K-3π *DECAY AND INTERACTIONS BETWEEN PIONS AT LOW ENERGIES*

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Submitted to JETP editor January 14, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 240-247 (July, 1964)

The Skornyakov–Ter-Martirosyan integral equation for the K-3π decay is solved by an iteration procedure for the case when a_0 is large and $a_2 = 0$ (a_T is the scattering length for mesons in a state with isotopic spin T). The energy and angular distributions obtained in this way for the $K^\pm \to 2\pi^\pm + \pi^\mp$ reactions are compared with the experimental data [6] for $a_0 = 1$, 2, 3. It is found that the experimental data can be satisfactorily described by the theoretical curves only if $a_0 = 1$. This should indicate that a_0 is of the order of unity or smaller (it was shown in [5] that the experimental data on K-3π decay are compatible with the assumption that a_0 is small).

INTRODUCTION

At present there are experimental data indicating that at low energies the interaction between pions is resonant (ABC resonance [1]). This resonance, however, was not observed in many investigations [2]. A study of K-3π reactions is of interest because of the possibility of explaining the character of interaction of pions at low energies.

Gribov et al. [3,4] wrote down an expression for the decay amplitude under the assumption that the pion scattering lengths are small (the pion interaction is nonresonant at low energies). A comparison of the obtained formulas with experiment does not make it possible, however, to determine the pion scattering lengths, owing to the large experimental errors. All that follows from this comparison is that $|a_0a_2| \leq 0.25$ [5], (a_T —pion scattering length in the state with isotopic spin T, in $\hbar/\pi c$ units). It is therefore of interest to obtain an expression for the K-3π decay amplitude under the assumption that a_0 is large (ABC resonance) and to see whether these formulas contradict the experimental data.

The author has obtained previously [5], from an examination of the diagrams, a dispersion relation for the K-3π decay. In the present paper we solve this dispersion relation with the aid of an iteration procedure.

We recall first, however, the considerations on the basis of which we analyze reactions in which three particles are produced near threshold (or decays into three particles with low kinetic-energy release). The decay amplitude (or the amplitude for the production of several particles) is expanded in a series of the squares k_{il}^2 of the momenta of the relative motion of the produced particles (the subscripts i and l denote the numbers of the produced particles—see Fig. 1). Near the physical region of the decay, the amplitude has singularities which must be separated in this expansion.

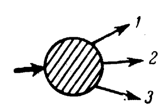


FIG. 1

Figure 2 shows the physical region of the decay (shaded) and some of the singularities closest to it. For example, the amplitude has on the physical sheet a singularity with respect to k_{23}^2 at $k_{23}^2 = 0$ and $k_{23}^2 = 3$ (or $s_{23} = 4$ and $s_{23} = 16$, where s_{23} is the square of the total energy of particles 2 and 3,

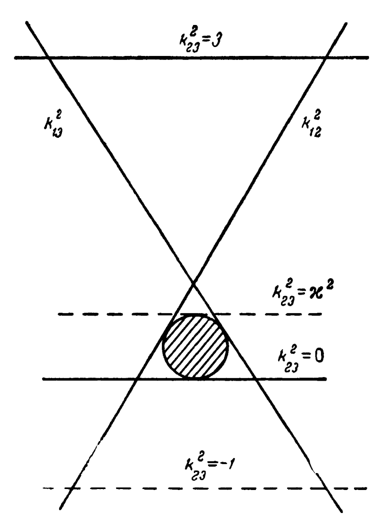
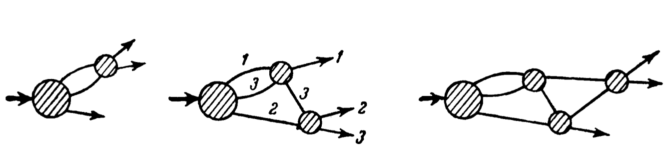


FIG. 2

and the pion mass is equal to unity). These singularities are shown in Fig. 2 by solid lines. In addition, near the physical region the amplitude has singularities on the second sheet at $k_{23}^2 = \kappa^2$ (κ^2 —total kinetic energy released in the decay) and at $k_{23}^2 = -1$ (the last singularity corresponds to $s_{23} = 0$). The position of these singularities is shown by the dashed lines. Analogous singularities exist, of course, also for the variables k_{12}^2 and k_{13}^2.

