DISPERSION OF SOUND IN METALS IN A MAGNETIC FIELD

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Dispersion of sound in a metal located in a magnetic field is investigated at T = 0. It is shown that when $k \perp H$, $r_H \ll \lambda$, and $\omega\tau \gtrsim 1$ the velocity of longitudinal sound changes considerably compared to that in the absence of a field. The analysis is carried out for a model in which an isotropic quadratic electron dispersion law is assumed.

As is well known, the interaction between a sound-wave field and conduction electrons influences appreciably the propagation of sound in a metal, causing strong absorption of the sound. It also leads the elastic properties of metals to deviate greatly from those of nonmetallic solids. In calculating the elastic moduli it is necessary to take into account the contribution from the conduction electrons, which turns out to be either predominant or at least of the same order of magnitude as the lattice contribution. In this case one speaks of "renormalization" of the lattice moduli of elasticity.

Since the electron energy spectrum is very sensitive to the application of an external magnetic field, the absorption of sound depends strongly on the magnetic field. This phenomenon has been treated in many papers; an analysis of the experimental dependence of the absorption of ultrasound on the magnetic field yields valuable information on the structure of the electron energy spectrum.

For the same reason, we should expect an appreciable magnetic-field dependence of the elastic properties of metals (i.e., velocity of the elastic waves). This paper is devoted to an investigation of this question. We shall show that a considerable (~10%) change in sound velocity takes place under certain conditions in a strong magnetic field perpendicular to the direction of propagation of the longitudinal sound[1)]. The qualitative explanation which we present below shows that this effect occurs for an arbitrary electron dispersion (for closed Fermi surfaces), and is connected with the anomaly of the absorption of longitudinal sound when $\mathbf{k} \perp \mathbf{H}$[1,2]. For simplicity, however, the analysis will be carried out in the free-electron model. A systematic study of the dispersion of sound for an arbitrary electron energy spectrum will be treated in a separate paper.

Following Silin[3] and Kontorovich[4], we shall describe sound oscillations in a metal with the aid of the elasticity equations in which account is taken of the presence of the force exerted on the lattice by the conduction electrons and by the electromagnetic fields produced by these oscillations. The solution of the combined system of elasticity equations, Maxwell's equations, and the kinetic equation leads in the case of a plane sound wave $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$ to a dispersion equation $f(\mathbf{k},\omega) = 0$. The solution of this equation $\omega = \omega(\mathbf{k}, \mathbf{H}) = \omega_0 + \Delta\omega - i\gamma$ enables us to find the dispersion $\Delta\omega$ and the sound damping coefficient γ.

In the isotropic case, in the free-electron model, this system of equations is of the form

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \left(e\mathbf{E} + \frac{e}{c}[\mathbf{vH}]\right)\frac{\partial f}{\partial \mathbf{p}} = -\hat{\nu}f; \qquad (1)^*$$

$$\rho\ddot{\mathbf{u}} = (\lambda + \mu)\nabla \operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} + \rho_l\mathbf{E} + \frac{1}{c}[\mathbf{j}_l\mathbf{H}] + \frac{m}{e}\nu(\mathbf{j}_e + \mathbf{j}_l); \qquad (2)$$

$$\operatorname{rot}\operatorname{rot}\mathbf{E} = -\frac{4\pi}{c^2}\frac{\partial}{\partial t}(\mathbf{j}_e + \mathbf{j}_l), \quad \rho_l + \rho_e = 0; \qquad (3)^\dagger$$

$$\rho_e = e\int f\,d\tau_l, \quad \mathbf{j}_e = e\int \mathbf{v}f\,d\tau_l, \quad \mathbf{j}_l = \rho_l\dot{\mathbf{u}}, \qquad (4)$$

where $f(\mathbf{r}, \mathbf{p}, t)$—electron distribution function, $\mathbf{u}(\mathbf{r}, t)$—ion displacement vector, j_e, j_l, ρ_e, ρ_l—electron and lattice currents and charge densities, λ and μ—lattice moduli of elasticity, ρ—density of metal, and $\nu = \tau^{-1}$—collision frequency. The collision integral $\hat{\nu}f$ is written in the relaxation-time approximation in the form

[1)]In other cases the change in sound velocity turns out to be considerably smaller, for example $\Delta s/s \sim 10^{-3}$.

