## *THE ORIGIN OF TURBULENCE*

A. A. VEDENOV and Yu. B. PONOMARENKO

Moscow Physico-technical Institute

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IT is well known that in the transition from laminar flow to turbulence, motion is established in a number of systems at small supercriticality with a definite frequency and wave vector (examples are the flow of liquid between rotating cylinders [1], strata in a gas discharge [3,4], screw instability in a gas discharge and in semiconductors [5-8], and convection between parallel planes: time-periodic motion arises also when a liquid flows around a solid [10]). The frequency and the wave number, and also the "waveform" of the oscillations, can be determined from the linear theory; to obtain the amplitude it is necessary to take nonlinear effects into account.

The equation for the square of the modulus of the amplitude $\eta$ of unstable perturbations only slightly above criticality is of the form

$$d\eta / dt = 2\eta(\gamma + a\eta + b\eta^2 + \ldots) \equiv 2\eta\gamma_{\text{unst}}$$

(the phase of the stationary solution is arbitrary). Here $\gamma$, a, b —functions of the parameters of the system $\lambda$ (temperature, geometric dimensions, electric and magnetic fields), $\gamma$ —increment of the linear theory, while the second and third terms in $\gamma_{\text{unst}}$ are connected with allowance for the nonlinear effects.

The critical parameters $\lambda_0$ are determined from the equation $\gamma(\lambda_0) = 0$; the equilibrium state $\eta = 0$ is unstable when $\gamma(\lambda) > 0$.

Let the system be such that $a(\lambda_0) \neq 0$ for any value of the parameter $\lambda_0$. If $a(\lambda_0) < 0$, then as the excess criticality $\lambda - \lambda_0$ is increased the amplitude of the stationary motion increases continuously from zero ("soft" mode); in this case we obtain from the equation $\gamma_{\text{unst}} = 0$ the expression $-(a^{-1}\partial\gamma/\partial\lambda)_0(\lambda - \lambda_0)$. If $a(\lambda_0) > 0$, then as $\lambda$ goes through the critical value $\lambda_0$ the amplitude changes jumpwise from zero to some finite value ("hard" mode) [2,4].

Let us consider systems for which the equalities $\gamma = a = 0$ are satisfied for certain values of the parameters $\lambda_0$. We introduce any two parameters $\lambda$ and $\mu$; the remaining parameters are fixed in such a way that the curves $a(\lambda, \mu) = 0$ and $\gamma(\lambda, \mu) = 0$ intersect (Fig. 1). We reckon $\lambda$ and $\mu$ from the point of intersection, and assume for concreteness that the region $\gamma > 0$ lies above the curve $\gamma = 0$, and the region $a > 0$ lies above the curve $a = 0$. When the parameter $\lambda$ is varied, the oscillations occur in soft fashion if $\mu < 0$ and in hard fashion if $\mu > 0$.

We put $\Lambda = \lambda - \lambda_0$; for small values of $|\Lambda|$ and $|\mu|$ we can assume that $\gamma = \gamma'\Lambda$, $a = a'(\Lambda - \Lambda_0)$, and $\Lambda = c\mu$, where c and the derivatives $\gamma'$ and $a'$ are taken at $\mu = \lambda = 0$. In the case considered in Fig. 1 we have $\gamma' > 0$, $a' > 0$, and $c < 0$. If $b \neq 0$ when $\lambda = \mu = 0$, then it follows from the equality $\gamma_{\text{unst}} = 0$ that

$$\eta = A(\Lambda - \Lambda_0) \pm (A^2(\Lambda - \Lambda_0)^2 + B\Lambda)^{1/2},$$
$$A = -a'/(2b), \qquad B = -4\gamma'/b.$$

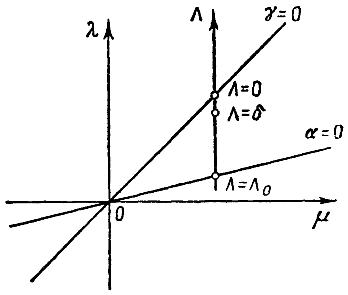


FIG. 1

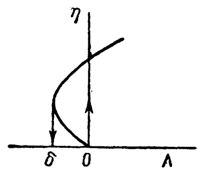


FIG. 2

Let us consider the case in which $b < 0$; then A and B are positive. Let $\Lambda$ change along the line $\mu = \text{const} > 0$ (here $\Lambda_0 = c\mu < 0$); then, for $\Lambda = +0$, the amplitude jumps from zero to the value $\eta_0 = -2A\Lambda \sim \mu$. If we now decrease $\Lambda$, then at some value $\Lambda = \delta < 0$, which is determined in equation $A^2(\delta - \Lambda_0)^2 + B\delta = 0$, a jump occurs in the amplitude from the value $\eta_0 = A(\delta - \Lambda_0)$ to zero (Fig. 2). Since $|\Lambda_0| \sim \mu$ is a small quantity, then

