_2, q_3, q_4) = -8\pi\left(\frac{ev}{c}\right)^4 T \sum_\omega \int (\mathbf{pa})^4 \times [\mathfrak{G}^4 + \mathfrak{F}^2\mathfrak{G}^2 + \mathfrak{G}^2\mathfrak{F}^2 + \mathfrak{F}^4 + \mathfrak{G}\mathfrak{F}^2\mathfrak{G} + \mathfrak{F}\mathfrak{G}^{+2}\mathfrak{F} - \mathfrak{G}\mathfrak{F}\mathfrak{G}^+\mathfrak{F} - \mathfrak{F}\mathfrak{G}^+\mathfrak{F}\mathfrak{G}]\frac{d\mathbf{p}}{(2\pi)^3}, \quad (4.2)$$

where **a** is a unit vector in the direction of **A**; the order of the arguments is determined by diagram a of Fig. 3. For example,

$$\mathfrak{G}^2\mathfrak{F}^2 = \mathfrak{G}(\mathbf{p})\,\mathfrak{G}(\mathbf{p} + \mathbf{q}_1)\,\mathfrak{F}(\mathbf{p} + \mathbf{q}_1 + \mathbf{q}_2)\,\mathfrak{F}(\mathbf{p} - \mathbf{q}_4).$$

Integrating over ξ and φ, we arrive at the expression

$$K_3(q_1, q_2, q_3, q_4) = -\frac{3}{2} imp_0\Delta^2\left(\frac{ev}{c}\right)^4 T \sum_\omega \frac{1}{\tilde{\omega}^2} \int_{-1}^{+1} (1 - \mu^2)\,d\mu \times \left(\left[1 - \frac{\Delta^2}{\tilde{\omega}^2}\right)(2i\tilde{\omega} - vq_1\mu)^{-1}(2i\tilde{\omega} + v(q_1 + q_2)\mu)^{-1} \times (2i\tilde{\omega} - vq_4\mu)^{-1} + (1234) + (13)(24) + (4321) + (123) + (432) + (12) + (34) - \Delta^2\tilde{\omega}^{-2}((2i\tilde{\omega} + vq_1\mu)^{-1} \times (2i\tilde{\omega} - vq_4\mu)^{-1}(2i\tilde{\omega} - vq_2\mu)^{-1} + (13))\right],$$
$$q_1 + q_2 + q_3 + q_4 = 0. \quad (4.3)$$

Here (abc) denotes permutation of the arguments, with the aid of which the appropriate term can be obtained from the first. For example, (123) denotes the substitution $q_1 \to q_2 \to q_3$, $q_3 \to q_1$. As we will see below, all arguments are equivalent in the expression for the correction to the penetration depth, so that the contributions of all the terms obtained by means of permutations are identical. As far as the kernel K_2 is concerned, it turns out to equal, to within a coefficient, the kernel L_2 [see Eq. (3.5)] with permutation of the arguments:

$$K_2(q_1, q_2, q_3) = \frac{mp_0}{2\pi^2} 16\pi L_2(q_3, q_2, q_1). \quad (4.4)$$

With the aid of Eqs. (4.1), (4.4), (2.4), and (2.6), it is easy to obtain the following expression for the correction δ_1 to the penetration depth:

$$\delta_1 = 8H^2\Big[\iiiint\limits_{-\infty}^{+\infty} K_3(q_1q_2q_3q_4)\,Y(q_1)\,Y(q_2)\,Y(q_3)\,Y(q_4)$$

$$\times\delta(q_1+q_2+q_3+q_4)\,\frac{dq_1\,dq_2\,dq_3\,dq_4}{(2\pi)^3}$$

$$+\frac{mp_0}{2\pi^2}16\pi\int\limits_{-\infty}^{+\infty}\frac{dq}{2\pi}\,\frac{1}{L(q)}\Big(\iint\limits_{-\infty}^{+\infty}L_2(q,q_1,q_2)\,Y(q_1)$$

$$\times Y(q_2)\,\delta(q+q_1+q_2)\,\frac{dq_1\,dq_2}{2\pi}\Big)^2\Big]. \tag{4.5}$$

In the derivation of Eq. (4.5), we used Eq. (3.3) for $\Delta_1(q)$.

Now let us consider the limiting cases.