*$[\mathbf{vH}] = \mathbf{v} \times \mathbf{H}$.

†rot = curl.

$$\hat{\nu} f = \nu\,[f - f_0(\varepsilon - \mathbf{pu} - \delta\zeta)] \tag{5}$$

($f_0(\epsilon)$—equilibrium Fermi function and $\delta\zeta$—change in chemical potential under lattice vibrations).

In writing down Eqs. (1)—(5) we assume that the electron dispersion law is not changed by the lattice vibrations. An account of the deformation interaction, which causes the electron energy to change by $\delta\epsilon = \lambda_{ik}u_{ik}$, does not change the results obtained below. It corresponds to some additional renormalization of the velocity of sound, which does not depend on the frequency and on the magnetic field, and therefore cannot be observed experimentally.

Equations (1)—(5) can be greatly simplified first, by linearization and second, by assuming cancellation not only of the electron and lattice charges, $\rho_e + \rho_l = 0$, but also of the currents, $\mathbf{j}_e + \mathbf{j}_l = 0$. Such a case is realized for sufficiently low frequencies ($\omega < 10^9 - 10^{10}\ \mathrm{sec}^{-1}$), at which the effective skin depth of the electromagnetic waves is smaller than the wavelength of sound[1,2]. By writing down the electron distribution function in the form $f = f_0(\epsilon) + \chi\partial f_0/\partial\epsilon$, we obtain

$$\frac{\partial\chi}{\partial t} + \mathbf{v}\frac{\partial\chi}{\partial\mathbf{r}} + e\mathbf{E}\mathbf{v} + \frac{\partial\chi}{\partial\tau} + \nu(\chi + \mathbf{p}\dot{\mathbf{u}} + \delta\zeta) = 0; \tag{6}$$

$$\rho\ddot{\mathbf{u}} = (\lambda+\mu)\,\nabla\operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} - Ne\mathbf{E} - Nm[\dot{\mathbf{u}}\boldsymbol{\Omega}],$$
$$\boldsymbol{\Omega} = e\mathbf{H}/mc; \tag{7}$$

$$\int \chi\mathbf{v}\frac{\partial f_0}{\partial\varepsilon}\,d\tau_l = N\dot{\mathbf{u}}, \quad \delta\zeta = \Lambda\operatorname{div}\mathbf{u}, \quad \Lambda = -\frac{1}{3}mv_F^2 \tag{8}$$

(τ—variable of the electron travel time along the orbit, N—electron concentration, v_F—Fermi velocity).

The kinetic equation (6) has a solution which is periodic in τ, with period $T = 2\pi/\Omega$, in the form (for a plane sound wave)

$$\chi = -\left(1 - e^{-i(\mathbf{k}\bar{\mathbf{v}}-\omega-i\nu)T}\right)^{-1} e^{-i\lambda(\tau)} \times \int_{\tau-T}^{\tau} e^{i\lambda(\tau')}\{i\nu\Lambda\,\mathbf{ku} + (e\mathbf{E} - i\omega\nu m\mathbf{u})\,\mathbf{v}(\tau')\}\,d\tau', \tag{9}$$

where $\lambda(\tau) = \mathbf{k}\cdot\mathbf{r}(\tau) - (\omega + i\nu)$ and $\bar{\mathbf{v}}$—average electron drift velocity.