$$\delta \approx -(A\Lambda_0)^2/B = -(Ac)^2\mu^2/B \sim \mu^2;$$

inasmuch as $|\delta| \ll |\Lambda_0|$ for small $\mu$, then $\eta_0 = -A\Lambda_0$, and

$$\eta_0 / \eta_\delta = 2. \qquad (1)$$

In the region $\delta < \Lambda < 0$ there exist three stationary solutions

$$\eta = 0, \qquad \eta_\pm = -A\Lambda_0(1 \pm \sqrt{1 - \Lambda/\delta});$$

the values $\eta = 0$ and $\eta = \eta_+$ correspond to stable motion (as observed by experiment). For $\delta < \Lambda < 0$, a system found initially in the state $\eta = 0$ can undergo transition to the state $\eta = \eta_+$ if we impose a perturbation of sufficiently large amplitude. [2,4]

The slope of the curve $\eta = \eta(\Lambda)$ at the point

$\Lambda = +0$ is equal to $d\eta\, d\Lambda = D/|\mu|$, $D > 0$; this expression is also valid for the soft mode.

The frequency $\omega$ and the mean value of any observed quantity x (the mean temperature, magnetic induction, electric current, particle flow, etc.) change upon the change of the amplitude of the stationary motion; for small amplitudes of motion, the corresponding dependences have the form

$$x = x_{eq} + x_1\eta + \ldots, \qquad \omega = \omega_{eq} + \omega_1\eta + \ldots \qquad (2)$$

(if the stationary motion is periodic in space, then a similar relation holds for the wave number: $k = k_{eq} + k_1\eta + \ldots$). The quantities x, $\omega$ on the left hand sides of the equations are functions of $\Lambda$; $x_p$ corresponds to the equilibrium stated in this system $\eta = 0$, $\omega_{eq}$ is determined from linear theory. Taking it into account that $|\delta| \sim \mu^2$, we get

$$\omega_0 - \omega_\delta = (\omega_{eq} + \omega_1\eta)_0 - (\omega_{eq} + \omega_1\eta)_\delta \approx (\omega_1)_0(\eta_0 - \eta_\delta) \sim \mu.$$

A similar relation $x_0 - x_\delta \sim \mu$ is obtained in the presence of stationary motion for the mean value of the observed x; if the motion is absent ($\eta_0 - \eta_\delta = 0$), then $x_0 - x_\delta = (x_{eq})_0 - (x_{eq})_\delta \sim \delta \sim \mu^2$.

For $\Lambda = 0$ and $\Lambda = \delta$ the mean value of any observed quantity changes discontinuously. Denoting the amplitude of the jump (the difference between the value of x in the presence of oscillations and in their absence or for fixed $\Lambda$) by $\Delta x$, we have for small $\mu$, by virtue of (1) and (2),

$$(\Delta x)_0 / (\Delta x)_\delta = 2.$$

Thus the amplitudes of the jumps of the mean values of $\Delta x$ and the square of the variable quantities $x^2_\sim$ of the observed values should satisfy the relations

$$(x^2_\sim)_0 / (x^2_\sim)_\delta = (\Delta x)_0 / (\Delta x)_\delta = 2. \qquad (3)$$

The dependence of x on the parameter $\Lambda$ can be of three types: for $\Lambda = 0$, one can observe: a) a

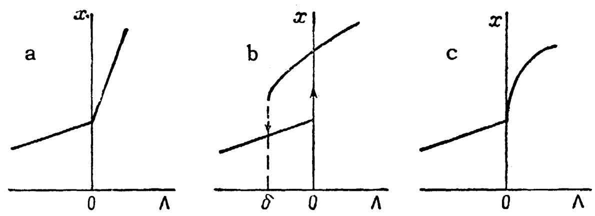


FIG. 3

kink;[8] b) a jump;[6,7] c) a root singularity (Fig. 3; case c) corresponding to a change of $\Lambda$ along the line $\mu = 0$.

The experimental study of the transition from the soft excitation of turbulence to the hard mode is of interest; a test of the relations (3) and a study of the various types of the dependence $x = x(\Lambda)$ shown in Fig. 3 would be of particular value.


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