In expanding the amplitude we wish to take into account the contribution from the nearest singularities of the amplitude—from the singularities at $k_{il}^2 = 0$ and $k_{il}^2 = \kappa^2$. The singularities at $k_{il}^2 = 0$ have diagrams of the type shown in Fig. 3a, the singularities at $k_{il}^2 = 0$ and at $k_{il}^2 = \kappa^2$ —diagrams of type 3b and 3c, or diagrams with even a larger number of scatterings of the produced particles by one another. The parts of the diagrams which have singularities near the physical region of the decay are expressed in terms of the same decay amplitude and in terms of the scattering amplitudes of the produced particles. The parts of the amplitude which have far singularities (singularities at $k_{il}^2 = -1$, $k_{il}^2 = 3$, etc.) are expanded in a series in k_{il}^2: $\lambda + C_1k_{23}^2 + C_2k_{13}^2 + C_3k_{12}^2 + \ldots$, where all the coefficients are unknown constants. All that we can say about the coefficients C_i (and also about the other coefficients of the higher powers of k_{il}^2) is that they should be, generally speaking, of the order of λ. This follows from the fact that the distances from the far singularities to the physical region are of the order of unity or more. Therefore this series converges rapidly if the kinetic energy of the produced particles is small.

The amplitude terms with singularities near threshold, as already mentioned, result from diagrams of the type shown in Fig. 3. The singular terms that result from diagram 3a are of the order of $\lambda\kappa a_T$, those from 3b are of the order of $\lambda(\kappa a_T)^2$, and those from 3c —of the order $\lambda(\kappa a_T)^3$. If $a_T < 1$, the contribution to the non-adiabatic parts of the amplitude from the diagrams with a large number of scatterings of the produced particles is small. On the other hand, however, if for example a_0 is large (resonance interaction), then the diagrams 3a, b, c, etc., give identical contributions



a b c

FIG. 3

to the part of the amplitude which is non-analytic near threshold. Summation of all diagrams of this type leads to an integral equation for the decay amplitude [5].

Since there is experimental evidence in favor of assuming a_2 to be small, we shall solve in this paper the dispersion relation for the decay amplitude with $a_2 = 0$. In the first section we consider the dispersion relation for the decay amplitude in the case when the coefficients C_i (and also the coefficients of the higher powers of k_{il}^2) vanish. In the second section we consider the equation in which the coefficients of the first powers of k_{il}^2 (C_1, C_2, and C_3) are different from zero, and the coefficients of the higher powers of k_{il}^2 are equal to zero.

1. SOLUTION OF THE EQUATION IN THE CASE WHEN $C_i = 0$

If $a_2 = 0$ and $C_i = 0$, then the amplitude $A(k_{12}k_{13}k_{23})$ of the processes $K^\pm \to 2\pi^\pm + \pi^\pm$ is determined by the following relations:

$$A(k_{12}k_{13}k_{23}) = d(k_{13}) + d(k_{23}), \tag{1a}$$

$$d(k) = \lambda(1 - ika_0)^{-1} + \frac{2k^2}{3\pi}(1 - ika_0)^{-1}\int_0^\infty dk'^2\, \frac{k'a_0\langle d(k'')\rangle}{k'^2(k'^2 - k^2 - i\varepsilon)}, \tag{1b}$$

$$\langle d(k'')\rangle = \frac{2}{\sqrt{3}k'(\varkappa^2 - k'^2)^{1/2}}\int_{k_-}^{k_+} k''d(k'')\,dk'',$$

$$k_\pm = \pm\frac{1}{2}k' + \frac{\sqrt{3}}{2}(\varkappa^2 - k'^2)^{1/2}. \tag{1c}$$