A. London case. In analogy to what was done in the calculation of the "gap" it is legitimate, by virtue of the weak momentum dependence of the kernels K_3 and L_2 (in comparison with Y) to set all the arguments in Eq. (4.5) equal to zero. Then

$$K_3(0,0,0,0) = \frac{16}{15\pi}mp_0\Delta^2\left(\frac{ev}{c\Delta_0}\right)^4\left(S_5-\left(\frac{\Delta}{\Delta_0}\right)^2S_7\right), \tag{4.6}$$

$$L_2(0,0,0) = \frac{\Delta}{3}\left(\frac{ev}{c}\right)^2\frac{1}{\Delta_0^2}\left(S_3-\left(\frac{\Delta}{\Delta_0}\right)^2S_5\right). \tag{4.7}$$

With account of (3.8), after elementary integrations in (4.5) we find

$$\frac{\delta_1}{\delta} = \frac{4}{3\pi}mp_0\left(\frac{ev\delta}{c\Delta_0}\right)^4\Delta^2\Big[\frac{1}{5}\left(S_5-\left(\frac{\Delta}{\Delta_0}\right)^2S_7\right)$$

$$+\frac{1}{2\sqrt{3}}\,\frac{N}{N_s}\,\frac{\sqrt{3}\Delta/v+2/\delta}{\sqrt{3}\Delta/v+1/\delta}\left(S_3-\left(\frac{\Delta}{\Delta_0}\right)^2S_5\right)^2\Big]H^2. \tag{4.8}$$

Near T_c the correction to the penetration depth has the form

$$\frac{\delta_1}{\delta} = \frac{\varkappa}{8}\,\frac{\varkappa+2\sqrt{2}}{(\varkappa+\sqrt{2})^2}\left(\frac{H}{H_c}\right)^2, \tag{4.9}$$

which agrees, of course, with the result of the phenomenological GL theory.

At temperatures near absolute zero we have, with exponential accuracy $S_3 \approx S_5 \approx S_7 \approx 1$, $\Delta \approx \Delta_0$, so that (4.8) vanishes. Therefore, for the London case near $T = 0$, it is necessary to consider the next higher terms of the expansion in powers of ξ_0/δ. In this connection it turns out that upon integration over the momenta in (4.1), values of $q \sim \xi_0^{-1}$ begin to play a role, i.e., nonlocal effects appear and the relative correction in units of $(H/H_c)^2$ at absolute zero is approximately κ^4 times smaller than at T_c:

$$\delta_1/\delta \sim \varkappa^{-4}(H/H_c)^2. \tag{4.10}$$

Superconductors belonging to the limiting London case for $T = 0$ apparently do not exist in nature. Therefore, we shall not study this problem in more detail.

B. The Pippard case. In this limiting case we proceed in a fashion similar to the determination of the kernel L_2 (see the preceding section). Thus, for the kernel K_3 we obtain

$$K_3(q_1,q_2,q_3,q_4) = -\frac{3\pi}{32}\,\frac{p_0{}^2\Delta^2}{\Delta_0{}^3}\left(\frac{e^2v}{c^2}\right)^2\Big[\left(S_4-\frac{3}{4}\left(\frac{\Delta}{\Delta_0}\right)^2S_6\right)$$

$$\times\left(\frac{|q_1|}{q_2(q_1+q_4)}+\frac{|q_4|}{q_3(q_1+q_4)}+\frac{|q_1+q_2|}{q_2q_3}\right.$$

$$+\text{ the permutations indicated in Eq. (4.3)}$$

$$-\frac{3}{4}\left(\frac{\Delta}{\Delta_0}\right)^2S_6\,\frac{|q_1|+|q_3|-|q_2|-|q_4|}{(q_1+q_2)(q_2+q_3)}\Big]. \tag{4.11}$$

Substituting this expression and the asymptotic form (3.15) for L_2 into the general expression for the correction δ_1, and using formula (2.15) for Y, we find

$$\delta_1 = \frac{12}{\pi^5}\,\frac{v}{\Delta_0}\,\frac{1}{\mathrm{th}^2(\Delta/2T)}$$

$$\times\Big[\left(S_4-\frac{3}{4}\left(\frac{\Delta}{\Delta_0}\right)^2S_6\right)I_1-\frac{3}{16}\left(\frac{\Delta}{\Delta_0}\right)^2S_6I_2\Big]\left(\frac{H}{H_{c0}}\right)^2$$

$$+\frac{8}{\pi^3}\,\frac{1}{q_0\,\mathrm{th}^2(\Delta/2T)}\int\limits_0^\infty\frac{dx}{L(q_0x)}\Big[\left(S_2-\frac{1}{2}\left(\frac{\Delta}{\Delta_0}\right)^2S_4\right)F_1(x)$$

