EQUATIONS FOR REGULARIZED GREEN'S FUNCTIONS

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For the case of a self-interacting scalar field, summation of the perturbation theory series yields a functional equation for a certain quantity, which allows one to compute easily the many-particle Green's function. Imposing a series of supplementary conditions on the coefficients of this equation, one obtains regularized values for the Green's functions. Although the equation has been obtained starting from perturbation theory, neither the equation itself not the supplementary conditions contains explicitly the perturbation theory in its formulation.

In connection with intensive investigations of the theory of strong interactions, methods of summation of various sequences of matrix elements in perturbation theory have attracted much interested recently. Assuming first that the coupling constants are small, one succeeds in deriving equations which involve quantities represented by infinite series in the coupling constant. Having obtained such equations, one hopes that they will remain valid also in the case when the coupling constant is no longer small. Examples of such equations are the Bethe-Salpeter equation [1] and Schwinger's equations for the Green's functions [2]. One can assume that it will be possible to extract from such equations some information for the theory of strong interactions.

However, one among many serious obstacles in carrying this through is the fact that divergences appear when solving the formally constructed equations by means of perturbation theory. In order to make the equations convergent it is necessary to utilize some regularization method, e.g. the R-operation of Bogolyubov [3,4]. However, the formulation of regularization methods, in particular the R-operation, is essentially connected with perturbation theory. Thus, by means of the R-operation we can obtain expressions for the perturbation-theory solutions of the mentioned equations, but it is not at all clear how to include the regularization into the equations themselves, without introducing meaningless infinite quantities.

Recently, the author of the present paper has proposed a method of regularization for the S-matrix elements [5], the formulation of which is not directly connected with perturbation theory, although the proof of its validity has been carried through only within the framework of perturbation theory. This method of regularization can be directly included in the equations that couple various many-particle Green's functions.

The essence of the proposed regularization method consists in the following. One introduces for the contractions of field operators the so-called "limiting representation," constructed according to a Pauli-Villars type regularization. All field contractions are considered as weak limits of certain continuous functions, for instance, the chronological contraction of the scalar field is represented in the form

$$D^{(c)}(x) = \lim_{\mu\to 0} \Delta^{(c)}(x; \mu), \tag{1}$$

where

$$\Delta^{(c)}(x; \mu) = \int dk e^{ikx} \sum_i \frac{C_i(\mu)}{m^2\mathfrak{M}_i(\mu) - k^2 - i\varepsilon}. \tag{2}$$

In Eq. (2) the coefficients C_i are bounded and all $\mathfrak{M}_i(\mu)$ go to infinity as $\mu \to 0$, except $\mathfrak{M}_0(\mu)$, the limit of which is one. If one imposes on the coefficients C_i and $\mathfrak{M}_i$ the usual Pauli-Villars conditions

$$\sum C_i \mathfrak{M}_i^\alpha = 0, \qquad \alpha = 0; 1, \tag{3}$$

the function $\Delta^{(c)}(x; \mu)$ will be continuous in x for $\mu \neq 0$.

By means of the "limiting representation" one can formulate a rule of multiplication of contractions which guarantees the finiteness of the matrix elements of the S-matrix. Namely, we define $\prod_{r,s} D^{(c)}(x_r - x_s)$ as that generalized function, for which the integral with a sufficiently smooth and rapidly decreasing (at infinity) trial function $g(x_1, \ldots, x_n)$ is defined as follows:

$$\int dx_1 \ldots dx_n g(x_1, \ldots x_n) \prod_{r,s} D^{(c)}(x_r - x_s)$$

$$= \lim_{\mu_1\to 0} \ldots \lim_{\mu_N\to 0} \frac{1}{N!} P(\mu_1, \ldots, \mu_N) \int dx_1 \ldots dx_n$$

$$\times g(x_1, \ldots, x_n) \prod_{r,s} \Delta^{(c)}(x_r - x_s;\ \mu_j). \tag{4}$$

Here $P(\mu_1, \ldots, \mu_N)$ is the sum over all permutations of the parameters

As has been proved previously [5], Eq. (4) together with the supplementary conditions on the coefficients C_i, $\mathfrak{M}_i$

$$\lim_{\mu\to 0} \sum_i C_i(\mu) \mathfrak{M}_i^{\alpha}(\mu) \ln^{\beta} \mathfrak{M}_i(\mu) = A_{\alpha\beta}, \tag{5}$$

where $\alpha = 0, 1$; $\beta = 1, 2, \ldots, n$; and $|A_{\alpha\beta}| < \infty$, yields finite values for the matrix elements of the S-matrix up to the n-th order of perturbation theory (for a renormalizable type of self-interacting scalar field), values which coincide exactly with those obtained by regularizing by means of the R-operation 1). It is worth mentioning that in the conditions (5) one can go to the limit $n \to \infty$.

