ENERGY OF QUADRUPOLE STATES OF STRONGLY DEFORMED EVEN-EVEN NUCLEI

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Energies of the β- and γ-vibrational states of even-even strongly deformed nuclei with $152 \leq A \leq 186$ and $228 \leq A \leq 254$ are calculated. Satisfactory agreement between the calculations and corresponding experimental data is found in the case when $\kappa_n = \kappa_p = \kappa_{np} \equiv \kappa$ where $\kappa = 10 A^{-4/3} \hbar\omega_0^0$ in the first region and $\kappa = 12 A^{-4/3} \hbar\omega_0^0$ in the second. In agreement with the experimental data, it is shown that for the Dy and Er isotopes the energies of the γ-vibrational states lie below those of the β-vibrational states. It is noted that in most cases the lowest states with $K\pi = 2^+$ and $K\pi = 0^+$ possess distinct collective properties and their energies are much below those of the nearest poles in the secular equations. In a number of cases the energies of the quadrupole states lie near the poles and their wave functions are very close to the two-particle ones. The calculated E2 transition probabilities do not contradict the experimental data.

1. The study of the collective excited states of even-even nuclei has been the subject of many theoretical and experimental papers. A sufficiently large amount of experimental material has been accumulated on the energies of the β- and γ-vibrational states in even-even strongly-deformed nuclei, and on the reduced probabilities of the electromagnetic transitions. The theoretical investigations of the microscopic nature of the collective states are based on the approximate second quantization proposed by Bogolyubov in 1947 [1] and on the mathematical methods developed for superconductivity theory [2]. The most thoroughly studied were the collective states of spherical nuclei [3-5]. In the region of strongly deformed nuclei, investigations [6] made prior to 1963 are limited to the presentation of the fundamental equations and to the elimination of the ghost state. In [7,8] were calculated the energies of the octopole states of strongly-deformed even-even nuclei, and sufficiently good agreement was obtained with the corresponding experimental data. In [9] were calculated the energies of the γ-vibrational states and the probabilities of the electromagnetic E2 transitions. However, the energies of the β oscillations were not calculated in that paper, and the energies of the γ oscillations, calculated using the asymptotic wave functions of the Nilsson potential, were approximately one and a half times larger than the experimental values.

It is of interest to calculate the energies of the β- and γ-vibrational states at the same value of the quadrupole-quadrupole interaction constants, using the exact wave functions of the Nilsson potential. The present paper is devoted to such calculations and to a comparison of theory with experiment.

2. The energies of the collective states will be calculated on the basis of the superfluid model of the nucleus. In this model the Hamiltonian for the interaction between the nucleons in the nucleus is written in the form of three terms

$$H = H_{av} + H_{pair} + H_{coll}, \tag{1}$$

which describe the average field of the nucleus, the interactions that lead to pair correlations of the superconducting type, and interactions responsible for the collective effects. In calculating the energies we take the quadrupole collective states H_{coll} in the form

$$H_{coll} = -\sum_{\mu=0,2} \left\{ \frac{\varkappa_n^{(2)}}{2} Q_{2\mu}^+(n) Q_{2\mu}(n) + \frac{\varkappa_p^{(2)}}{2} Q_{2\mu}^+(p) Q_{2\mu}(p) + \frac{\varkappa_{np}^{(2)}}{2} \left(Q_{2\mu}^+(n) Q_{2\mu}(p) + Q_{2\mu}^+(p) Q_{2\mu}(n) \right) \right\}, \tag{2}$$

where

$$Q_{2\mu}(n) = \sum_{\substack{ss' \\ \sigma\sigma'}} f_{\sigma\sigma'}^{2\mu}(ss') a_{s\sigma}^+ a_{s'\sigma},$$

$$f^{20} = r^2 Y_{20}, \qquad f^{22} = r^2 (Y_{22} + Y_{2,-2})/\sqrt{2},$$

$\kappa_n^{(2)}$, $\kappa_t^{(2)}$, and $\kappa_{np}^{(2)}$ are quadrupole-quadrupole interaction constants.