$$-\frac{1}{2}\left(\frac{\Delta}{\Delta_0}\right)^2S_4F_2(x)\Big]^2\left(\frac{H}{H_{c0}}\right)^2, \tag{4.12}$$

where I_1 and I_2 are numbers given by the following expressions:

$$I_1 = \iiint\limits_0^\infty\frac{x}{(x+y+z)(x+y)}\,[\tilde{Y}(x)\,\tilde{Y}(y)\,\tilde{Y}(z)\,\tilde{Y}(x+y+z)$$

$$+\tilde{Y}(x)\,\tilde{Y}(z)\,\tilde{Y}(x+y)\,\tilde{Y}(y+z)$$

$$+\tilde{Y}(y)\,\tilde{Y}(x+y)\,\tilde{Y}(y+z)\,\tilde{Y}(x+y+z)]\,dx\,dy\,dz,$$

$$I_2 = \iiint\limits_0^\infty\Big[\frac{x}{(x+y)(x+z)}\,\tilde{Y}(x)\,\tilde{Y}(y)\,\tilde{Y}(z)\,\tilde{Y}(x+y+z)$$

$$+\frac{1}{x+y+z}\,\tilde{Y}(y)\,\tilde{Y}(z)\,\tilde{Y}(x+z)\,\tilde{Y}(x+y)\Big]\,dx\,dy\,dz,$$

$$\tilde{Y}(x) = q_0{}^2Y(q_0x).$$

The values of these integrals were obtained by numerical integration: $I_1 \approx 0.056$, $I_2 \approx 0.069$.

At temperatures not very close to T_c (it will be evident from what follows that this is valid for $1-(T/T_c) \gg \kappa^{4/5}$), values of $x \sim 1$ make the main contribution to the integral in Eq. (4.12). In this region $L^{-1}(q_0x) \approx 1/\ln q_0\xi_0 x$ is a slowly varying function; therefore one can bring it outside the integral with logarithmic accuracy, after which the remaining integral reduces to a number. As the result of numerical integration, we find:

$$\int_0^\infty F_1^2(x)\,dx \approx 0.066, \qquad \int_0^\infty F_2^2(x)\,dx \approx 0.011,$$

$$\int_0^\infty F_1(x)\,F_2(x)\,dx \approx 0.010.$$

Taking what has been said into account, it is easy to write out a general expression for δ_1/δ, valid for the entire temperature range $1-(T/T_c) \gg \kappa^{4/5}$. Because of the cumbersome nature of this expression, however, we shall only consider limiting formulas. At absolute zero

$$\frac{\delta_1}{\delta} \approx \left[2.2\cdot 10^{-4}\varkappa^{-2/3} + \frac{7.2\cdot 10^{-3}}{\ln(1/\varkappa)}\right]\left(\quad\right) \tag{4.13}$$

The origin of each of the terms inside the square brackets of this formula is different. The first term is obtained from diagram a of Fig. 3, which does not take the change in the gap into account. The contribution to the penetration depth, related to the change in the gap (Fig. 3b), is given by the second term. Although the second ("gap") term is formally smaller than the first in the limiting case $\kappa \ll 1$, owing to the (numerically) large coefficient, it is just this term which gives the essential contribution for real superconductors.

For temperatures $\kappa^{4/5} \ll 1-(T/T_c) \ll 1$

$$\frac{\delta_1}{\delta} = \left[0.95\cdot 10^{-2}\,\frac{(1-T/T_c)^{4/3}}{\varkappa^{2/3}} + 6.6\cdot 10^{-2}\left(1-\frac{T}{T_c}\right)\Big/\ln\frac{1-T/T_c}{\varkappa^2}\right]\left(\frac{H}{H_c}\right)^2. \tag{4.14}$$

As already noted earlier [see Eq. (3.9) of the preceding section], for small values of x ($x \ll (q_0\xi_0)^{-1} \ll 1$) the function $L^{-1}(q_0x)$ increases like $\{1-(T/T_c)\}^{-1}$ as the temperature approaches T_c. Because of this, small values of x give the major contribution to the integral in the immediate neighborhood of T_c. In this region, as a consequence of the slow variation of $F_1(x)$, its argument may be set equal to zero. Then using expansion (3.9) for $L(x)$, in view of the convergence of the integral, the upper limit of integration can be extended to infinity. After elementary calculations we arrive at the result

$$\frac{\delta_1}{\delta} = \left[0.95\cdot 10^{-2}\,\frac{(1-T/T_c)^{4/3}}{\varkappa^{2/3}} + 0.046\,\varkappa^{2/3}\left(1-\frac{T}{T_c}\right)^{1/6}\right] \times\left(\frac{H}{H_c}\right)^2. \tag{4.15}$$