Thus Eqs. (4) and (5) yield for the S-matrix a regularization method equivalent to the R-operation. In this connection one remark is in order. In the preceding paper on regularization [5] the author has made the erroneous assertion that in utilizing the R-operation one must introduce diverging expressions and that therefore the R-operation is not completely consistent. In reality, as has been shown by Parasyuk [4], a careful use of intermediate regularization allows one to avoid divergent expressions in all steps of the construction of the S-matrix by means of the R-operation. Thus the use of the R-operation is mathematically completely correct.

The difference of Eqs. (4) and (5) from the R-operation consists in the following. For a complete formulation of the R-operation it is necessary to define a sequence of counterterms, which can be constructed only after solving the problem of constructing the matrix elements of the S-matrix according to perturbation theory. In regularizing by means of the "limiting representation" the sequence of counterterms is replaced by the sequence of conditions (5), which can be formulated independently of finding the matrix elements of the S-matrix by perturbation theory. This fact will prove to be essential for our further purpose, consisting in constructing a system of equations for many-particle Green's functions. We wish to construct such equations for which the solutions should lead to regularized expressions of the Green's functions, at least in perturbation theory.

1)The conditions (5) do not take into account the vacuum diagrams. In order to take these into account, one must require (5) to hold for $\alpha = 2$. The same refers to the condition (3).

We consider the theory with the interaction Lagrangian

$$L(x) = g{:}\varphi^4(x){:}, \tag{6}$$

where $\varphi(x)$ is a scalar field. The Green's function G_m will be defined as follows:

$$G_m(p_1, \ldots, p_m) = \left\langle 0 \left| \frac{\delta^m S}{\delta\varphi(p_1)\ldots\delta\varphi(p_m)} \right| 0 \right\rangle. \tag{7}$$

We will connect G_m with the function G_{m+2}. We first find this connection in n-th order perturbation theory. For this we expand G_m in powers of the coupling constant

$$G_m(g) = \sum_{n \geqslant 4m}^{\infty} g^n G_m^{(n)}. \tag{8}$$

Here

$$G_m^{(n)} = g^{-n} \left\langle 0 \left| \frac{\delta^m S_n}{\delta\varphi(p_1)\ldots\delta\varphi(p_m)} \right| 0 \right\rangle. \tag{9}$$

Since we wish to deal from the very beginning with regularized Green's functions, we will use the definition of the product of chronological contractions in reducing the T-product to normal products, i.e., we write S_n in the form

$$S_n = \frac{1}{n!}(ig)^n \lim_{\mu_1\to 0} \ldots \lim_{\mu_N\to 0} \frac{1}{N!} P(\mu_1, \ldots, \mu_N)$$
$$\times \int dx_1 \ldots dx_n T_{reg}\Big(\prod_{i=1}^{n} {:}\varphi^4(x_i){:}\Big), \tag{10}$$

where T_{reg} is reduced according to the usual Wick theorem, with the chronological contractions replaced by the functions $\Delta^{(c)}(x; \mu)$.

Further, it is convenient to go over to momentum space. Then

$$R_n = \frac{1}{N!} P(\mu_1, \ldots, \mu_N) \int dx_1 \ldots dx_n T_{reg}\Big(\prod_{i=1}^{n} {:}\varphi^4(x_i){:}\Big)$$
$$= \frac{1}{(2\pi)^{4n}} \int dk_1 \ldots dk_n \widetilde{T}\Big(\prod_{i=0}^{n-1} {:}\varphi(k_{4i+1})\ldots\varphi(k_{4i+4})$$
$${:}\delta(k_{4i+1} + \ldots + k_{4i+4})\Big), \tag{11}$$

where $\widetilde{T}$ is reduced according to Wick's theorem with the contraction

$$\varphi(k)\varphi(p) = -i\delta(k+p)\Delta(k; \mu), \tag{12}$$

where

$$\Delta(k; \mu) = \sum_i \frac{C_i(\mu)}{m^2\mathfrak{M}_i(\mu) - k^2 - i\varepsilon}. \tag{13}$$