Substituting (9) in (8) we obtain the electric field $\mathbf{E}$; using now the expression for $\mathbf{E}$ and substituting in (7), we arrive at an elasticity equation containing no variables other than the displacement vector $\mathbf{u}$. The corresponding equation is

$$\Lambda_{ij}u_j = 0, \tag{10}$$

where the dynamic matrix Λ_{ij} is represented in the form

$$\Lambda_{ij} = -\rho\omega^2\delta_{ij} + (\lambda+\mu)k_ik_j + \mu k^2\delta_{ij} + i\omega\nu mN\delta_{ij} - imN\omega\varepsilon_{ijk}\Omega_k + \frac{1}{3}Nmv_F^2\frac{i\nu}{\omega+i\nu}k_ik_j + \frac{Nm\omega\Omega}{3\pi i}F_{ij}^{-1}, \tag{11}$$

with

$$F_{ij} = \frac{1}{p_F v_F^2 T^2}\int dp_z\left(1 - e^{-i(\mathbf{k}\bar{\mathbf{v}}-\omega-i\nu)T}\right)^{-1}\int_0^T d\tau\int_{\tau-T}^{\tau} d\tau' \times e^{i(\omega+i\nu)(\tau-\tau')}\,e^{-i\mathbf{k}(\mathbf{r}(\tau)-\mathbf{r}(\tau'))}\,v_i(\tau)\,v_j(\tau'). \tag{12}$$

The two last members of (11) contain factors corresponding to "renormalization" of the velocity of sound. The first is important when $\omega\tau \ll 1$, and the second when $\omega\tau \gg 1$. When $\omega\tau \ll 1$ the longitudinal sound velocity is determined by the expression

$$\rho s^2 = \lambda + 2\mu + {}^1\!/_3 Nmv_F^2 \tag{13}$$

(and the transverse sound velocity is not renormalized). When $\omega\tau \gg 1$, the renormalization of the velocity of sound is determined by the last term of (11). Generally speaking, it is different from (13).

To prove this, let us consider the case[2)] $\mathbf{k} \perp \mathbf{H}$. In this case we have $\mathbf{k}\cdot\bar{\mathbf{v}} = 0$ in (12). Directing $\mathbf{H}$ along the z axis and $\mathbf{k}$ along the x axis, we obtain ($\varphi = \Omega\tau$)

$$F_{ij} = \frac{(1 - e^{-2\pi\beta})^{-1}}{p_F v_F^2 T^2}\int dp_z\int_0^{2\pi} d\varphi\int_{\varphi-2\pi}^{\varphi} d\varphi' e^{-\beta(\varphi-\varphi')} \times\exp\{-i\alpha(\sin\varphi - \sin\varphi')\,v_\perp/v_F\}\,v_i(\varphi)\,v_j(\varphi'), \tag{14}$$

where we introduce the parameters

$$\alpha = kv_F/\Omega = kr_H, \qquad \beta = (\nu - i\omega)/\Omega. \tag{15}$$

Of greatest interest is the case of a strong field, when $|\beta| \ll 1$. In this case expression (14) can be calculated by expanding $\exp[-\beta(\varphi - \varphi')]$ in powers of β. As $\beta \to 0$ the asymptotic value of F_{ij} (for arbitrary α) takes the form

$$F_{ij} \approx \frac{1}{2\pi}\begin{pmatrix} 2\beta\alpha^{-2}\varphi(\alpha) & \alpha^{-1}\varphi'(\alpha) & 0 \\ -\alpha^{-1}\varphi'(\alpha) & 2\beta^{-1}\psi(\alpha) & 0 \\ 0 & 0 & 2\beta^{-1}\eta(\alpha)\end{pmatrix}, \tag{16}$$

where the functions $\varphi(\alpha)$, $\psi(\alpha)$, and $\eta(\alpha)$ are defined by

$$\varphi(\alpha) = \int_0^1 [1 - J_0^2(\alpha\sqrt{1-x^2})]\,dx, \tag{17}$$

$$\psi(\alpha) = \int_0^1 (1-x^2)\,J_1^2(\alpha\sqrt{1-x^2})\,dx, \tag{18}$$

$$\eta(\alpha) = \int_0^1 x^2 J_0^2(\alpha\sqrt{1-x^2})\,dx. \tag{19}$$

Formula (16) is valid for arbitrary α satisfying the condition $\alpha \gg \beta$ (it is necessary in particular

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2)These formulas are valid if the angle of deviation of $\mathbf{k}$ from the direction perpendicular to $\mathbf{H}$ does not exceed $s/v_F \sim 10^{-3}$.