From a comparison of (4.14) and (4.15), it follows that the domain of applicability of the latter is given by the inequality $\kappa^2 \ll 1-(T/T_c) \ll \kappa^{4/5}$, which justifies the assumption made above. From Eq. (4.15) one can see, in addition, that in the region $\kappa^2 \ll 1-(T/T_c) \ll \kappa^{8/7}$ the second term is the principal term, which for $1-(T/T_c) \sim \kappa^2$ naturally agrees with the Ginzburg-Landau result [see Eq. (4.9) for $\kappa \ll 1$]:

$$\delta_1/\delta = 0.18\,\varkappa(H/H_c)^2, \qquad \varkappa \ll 1. \tag{4.16}$$

Up to the present time, the dependence of the penetration depth on a static magnetic field has apparently been determined only for tin in the experiment by Sharvin and Gantmakher.[9] However, it is impossible to make a quantitative comparison with the limiting formulas we have obtained, since tin belongs to the intermediate case ($\kappa \approx 0.16$). Therefore, measurements of the penetration depth and its dependence on the field for strongly pronounced Pippard metals (for example, for Al we have $\kappa \sim 0.01$) would be of definite interest, since for such metals the nonlocality that is characteristic of the BCS theory appears over the entire temperature interval, except for an extremely narrow region at T_c.

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APPENDIX

Temperature coefficients of the type $T\Sigma(\tilde{\omega})^{-n}$ (where n is an integer) are often encountered in various problems in the theory of superconductivity. In this connection, it is convenient to introduce dimensionless temperature functions $S_n(T,\Delta)$ in such a way that S_n goes to unity at absolute zero:

$$S_n = \frac{(n-2)!!}{(n-3)!!}\Delta_0^{n-1}T\sum_\omega \frac{1}{\tilde{\omega}^n}\begin{cases}\pi, & n - \text{odd}\\ 2, & n - \text{even}\end{cases}. \tag{A.1}$$

It is easy to express S_n in terms of well-known functions, using the recurrence relation

$$S_n = -(n-3)\frac{\Delta_0^2}{\Delta}\frac{\partial S_{n-2}}{\partial\Delta} \tag{A.2}$$

and the known values for $n = 2$ and $n = 3$:

$$S_2 = \frac{\Delta_0}{\Delta}\,\mathrm{th}\,\frac{\Delta}{2T}, \qquad S_3 = \left(\frac{\Delta_0}{\Delta}\right)^2\frac{N_s}{N}. \tag{A.3}$$

We present the values (used in the text) of S_n for $T = T_c$:

$$S_n(T_c) = \frac{(n-2)!!}{(n-3)!!}\frac{2\zeta(n)}{\gamma^{n-1}}(1-2^{-n})\begin{cases}1, & n - \text{odd}\\ 2/\pi, & n - \text{even}\end{cases}, \tag{A.4}$$

where $\ln\gamma = C$ (C is Euler's constant), $\zeta(n)$ is the Riemann ζ-function.

In particular, for the first few values of n we have

$$S_2(T_c) = \frac{\pi}{2\gamma} \approx 0.88, \qquad S_3(T_c) = \frac{7\zeta(3)}{4\gamma^2} \approx 0.66,$$

$$S_4 = \frac{1}{12}\left(\frac{\pi}{\gamma}\right)^3 \approx 0.55, \qquad S_5(T_c) = \frac{93}{32}\frac{\zeta(5)}{\gamma^4} \approx 0.30.$$

[1] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] L. P. Gor'kov, JETP **36**, 1918 (1959), Soviet Phys. JETP **9**, 1364 (1959).

[3] A. A. Abrikosov and L. A. Fal'kovskiĭ, JETP **40**, 262 (1961), Soviet Phys. JETP **13**, 179 (1961).

[4] V. L. Ginzburg and L. D. Landau, JETP **20**, 1064 (1950).

[5] Abrikosov, Gor'kov, and Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz, 1962, pp. 401-402.

[6] Y. Nambu and S. F. Tuan, Phys. Rev. **128**, 2622 (1962).

[7] T. Tsuzuki, preprint, Osaka University, Japan, 1963.

[8] V. L. Ginzburg, Supplement to Physica **24**, S42 (1958).

[9] Yu. V. Sharvin and V. F. Gantmakher, JETP **39**, 1242 (1960), Soviet Phys. JETP **12**, 866 (1961).

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