In the microscopic treatment of the superfluid model of the nucleus, the wave functions of the

collective states are superpositions of the wave functions of two quasiparticle states. The collective states are considered alongside with the two quasiparticle states, and there are no limitations here on the energies of the collective states. The terms β-vibrational and γ-vibrational states must be regarded as arbitrary, the β-vibrational states being the lowest excited states with $K\pi = 0^+$, that is, with zero projection of the angular momentum on the nuclear symmetry axis and with positive parity (different from the rotational states), while the γ-vibrational states are the lowest states with $K\pi = 2^+$.

3. Let us consider the γ-vibrational states. In the study of these states it must be taken into account that along with the matrix elements $f^{22}_{\sigma\sigma}(s_1, s_2) \equiv f(s_1, s_2)$, where $K_1 \pm 2 = K_2$ (K_1 and K_2 are the momentum projections of the two states) it is necessary to take into account the matrix elements $f^{22}_{\sigma,-\sigma}(s_1, s_2) \equiv \bar{f}(s_1, s_2)$ with $K_1 + K_2 = \pm 2$, elements not taken into account in many papers.

On the basis of the variational principle we obtain, within the framework of the approximate second quantization method, a secular equation that determines the energies ω_i of the excited states with $K\pi = 2^+$, in the form

$$1 = 2\varkappa_n^{(2)} \sum_{ss'} \frac{[f(ss')^2 + \bar{f}(ss')^2]\, u_{ss'}^2}{\Omega(ss')} + 2\varkappa_p^{(2)} \sum_{\nu\nu'} \frac{[f(\nu\nu')^2 + \bar{f}(\nu\nu')^2]\, u_{\nu\nu'}^2}{\Omega(\nu\nu')} + 4\left[(\varkappa_{np}^{(2)})^2 - \varkappa_n^{(2)}\varkappa_p^{(2)}\right] \sum_{ss'} \frac{[f(ss')^2 + \bar{f}(ss')^2]\, u_{ss'}^2}{\Omega(ss')} \times \sum_{\nu\nu'} \frac{[f(\nu\nu')^2 + \bar{f}(\nu\nu')^2]\, u_{\nu\nu'}^2}{\Omega(\nu\nu')}, \quad (3)$$

where

$$\Omega(ss') = \varepsilon(s) + \varepsilon(s') - \omega_i^2 / [\varepsilon(s) + \varepsilon(s')]$$

and analogously for $\Omega(\nu\nu')$. The summation of $ss'(\nu\nu')$ is carried out over the single-particle levels of the average field of the neutron (proton) system;

$$\varepsilon(s) = \sqrt{C_n^2 + \{E(s) - \lambda_n\}^2}, \quad \varepsilon(\nu) = \sqrt{C_p^2 + \{E(\nu) - \lambda_p\}^2},$$
$$u_{ss'} = u_s v_{s'} + u_{s'} v_s.$$

The subscript i denotes the first, second, etc., root of the secular equation.

In the case $\kappa_n^{(2)} = \kappa_p^{(2)} = \kappa_{np}^{(2)} \equiv \kappa$, to which we confine ourselves henceforth, the secular equation assumes a simpler form, namely

$$\frac{1}{2\varkappa} = \sum_{ss'} \frac{[f(ss')^2 + \bar{f}(ss')^2]\, u_{ss'}^2}{\Omega(ss')} + \sum_{\nu\nu'} \frac{[f(\nu\nu')^2 + \bar{f}(\nu\nu')^2]\, u_{\nu\nu'}^2}{\Omega(\nu\nu')}. \quad (4)$$

The wave function of the i-th state with $K\pi = 2^+$ is of the form $Q_i^+ \Psi$, where the operator Q_i is equal to

$$Q_i = \frac{1}{2}\Big\{\sum_{ss'} [\psi_{ss'}^i A(ss') - \varphi_{ss'}^i A(ss')^+ + \bar{\psi}_{ss'} \mathfrak{A}(ss') - \bar{\varphi}_{ss'}^i \mathfrak{A}(ss')^+] + \sum_{\nu\nu'} [\psi_{\nu\nu'}^i A(\nu\nu') - \varphi_{\nu\nu'}^i A(\nu\nu')^+ + \bar{\psi}_{\nu\nu'}^i \mathfrak{A}(\nu\nu') - \bar{\varphi}_{\nu\nu'}^i \mathfrak{A}(\nu\nu')^+]\Big\}, \quad (5)$$

with

$$A(ss') = \frac{1}{\sqrt{2}} \sum_\sigma \sigma \alpha_{s'\sigma}\, \alpha_{s,-\sigma}, \qquad \mathfrak{A}(ss') = \frac{1}{\sqrt{2}} \sum_\sigma \alpha_{s\sigma}\, \alpha_{s'\sigma},$$

where $\alpha_{s\sigma}$, $\alpha_{\nu\tau}$ —quasiparticle operators.