Applying Wick's theorem, one easily obtains for $\widetilde{T}$ the expression:

$$\widetilde{T}\Big(\prod_{i=0}^{n-1} {:}\varphi(k_{4i+1})\ldots\varphi(k_{4i+4}){:}\Big) = \sum_{r=0}^{2n} (2i)^{-r} \frac{1}{r!} \frac{1}{(4n-2r)!}$$

$$\times P(k_1, \ldots, k_{4n}) \prod_{i=1}^{r} \delta(k_i + k_{i+r})$$

$$\times \Delta'(k_i; \mu_i) : \varphi(k_{2r+1}) \ldots \varphi(k_{4n}) :. \tag{14}$$

Here $\Delta'(k_i, \mu_i)$ coincides with $\Delta(k_i, \mu_i)$ if k_i and k_{i+r} are the momenta of fields not contained in the same normal product inside the $\widetilde{T}$-product, and vanishes otherwise. Since

$$\frac{1}{r!} P(\mu_1, \ldots, \mu_r) P(k_1, \ldots, k_{4n}) = P(k_1, \ldots, k_{4n}), \tag{15}$$

we have

$$R_n = (2\pi)^{-4n} \sum_{r=0}^{n} (2i)^{-r} \frac{1}{r!} \frac{1}{(4n-2r)!} \int dk_1 \ldots dk_{4n}$$

$$\times \prod_{i=0}^{n-1} \delta(k_{4i+1} + \ldots + k_{4i+4}) \, P(k_1, \ldots, k_{4n}) \prod_{j=1}^{r} \delta(k_j + k_{j+r})$$

$$\times \Delta'(k_j; \mu_j). \tag{16}$$

Substituting (16) into (10) we obtain

$$G_m^{(n)}(p_1, \ldots, p_m) = (i)^{-n} (2\pi)^{-4n} \frac{1}{n!} \frac{1}{r!} (2i)^{-r} \lim_{\mu_1 \to 0} \ldots$$

$$\lim_{\mu_r \to 0} \int dk_1 \ldots dk_{4n} \prod_{i=0}^{n-1} \delta(k_{4i+1} + \ldots + k_{4i+4}) \, P(k_1, \ldots, k_{4n})$$

$$\times \prod_{l=1}^{m} \delta(p_l - k_{2r+l}) \prod_{j=1}^{r} \delta(k_j + k_{j+r}) \Delta'(k_j; \mu_j), \tag{17}$$

where $r = 2n - m/2$.

It is easy to show that if we substitute $\Delta(k_j; \mu_j)$ for $\Delta'(k_j; \mu_j)$ in (17), terms will be added to $G_m^{(n)}$ each of which is proportional to

$$\lim_{\mu \to 0} \int dk \Delta(k; \mu) = \pi i m^2 \lim_{\mu \to 0} \sum_i C_i(\mu) \mathfrak{M}_i(\mu) \ln \mathfrak{M}_i(\mu). \tag{18}$$

Thus, if we require that

$$\lim_{\mu \to 0} \sum_i C_i(\mu) \mathfrak{M}_i(\mu) \ln \mathfrak{M}_i(\mu) = A_{11} = 0, \tag{19}$$

then we can replace Δ' by Δ in (17). We note that the condition (19) is completely compatible with the regularization scheme that is being used.

We consider the expression

$$B(p_1, \ldots, p_m) = \lim_{\mu \to 0} \int dp \Delta(p; \mu) G_m^{(n)}(p, -p, p_1, \ldots, p_m). \tag{20}$$

Using (17) and (19), we can write $B(p_1, \ldots, p_m)$ in the form

$$B(p_1, \ldots, p_m) = \lim_{\mu_1 \to 0} \int dp \Delta(p; \mu_1) (i)^{-n} (2\pi)^{-4n} (2i)^{-r+1} \frac{1}{n!}$$

$$\times \frac{1}{(r-1)!} \lim_{\mu_2 \to 0} \ldots \lim_{\mu_r \to 0} \int dk_1 \ldots dk_{4n}$$

$$\times \prod_{i=0}^{n-1} \delta(k_{4i+1} + \ldots + k_{4i+4}) P(k_1, \ldots, k_{4n}) \prod_{l=1}^{m} \delta(p_l - k_{2r+l})$$

$$\times \prod_{j=1}^{r-1} \delta(k_j + k_{j+r-1}) \Delta(k_j; \mu_j) \delta(k_{2r-1} - p) \delta(k_{2r} + p), \tag{21}$$

where $r = 2n - m/2$. Integrating with respect to p (according to the conditions (3) we can integrate with respect to p under the limit sign), we obtain

$$B(p_1, \ldots, p_m) = i(4n - m)(i)^{-n} (2\pi)^{-4n} (2i)^{-r} \frac{1}{n!} \frac{1}{r!}$$

$$\times \lim_{\mu_1 \to 0} \ldots \lim_{\mu_r \to 0} \int dk_1 \ldots dk_{4n} \prod_{i=0}^{n-1} \delta(k_{4i+1} + \ldots + k_{4i+4})$$

$$\times P(k_1, \ldots, k_{4n}) \prod_{l=1}^{m} \delta(p_l - k_{2r+l}) \prod_{j=1}^{r} \delta(k_j + k_{j+r}) \Delta(k_j; \mu_j). \tag{22}$$