to have $kl \gg 1$). Substituting (16) in (11) we obtain ($\beta \ll 1$)

$$\Lambda_{xx} = -\rho\omega^2 + (\lambda+2\mu)k^2 + {}^1/_3 Nmv_F{}^2k^2\left[\frac{i\nu}{\omega+i\nu} + \frac{\omega}{\omega+i\nu}\psi(\alpha)\chi^{-1}(\alpha)\right]; \quad (20)$$

$$\Lambda_{yy} = -\rho\omega^2 + \mu k^2 + i\omega\nu mN - {}^1/_3 Nm\omega(\omega+i\nu)\varphi(\alpha)\chi^{-1}(\alpha); \quad (21)$$

$$\Lambda_{xy} = -\Lambda_{yx} = -imN\omega\Omega\,[1 - {}^1/_6\alpha\varphi'(\alpha)\chi^{-1}(\alpha)]; \quad (22)$$

$$\Lambda_{zz} = -\rho\omega^2 + \mu k^2 + i\omega\nu mN - {}^1/_3 Nm\omega(\omega+i\nu)\eta^{-1}(\alpha); \quad (23)$$

$$\Lambda_{xz} = \Lambda_{yz} = \Lambda_{zy} = \Lambda_{zx} = 0;$$

$$\chi(\alpha) = \varphi(\alpha)\psi(\alpha) + {}^1/_4(\varphi'(\alpha))^2. \quad (24)$$

The dispersion equation for the sound has the form det $\Lambda_{ij} = 0$, which yields

$$\Lambda_{xx}\Lambda_{yy} + \Lambda^2_{xy} = 0; \qquad \Lambda_{zz} = 0.$$

With the aid of (20)—(23) we can obtain the dispersion of both longitudinal and transverse sound. We are interested, however, only in longitudinal sound, for which the change in velocity can be large. It can be shown that the role of the off-diagonal elements Λ_{xy} and Λ_{yx} is small (they lead only to small corrections to the frequency, which can be readily estimated). Neglecting Λ_{xy} and Λ_{yx}, we represent the dispersion equation of longitudinal sound in the form

$$\Lambda_{xx} = -\rho\omega^2 + (\lambda+2\mu)k^2 + {}^1/_3 Nmv_F{}^2k^2 \times\left\{\frac{i\nu}{\omega+i\nu} + \frac{\omega}{\omega+i\nu}\psi(\alpha)\chi^{-1}(\alpha)\right\} = 0. \quad (25)$$

Solving this equation and writing its solution in the form $\omega = sk - i\gamma$, where s—phase velocity of sound and γ—its attenuation, we obtain for $\omega \ll \nu$ and for $\omega \gg \nu$:

$$1)\ \omega\tau \ll 1: \quad \rho s^2 = \lambda + 2\mu + {}^1/_3 Nmv_F{}^2; \quad \gamma = \frac{Nm}{3\rho}\left(\frac{v_F}{s}\right)^2\omega^2\tau f(\alpha); \quad (26)$$

$$2)\ \omega\tau \gg 1: \quad \rho s^2 = \lambda + 2\mu + {}^1/_3 Nmv_F{}^2(1+f(\alpha)); \quad \gamma = \frac{Nm}{3\rho}\left(\frac{v_F}{s}\right)^2\frac{1}{\tau}f(\alpha); \quad (27)$$

$$f(\alpha) = \psi(\alpha)\chi^{-1}(\alpha) - 1. \quad (28)$$

The absorption described by (26) and (27) is quite large. Formula (26) has the same form (apart from a numerical factor) as in the absence of a field for $kl \ll 1$ (see, for example, [5]). As such, nonetheless, it pertains to the case $kl \gg 1$, and this means that the absorption of the sound in a magnetic field, described by this formula, is appreciably larger than without the field (by approximately a factor kl, where $kl < v_F/s$. With further increase in kl, when $\omega\tau$ becomes larger than unity, the absorption begins to decrease with increasing τ [formula (27)].