A derivation of these relations was presented earlier [5]. The dispersion relation (1b) can be reduced to an equation of the Skornyakov–Ter-Martirosyan type:

$$d(k) = \lambda(1 - ika_0)^{-1} + \frac{2}{3}a_0(1 - ika_0)^{-1}\int_{\varkappa^2}^{-\infty} dk'^2 d(k')\,[L(k'k) - L(k'0)],$$

$$L(k'k) = [\sqrt{3}\pi(\varkappa^2 - k^2)^{1/2}]^{-1} \times \ln\frac{[-{}^1\!/_2(\varkappa^2 - k'^2)^{1/2} + (\varkappa^2 - k^2)^{1/2}]^2 - 3k'^2/4}{[{}^1\!/_2(\varkappa^2 - k'^2)^{1/2} + (\varkappa^2 - k^2)^{1/2}]^2 - 3k'^2/4}. \tag{2}$$

To find the location of the singularities relative to the contour of integration it is necessary to assume that k^2 has in the integrand of (2) a positive imaginary addition $i\epsilon$.

When $\kappa^2 \le 0$, Eq. (2) can be solved by successive iterations: in the zeroth approximation $d(k)$ is equal to $\lambda(1 - ika_0)^{-1}$, the first correction is determined by the second term in (2), in which the amplitude $d(k')$ under the integral sign is replaced by its zeroth approximation, etc. The possibility of such an iteration procedure follows from the fact that if $a_0 > 0$ (we are interested in just positive a_0), then the inequality $(1 - ika_0)^{-1} \leq 1$ holds when $k^2 < 0$, and the quantity $L(k'k) - L(k'0)$ does not reverse sign on the integration contour. In this case we can readily show that the ratio of the modulus of any correction to that of the preceding one should be smaller than or of the order of $2/3$. Since in fact $(1 - ika_0)^{-1}$ is appreciably smaller than unity over almost the entire integration contour, this is a slight overestimate. It can be thought that a similar iteration procedure can be used to find $d(k)$ also when $\kappa^2 > 0$. If we make one iteration, we obtain

$$d(k) = \lambda(1 - ika_0)^{-1}[1 + {}^2/_3F(ka_0)], \qquad (3a)$$

$$F(ka_0) = a_0 \int\limits_{\kappa^2}^{-\infty} dk'^2 (1 - ik'a_0)^{-1} [L(k'k) - L(k'0)]. \qquad (3b)$$

For the real case $\kappa^2 = 0.56$ (the pion mass is equal to unity) we can obtain $F(ka_0)$ by numerical integration.

The values of $F(ka_0)$ for $a_0 = 1, 2,$ and 3, and for k^2 in the interval between zero and κ^2 are listed in the table. It is seen from the table that the ratio of the moduli of the correction terms to the modulus of the zeroth approximation is smaller than or of the order of 0.4. This fact serves as some confirmation of our hope that $d(k)$ can also be obtained with the aid of the iterations described above in the case $\kappa^2 > 0$.

k^2/κ^2	$F(ka_0)$		
	$a_0=1$	$a_0=2$	$a_0=3$
0	0	0	0
0.25	$-0.15+i\,0.19$	$-0.29+i\,0.17$	$-0.35+i\,0.13$
0.5	$-0.22+i\,0.26$	$-0.40+i\,0.24$	$-0.47+i\,0.19$
0.75	$-0.28+i\,0.30$	$-0.47+i\,0.28$	$-0.54+i\,0.22$
1	$-0.32+i\,0.34$	$-0.49+i\,0.29$	$-0.55+i\,0.24$

The meaning of these iterations consists in the following. The quantity $\lambda(1 - ika_0)^{-1}$ is represented by diagrams of the type 3a. If diagram 3a is integrated once, the result is represented by diagrams of type 3b. The smallness is the result of the fact that in the second term in the right side of (2) there is a factor $2/3$, and not some large quantity (for example, were we to consider the case $a_0 = a_2$, we would have a factor 2 in the analogous equation for the decay amplitude. If the particles interact only in a state with isotopic spin T = 0, then it is necessary to project in the diagram of Fig. 3b the state in which particles 1 and 3 have T = 0 on the state in which particles 2 and 3 have T = 0. This leads to the appearance of the factor $2/3$.