The functions

$$\psi_{ss'}^i,\ \varphi_{ss'}^i,\ \bar{\psi}_{ss'}^i,\ \bar{\varphi}_{ss'}^i,\ \psi_{\nu\nu'}^i,\ \varphi_{\nu\nu'}^i,\ \bar{\psi}_{\nu\nu'}^i \text{ and } \bar{\varphi}_{\nu\nu'}^i$$

are written in the form

$$\psi_{ss'}^i = {}^1/_2 (g_{ss'}^i + w_{ss'}^i), \qquad \bar{\psi}_{ss'}^i = {}^1/_2 (\bar{g}_{ss'}^i + \bar{w}_{ss'}^i),$$
$$\varphi_{ss'}^i = {}^1/_2 (g_{ss'}^i - w_{ss'}^i), \qquad \bar{\varphi}_{ss'}^i = {}^1/_2 (\bar{g}_{ss'}^i - \bar{w}_{ss'}^i), \quad (6)$$

where

$$w_{ss'}^i = \frac{\omega_i}{\varepsilon(s) + \varepsilon(s')} g_{ss'}^i, \qquad \bar{w}_{ss'}^i = \frac{\omega_i}{\varepsilon(s) + \varepsilon(s')} \bar{g}_{ss'}^i; \quad (7a)$$

$$g_{ss'}^i = \left[\frac{2}{Y_n^i + \bar{Y}_n^i + Y_p^i + \bar{Y}_p^i}\right]^{1/2} \frac{f(ss')\, u_{ss'}}{\Omega(ss')},$$

$$\bar{g}_{ss'}^i = \left[\frac{2}{Y_n^i + \bar{Y}_n^i + Y_p^i + \bar{Y}_p^i}\right]^{1/2} \frac{\bar{f}(ss')\, u_{ss'}}{\Omega(ss')}, \quad (7b)$$

with

$$Y_n^i + \bar{Y}_n^i = \sum_{ss'} \frac{[f(ss')^2 + \bar{f}(ss')^2]\, u_{ss'}^2\, \omega_i [\varepsilon(s) + \varepsilon(s')]}{[(\varepsilon(s) + \varepsilon(s'))^2 - \omega_i^2]^2}.$$

Thus, solving the secular equations (3) and (4), we obtain the energies of the states with $K\pi = 2^+$ and the corresponding wave functions.

4. Let us consider the β-oscillations, that is, excited states with characteristic $IK\pi = 00^+$. In this case there are diagonal and off-diagonal matrix elements of the quadrupole-moment operator, and $\bar{f}(ss') = 0$. Thus, the secular equation for a system consisting of one sort of particles can be written in the form

$$\frac{1}{2\varkappa} = \sum_{s \neq s'} \frac{f(ss')^2\, u_{ss'}^2}{\Omega(ss')} + \sum_s \frac{f(ss)^2\, c^2}{[4\varepsilon(s)^2 - \omega_i^2]\, \varepsilon(s)}, \quad (8)$$

with the contribution of the diagonal terms, calculated by this formula, reaching 50—70%.

We shall show that the diagonal terms should not play so important a role. Indeed, let us consider a case when all the diagonal terms are equal to one another, that is, $f(ss) = f_0$, and then the corresponding part H_{coll} is of the form

$$-{}^1/_2\varkappa f_0{}^2N^2,$$

where N —particle-number operator. Since the number of particles is the same in the ground and in the excited states, it follows that this term does not lead to a change in the energy difference between the β-vibrational and the ground states. If $f(ss) \neq f_0$, then the diagonal terms can lead to both a decrease and an increase in the energy of the excited states with $K\pi = 0^+$. Thus, the coherent effect that leads to a lowering of the energy of the states with $K\pi = 0^+$, relative to the nearest pole of the secular equation, is connected essentially with the off-diagonal matrix elements of the quadrupole-moment operator. From these arguments we see that the calculations of ω_i from (8) are very inaccurate.