It follows that for $m < 4n$

$$C_m^{(n)}(p_1, \ldots, p_m) = \frac{1}{i} \frac{1}{4n - m} \lim_{\mu \to 0} \int dp \, \Delta(p; \mu)$$

$$\times G_{m+2}^{(n)}(p, -p, p_1, \ldots, p_m). \tag{23}$$

For $m = 4n$ we obtain directly from (17) for $G_m^{(n)}$

$$G_{4n}^{(n)} = \frac{1}{n!} \frac{i^n}{(2\pi)^{4n}} P(p_1, \ldots, p_{4n}) \prod_{i=0}^{n-1} \delta(p_{4i+1} + \ldots + p_{4i+4}). \tag{24}$$

It is also clear that for $m > 4n$

$$G_m^{(n)} = 0. \tag{25}$$

We now multiply both sides of Eq. (23) by $g_1^{n-1-m/4}$ and integrate from 0 to g

$$g^n G_m^{(n)}(p_1, \ldots, p_m) = \frac{1}{4i} g^{m/4} \int_0^g dg_1 g_1^{-1-m/4}$$

$$\times \lim_{\mu \to 0} \int dp \Delta(p; \mu) \, g_1^n G_{m+2}^{(n)}(p, -p, p_1, \ldots, p_m). \tag{26}$$

Substituting (24), (25) and (26) in (8) we obtain

$$G_m(p_1, \ldots, p_m; g) = \frac{1}{4i} g^{m/4} \int_0^g dg_1 g_1^{-1-m/4}$$

$$\times \lim_{\mu \to 0} \int dp \Delta(p; \mu) G_{m+2}(p, -p, p_1, \ldots, p_m) + (ig)^{m/4}$$

$$\times \frac{\sigma(m)}{(2\pi)^m} \frac{1}{(m/4)!} P(p_1, \ldots, p_m) \prod_{i=0}^{m/4-1} \delta(p_{4i+1} + \ldots + p_{4i+4}), \tag{27}$$

$$\sigma(m) = \begin{cases} 1, & m = 4l \\ 0, & m \neq 4l \end{cases}, \text{ where } l = 1, 2, \ldots.$$

Equation (27) gives an infinite chain of coupled equations for the determination of the regularized many-particle Green's functions.

We call attention to the following fact. The system (27) has been obtained by perturbation theory, but the final result does not manifestly show that perturbation theory is involved. Therefore we may hope that a similar system of equations will also hold outside the framework of perturbation theory. Here, however, we must make one remark: we have not touched at all upon the problem of a possible non-analytic dependence of the Green's functions on the charge [6,7]. There is of course no way of exhibiting such a non-analyticity by starting from perturbation theory.

From the infinite system of equations (27) one can go over to one variational (functional) equation. For this we introduce the quantity

$$K_m(p_1, \ldots, p_m; g) = g^{-m/4} G_m(p_1, \ldots, p_m; g). \tag{28}$$

Then (27) goes over into

$$K_m(p_1, \ldots, p_m; g) = \frac{1}{4i}\int_0^g dg_1 g_1^{-1/2} \lim_{\mu\to 0} \int dp \Delta(p; \mu) \times K_{m+2}(p, -p, p_1, \ldots, p_m; g_1) + (i)^{m/4} \frac{\sigma(m)}{(2\pi)^m} \frac{1}{(m/4)!} \times P(p_1, \ldots, p_m) \prod_{i=0}^{m/4-1} \delta(p_{4i+1} + \ldots + p_{4i+4}). \tag{29}$$

We multiply both sides of (29) by $(m!)^{-1} f(p_1) \ldots f(p_m)$ where $f(p)$ is some function, and then integrate over $p_1, \ldots, p_m$ and sum over m. As a result we obtain

$$K(f; g) = \frac{1}{4i}\int_0^g dg_1 g_1^{-1/2} \lim_{\mu\to 0} \int dp \Delta(p; \mu) \times \frac{\delta^2 K(f; g_1)}{\delta f(p) \delta f(-p)} + N(f). \tag{30}$$