If, using (3), we obtain the energy distributions of the mesons produced in the reactions $K^{\pm} \to 2\pi^{\pm} + \pi^{\pm}$, then we see immediately that at large values of a_0 these energy distributions contradict the experimental data. The amplitude (3) gives pion energy-distribution curves with a slope opposite to that observed experimentally.

In (2), and consequently also in (3), no account was taken of the fact that the decay amplitude has the large remote singularities referred to in the introduction. The presence of these singularities leads effectively to a dependence of λ on k_{il}^2. The decay amplitude satisfies (2) only if $\lambda(k_{il}^2)$ varies little near the physical region of the decay. In the opposite case we must write dispersion equations that take into account the dependence of λ on k_{il}^2. If $\lambda(k_{il}^2)$ is expanded in powers of k_{il}^2 near the physical region of the decay, then, as already mentioned, the coefficients of this expansion should be of the order of unity. In the next section we shall consider the simplest case, when

$$\lambda(k_{il}^2) = \lambda + \sum_{i \neq l, m} C_i k_{lm}^2$$

(the constants C_i are unknown beforehand and must be determined from experiment). The amplitudes of the decays $K^{\pm} \to 2\pi^{\pm} + \pi^{\mp}$ contain only one of the three C_i, since $C_1 = C_2$ and the term $C_3k_{12}^2 = C_3(3\kappa^2/2 - k_{13}^2 - k_{23}^2)$ leads only to a redefinition of λ and C_1.

2. DECAY AMPLITUDE IN THE CASE $C_i \neq 0$

The dispersion relation for the amplitude $d(k)$, which has the properties described above, is obtained from (1b) by means of the substitution $\lambda \to \lambda + Ck^2$:

$$d(k) = \frac{\lambda + Ck^2}{1 - ika_0} + \frac{2k^2}{3\pi}(1 - ika_0)^{-1} \int\limits_0^{M^2} dk' \frac{k'a_0 \langle d(k'') \rangle}{k'^2(k'^2 - k^2 - i\varepsilon)}. \qquad (4)$$

We have introduced here a cutoff in the dispersion integral, since the latter diverges at infinity when $C \neq 0$. The divergence at the upper limits in the dispersion integrals can be eliminated by recognizing that the derivative of the function $d(k) - {}^2/_3 ika_0 \langle d(k'') \rangle_{k=0} - ika_0\lambda$ with respect to k^2 depends on the constant C. After redefining the unknown constant C, Eq. (4) can be written in a different form:

$$d(k) = (\lambda + Ck^2)(1 - ika_0)^{-1} + \frac{2k^2}{3\pi}(1 - ika_0)^{-1}\left\{\int_0^\infty dk'^2 \right.$$
$$\times \frac{k' a_0 \langle d(k'')\rangle}{k'^2(k'^2 - k^2 - i\varepsilon)} - \int_0^\infty dk'^2 a_0 k'^{-3}$$
$$\times [\langle d(k'')\rangle - \langle d(k'')\rangle_{k'=0}]. \tag{5}$$

We assume that this equation can be solved with the aid of the iteration procedure described in the first section. It will be shown later that the modulus ratio of the first correction term to the zeroth approximation $d_0(k) = (\lambda + Ck^2)(1 - ika_0)^{-1}$ is smaller than or of the order of 0.4 at the values of C of interest to us and for $a_0 = 1, 2, 3$. The smallness resulting from the iteration solution of (2) was of the same order of magnitude.