A method proposed in [4] for eliminating the ghost states makes it possible to improve greatly the approximation considered above. The secular equation has in this case a more complicated form, namely (for $\kappa_n^{(2)} = \kappa_p^{(2)} = \kappa_{np}^{(2)} \equiv \kappa$):

$$\begin{vmatrix} X_n{}^i + X_p{}^i - \dfrac{1}{2\varkappa} & V_n{}^i & V_p{}^i & W_n{}^i & W_p{}^i \\ V_n{}^i & \sum\limits_s \dfrac{1}{2\varepsilon(s)\Delta_s} & 0 & \sum\limits_s \dfrac{u_s{}^2 - v_s{}^2}{\Delta_s} & 0 \\ V_p{}^i & 0 & \sum\limits_\nu \dfrac{1}{2\varepsilon(\nu)\Delta_\nu} & 0 & \sum\limits_\nu \dfrac{u_\nu{}^2 - v_\nu{}^2}{\Delta_\nu} \\ W_n{}^i & \sum\limits_s \dfrac{u_s{}^2 - v_s{}^2}{\Delta_s} & 0 & \sum\limits_s \dfrac{\omega_i{}^2 - 4C_n{}^2}{2\varepsilon(s)\Delta_s} & 0 \\ W_p{}^i & 0 & \sum\limits_\nu \dfrac{u_\nu{}^2 - v_\nu{}^2}{\Delta_\nu} & 0 & \sum\limits_\nu \dfrac{\omega_i{}^2 - 4C_p{}^2}{2\varepsilon(\nu)\Delta_\nu} \end{vmatrix} = 0; \tag{9}$$

where we put

$$\Delta_s = 4\varepsilon(s)^2 - \omega_i{}^2, \quad \Delta_\nu = 4\varepsilon(\nu)^2 - \omega_i{}^2,$$

and

$$X_n{}^i = \sum_{ss'} \frac{f(ss')^2 u_{ss'}{}^2}{\Omega(ss')}, \qquad V_n{}^i = \sum_s \frac{f(ss)\, C_n}{\varepsilon(s)\,[4\varepsilon(s)^2 - \omega_i{}^2]},$$

$$W_n{}^i = \sum_s \frac{f(ss)\cdot 2C_n\,[E(s) - \lambda_n]}{\varepsilon(s)\,[4\varepsilon(s)^2 - \omega_i{}^2]}. \tag{9a}$$

We note that in the case when $f(ss') = f_0$ the diagonal terms drop out from (9). In this case therefore the energy of the excited state with $K\pi = 0^+$ does not depend on the diagonal terms, in agreement with the reasoning presented above. The role of the diagonal matrix elements in (9) was greatly reduced compared with (8).

The wave functions for the states with $K\pi = 0^+$ can be easily obtained by using the following expressions for $g_{ss'}^i$ and $w_{ss'}^i$:

$$g_{ss'}{}^i = \frac{\sqrt{2}}{\sqrt{Z_n{}^i + Z_p{}^i}} \left\{ \frac{f(ss')\, u_{ss'}}{\Omega(ss')} - \delta_{ss'} \frac{2C_n}{4\varepsilon(s)^2 - \omega_i{}^2} \frac{\Gamma_n{}^i(s)}{\gamma_n{}^i} \right\}, \tag{10}$$

$$w_{ss'}{}^i = \frac{\sqrt{2}}{\sqrt{Z_n{}^i + Z_p{}^i}} \left\{ \frac{f(ss')\, u_{ss'}\, \omega_i}{[\varepsilon(s) + \varepsilon(s')]^2 - \omega_i{}^2} - \delta_{ss'} \frac{\omega_i C_n}{\varepsilon(s)\,[4\varepsilon(s)^2 - \omega_i{}^2]} \frac{\Gamma_n{}^i(s)}{\gamma_n{}^i} - \delta_{ss'} \frac{C_n}{\varepsilon(s)\,\omega_i} \frac{\xi_n{}^i}{\gamma_n{}^i} \right\} \tag{11}$$