Here

$$K(f; g) = \sum_{m=0}^{\infty} \frac{1}{m!} \int dp_1 \ldots dp_m f(p_1) \ldots f(p_m) \times K_m(p_1, \ldots, p_m; g), \tag{31}$$

$$N(f) = \exp\left\{\frac{i}{(2\pi)^4} \int dp_1 \ldots dp_4 \delta(p_1 + \ldots + p_4) \times f(p_1) \ldots f(p_4)\right\}. \tag{32}$$

Equation (30) can be rewritten in the configuration space variables

$$K(f; g) = \frac{1}{4i}\int_0^g dg_1 g_1^{-1/2} \lim_{\mu\to 0} \int dx dy \Delta^{(c)}(x - y; \mu) \times \frac{\delta^2 K(f; g_1)}{\delta f(x) \delta f(y)} + N(f), \tag{33}$$

$$N(f) = \exp\left\{i \int f^4(x) dx\right\}.$$

Knowing the quantity $K(f; g)$ is of course equivalent to knowing all many-particle Green's functions, or, what is the same, knowing the S-matrix. The latter is expressed in an elementary way in terms of K:

$$S = \sum_{m=0}^{\infty} \frac{1}{m!} g^{m/4} \int dx_1 \ldots dx_m : \varphi(x_1) \ldots \varphi(x_m) : \left.\frac{\delta^m K(f; g)}{\delta f(x_1) \ldots \delta f(x_m)}\right|_{f=0}. \tag{34}$$

We have thus obtained an equation for a quantity that allows us to construct easily any element of the S-matrix, and all orders of perturbation theory are taken into account at once.

Equation (33) [or (30)] has been derived under the assumption of smallness for the coupling constant g. However, once the equation is established, one may assume that it will be valid even for values of g which are no longer small.

A peculiarity of Eq. (33) is that although it is an equation for renormalized quantities, it contains no infinite coefficients. This essentially distinguishes it from, say, the Schwinger equations for the Green's functions, in which infinite renormalization constants appear in the case of renormalized quantities.

By solving Eq. (33), we see readily that we deal with an equation for renormalized quantities. Neglecting in zeroth order the integral in the right side of (33), we have $K^{(0)}(f; g) = N(f)$. Substituting this $K^{(0)}$ in the right side of (33) we compute $K^{(1)}$ etc. As a result we obtain for $K(f; g)$ a series in powers of $g^{1/2}$. It is easy to verify that the S-matrix constructed according to Eq. (34) from this series coincides with the expression (10). But, as already indicated, Eq. (10) defines a regularized S-matrix if the conditions (5) are satisfied.

Thus, in order to obtain from (33) regularized values for the many-particle Green's functions, two facts are important: the presence of the limit with respect to μ in (33) and the supplementary conditions (5). In connection with the limiting process with respect to μ in (33) it is worth mentioning that regularized expressions for the Green's functions will be obtained only if the limit is taken after the integrations with respect to x and y are performed in (33) [or the integration with respect to p in (30)]. If we insert the limit sign under the integral signs, we obtain formal equations for the Green's functions. Solutions of these equations by perturbation theory will contain the usual ultraviolet divergences.

We now discuss the supplementary conditions (5) imposed on the coefficients C_i and $\mathfrak{M}_i$ in $\Delta(x; \mu)$. These conditions are obtained from the requirement that no divergences occur in the elements of the S-matrix constructed according to perturbation theory. Perturbation theory is reflected as follows in the formulation of these conditions. The conditions (5) guarantee the convergence of all matrix elements in n-th order perturbation theory, if the parameter β involved in them takes the values $\beta = 1, 2, \ldots, n$. But, as already mentioned, we can go to the limit

$n \to \infty$ in the conditions (5). This means that we can require from the very beginning that the conditions (5) be satisfied for all natural β. In this case the matrix elements of any arbitrarily high order in the coupling constant will be convergent. The quantities $A_{\alpha\beta}$ can be completely arbitrary, and define only the normalization point of the Green's functions (for renormalizable theories). Thus the conditions (5) can be reformulated so that they do not contain perturbation theory explicitly.

Of course, from the fact that the conditions (5) guarantee regularization up to any order in the coupling constant one cannot assert with complete confidence that they will guarantee regularization also outside the framework of perturbation theory. On the other hand, since there exists no regularization outside the framework of perturbation theory so far, it seems to us that the assumption that Eq. (33) together with the conditions (5) leads to regularized Green's functions, is a very modest one. How far this assumption is justifiable, can be tested, however, only if a method is found of solution of Eq. (33), other than expansion in powers of the coupling constant.

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