If we confine ourselves in the solution of (5) to a single iteration, then the expression for d(k) takes the form

$$d(k) = (\lambda + Ck^2)(1 - ika_0)^{-1}$$
$$+ {}^2/_3(1 - ika_0)^{-1}\{(1 - a_0^{-2}C)\,[F(ka_0) - f(a_0)k^2]$$
$$- a_0^{-1}Cik + C[-\sqrt{3}\pi^{-1}k(\varkappa^2 - k^2)^{-1/2}$$
$$\times (\varkappa^2 - {}^8/_9 k^2)\arccos(k/\varkappa) - \sqrt{3}k^2/\pi]\}. \tag{6}$$

where the constant $f(a_0)$ with $a_0 = 1, 2,$ and 3 has a value

$$f(1) = 0.36 - i\,0.05, \quad f(2) = 0.71 - i\,0.43,$$
$$f(3) = 1.1 - i\,0.5. \tag{7}$$

So far we have disregarded the mass difference of the neutral and charged mesons. In some cases, however, this mass difference can be appreciable. For example, in the reaction $K^+ \to 2\pi^+ + \pi^-$ the pion mass difference influences the energy distribution of the positive pions at $E^+ \sim \kappa^2$, and hardly affects the energy distribution of the negative pions. In order to take into account the mass difference of the neutral and charged pions, it is necessary to replace in the right side of (4) the factor $(1 - ika_0)^{-1}$ in the first and second terms by $[1 - ia_0(2k/3 + (k^2 - \Delta)^{1/2}/3)^{1/2})]^{-1}$ (here Δ — double the mass difference between the charged and neutral pions). We can estimate with sufficient accuracy the effect of the mass difference by replacing in the right side of (6) the factor $(1 - ika_0)^{-1}$ by $[1 - ia_0(2k/3 + (k^2 - \Delta)^{1/2}/3)]^{-1}$.

The energy and angular distributions of the produced pions depend on the unknown parameter C/λ. Figures 4 and 5 show the distributions of the pions in the reactions $K^\pm \to 2\pi^\pm + \pi^\mp$, obtained by starting from formulas (6) with $a_0 = 1$, $C/\lambda = 1.7$; $a_0 = 2$, $C/\lambda = 2.7$; $a_0 = 3$, $C/\lambda = 3.8$. The dashed line in Fig. 5 shows the behavior of the energy distribution without account of the meson mass difference.

The experimental data on the reactions $K^\pm \to 2\pi^\pm + \pi^\mp$ can be found in the paper by Ferro-Luzzi et al.[6]:

$$w(\varepsilon_u) = 1 + (\varepsilon_u - {}^1/_2)(0.53 \pm 0.07),$$
$$w(\varepsilon_l) = 1 - (\varepsilon_l - {}^1/_2)(0.26 \pm 0.09). \tag{8}$$

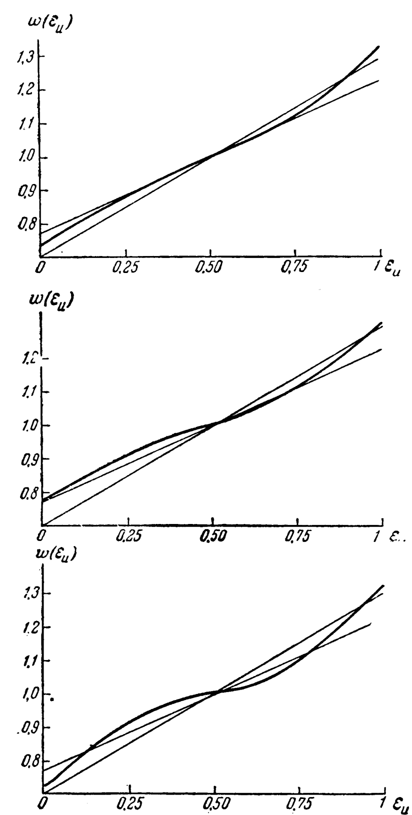


FIG. 4

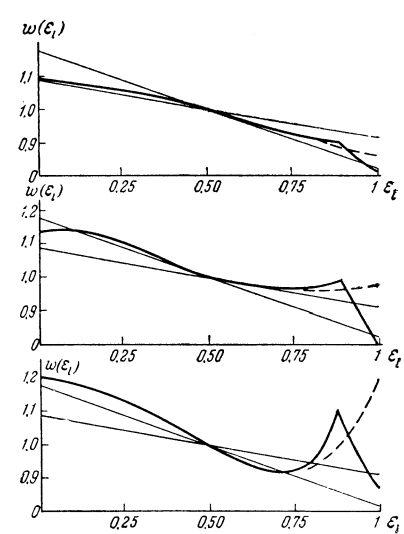


FIG. 5

Here ϵ_l —energy of the identical mesons (π^+ in the reaction $K^+ \to 2\pi^+ + \pi^-$), divided by its maximum value, and ϵ_u —energy of the oppositely charged meson. Figures 4 and 5 show the limiting values of the experimental straight lines (8):