Of greatest interest is the expression for the velocity of sound s. In formula (26), which is valid for $\omega\tau \ll 1$, $(\lambda + 2\mu)$—modulus of elasticity of the lattice, $Nmv_F^2/3$—that of the electrons (see [6]), and usually $Nmv_F^2/3 \gtrsim (\lambda + 2\mu)$. When $\omega\tau > 1$ the expression for the velocity of sound has a different form [formula (27)]. Since the function $f(\alpha)$ describing the sound absorption should be positive (as is confirmed by direct calculation), the velocity of sound is larger when $\omega\tau > 1$ than when $\omega\tau < 1$ (Fig. 1). In a strong field $\alpha \ll 1$ (the radius r_H of the electron orbit is much smaller than the wavelength of sound λ) we obtain $f(\alpha) = 1/5$ [see (17), (18), (24), (28); we note that $f(\alpha) \to 0$ as $\alpha \to \infty$]. Consequently, the sound velocity can apparently experience a relative change on the order of 10—20% when $\omega\tau \gtrsim 1$ and $r_H \ll \lambda$, compared with its value in the absence of a field or when $\omega\tau \ll 1$.3) When $r_H \gtrsim \lambda$, strong periodic variations of the velocity and of the coefficient of absorption of sound will be observed (magnetoacoustic oscillations), since $f(\alpha)$ is an oscillating function of α (see Fig. 2). The reason for these effects can be explained in the following fashion. Figure 3 shows the projection of the elec-

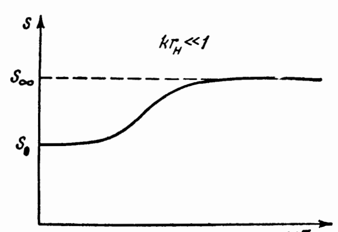

FIG. 1

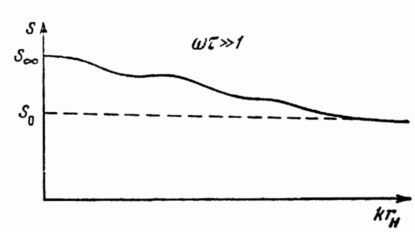

FIG. 2

3)We note that this strong change in the velocity of longitudinal sound, to which apparently no attention was paid previously, can also be obtained from the formulas of Gurevich and Kaner[1,2].

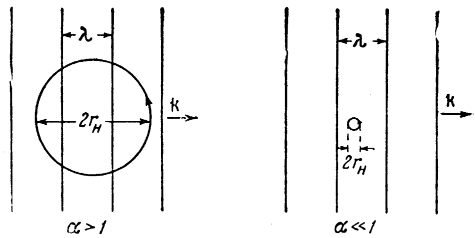

FIG. 3

tron trajectory on a plane perpendicular to **H** (which contains the sound wave vector **k**). The parallel lines are the intersections of this plane with the equal-phase planes of the sound wave. When the diameter of the orbit is a multiple of the wavelength of sound, magnetoacoustic "resonance" oscillations set in. In a strong field ($\alpha \ll 1$) the radius of the electron trajectory is small compared with the wavelength; in this case the electron is practically at one point of the sound field, and therefore its interaction with the sound field is different than if $\alpha \gg 1$. In this case strong absorption of sound sets in (at $\omega\tau \sim 1$) and with it strong dispersion of the sound.

Summarizing we can state that in a strong magnetic field ($r_H \ll \lambda$) at sufficiently high frequencies ($\omega\tau > 1$) an anomaly should take place in the dispersion of the longitudinal sound propagating perpendicular to the magnetic field **H**. Observation of a similar effect makes it possible to separate the lattice and electronic contributions to the modulus of elasticity of the metal. It also makes it possible to reconstitute some characteristics of the Fermi surface (for this purpose, however, it is necessary to make calculations that employ no prescribed concrete model of the surface).

The observation of the effect is hindered not only by the indicated need for obtaining a strong field and high frequencies, but also by special requirements imposed on the adjustment of the samples and homogeneity of the magnetic field (the angle of deviation from perpendicularity of **k** and **H**, and also the angle of divergence of the force lines of the field, must not exceed s/v_F, i.e., fractions of a degree), and also by the large sound absorption under these conditions.

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