$$w(\varepsilon_u) = 1 + 0.6(\varepsilon_u - {}^1\!/_2), \quad w(\varepsilon_u) = 1 + 0.46(\varepsilon_u - {}^1\!/_2),$$
$$w(\varepsilon_l) = 1 - 0.35(\varepsilon_l - {}^1\!/_2), \quad w(\varepsilon_l) = 1 - 0.17(\varepsilon_l - {}^1\!/_2).$$

The energy distribution with respect to ϵ_l shows for $a_0 = 2$ and $a_0 = 3$ a noticeable hump at $\epsilon_l \sim 0.9$, which has not been observed experimentally, although the presently attainable experimental accuracy should have made this possible. It is important to emphasize the following circumstance: This hump is only the result of the fact that the expression for $d(k)$ contains the function $(1 - ika_0)^{-1}$ as a common denominator (more accurately, the function $[1 - ia_0(2k/3 + (k^2 - \Delta)^{1/2}/3]^{-1}$). At large a_0 this function vanishes on the unphysical sheet near $k = 0$. In order for the energy distribution with respect to ϵ_l not to have a hump it is necessary that the numerator of the expression for the amplitude also vanish at this point. If, for example, we consider for simplicity the amplitude in the zeroth approximation $d_0(k) = (\lambda + Ck^2) \times (1 - ika_0)^{-1}$, then in order for no hump to appear it is necessary to have $C/\lambda \approx a_0^2$. Such a large value of C/λ leads to pion energy distributions that differ from those observed in experiment.

In the calculations presented above the ratio C/λ was assumed real. The imaginary parts of C and λ are due to diagrams of the type shown in Figs. 3b and c. A contribution to the imaginary part of λ is made by diagrams of the type shown in Figs. 3b and c with $k = 0$, while a contribution to the imaginary part of C is made by the second derivatives with respect to k of these diagrams at $k = 0$. These imaginary parts are utterly different and the ratio C/λ should not be real at all. However, the humps in the energy distribution with respect to ϵ_l remain even for complex C/λ. In fact, for example in the zeroth-approximation amplitude, the zero of the numerator goes over into the complex plane and no cancellation of the zeroes of the numerator and denominator will take place. It is unlikely that a more accurate solution of (5) (inclusion of the next iterations) would make it possible to cancel simultaneously the zeroes of the numerator and of the denominator of the amplitude and to obtain pion energy distribution curves with relatively small slopes, as are observed in experiment.

In the above calculations it was assumed that $a_2 = 0$. However, it is clear that if a_2 is small, then an account of the pion interaction with $a_2 \neq 0$ will not change the energy distribution with respect to ϵ_l appreciably. In order for (6) to give the correct energy distribution with respect to ϵ_u for $a_0 = 2.3$, it would be necessary to assume an unnaturally large C (on the order of λ). The humps in the energy distribution can apparently be eliminated by assuming that the coefficients of the higher powers of k_{il}^2 in the expansion of λ in powers of k_{il}^2 are likewise anomalously large. Such an assumption, however, is quite unlikely.

The foregoing analysis indicates that a_0 is of the order of unity or less (the author has shown[5] that the experimental data on K-3π decay are in good agreement with the assumption that a_0 is small). More accurate information on the character of the interaction of the pions at low energies based on the formulas considered above (or on the formulas given in [5]) can be obtained only if the experimental accuracy is increased.

The author is deeply grateful to A. A. Ansel'm and G. S. Danilov for useful discussions and to T. Yu. Andrievskaya and N. V. Koroleva for the numerical calculations.

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Translated by J. G. Adashko
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