and analogous expressions for the proton system, with

$$Z_n{}^i = Y_n{}^i + 2\frac{\omega_i C_n{}^2}{(\gamma_n{}^i)^2} \sum_s \frac{[\Gamma_n{}^i(s)]^2}{\varepsilon(s)\,[4\varepsilon(s)^2 - \omega_i{}^2]} - 4\frac{\omega_i C_n{}^2}{\gamma_n{}^i} \sum_s \frac{f(ss)\,\Gamma_n{}^i(s)}{\varepsilon(s)\,[4\varepsilon(s)^2 - \omega_i{}^2]^2},$$

$$\Gamma_n{}^i(s) = \sum_{s_2 s_2'} \frac{f(s_2 s_2)}{\varepsilon(s_2)\,[4\varepsilon(s_2)^2 - \omega_i{}^2]} \times \frac{4\,[E(s_2') - \lambda_n][E(s_2) - \lambda_n] - 4\,[E(s) - \lambda_n][E(s_2) - \lambda_n]}{\varepsilon(s_2')\,[4\varepsilon(s_2')^2 - \omega_i{}^2]} + \frac{4\,[E(s) - \lambda_n]\,[E(s_2') - \lambda_n] + 4C_n{}^2 - \omega_i{}^2}{\varepsilon(s_2')\,[4\varepsilon(s_2')^2 - \omega_i{}^2]},$$

$$\gamma_n{}^i = \sum_{ss'} \frac{4\,[E(s) - \lambda_n]\,[E(s') - \lambda_n] + 4C_n{}^2 - \omega_i{}^2}{\varepsilon(s)\,[4\varepsilon(s)^2 - \omega_i{}^2]\,\varepsilon(s')\,[4\varepsilon(s')^2 - \omega_i{}^2]},$$

$$\xi_n{}^i = \sum_{ss'} \frac{f(ss)}{\varepsilon(s)\,[4\varepsilon(s)^2 - \omega_i{}^2]} \times \frac{4C_n{}^2 - \omega_i{}^2 + 4\,[E(s) - \lambda_n]\,[E(s') - \lambda_n]}{\varepsilon(s')\,[4\varepsilon(s')^2 - \omega_i{}^2]}.$$

The roots of the secular equation (9) were obtained, as for (4), with an electronic computer. A determinant of sixth order was expanded (the program was set up for the general case of arbitrary values of κ_n, κ_p, and κ_{np}) and was so that the dependence on κ_n, κ_p, and κ_{np} became explicit. The first root was then sought in the interval from $\omega = 0$ to the nearest pole $X_n(\omega)$ and $X_p(\omega)$, the second root between the first and second poles, etc. The roots were obtained by dividing the seg-

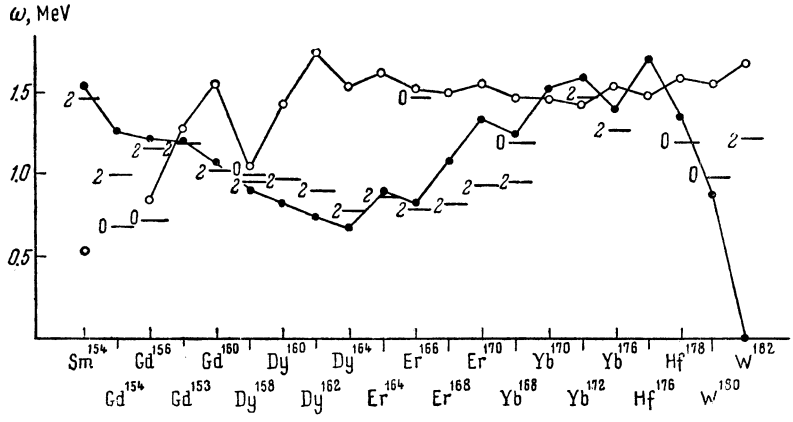

FIG. 1. Energies of β- and γ-vibrational states for $\kappa^{(2)} = 10A^{-4/3}\hbar\omega_0^0 = 410A^{-5/3}$ MeV. Points – experimental data (○ – $K\pi = 0^+$. ● – $K\pi = 2^+$), horizontal segments – calculated data (the value of K is indicated, the parity is positive everywhere).

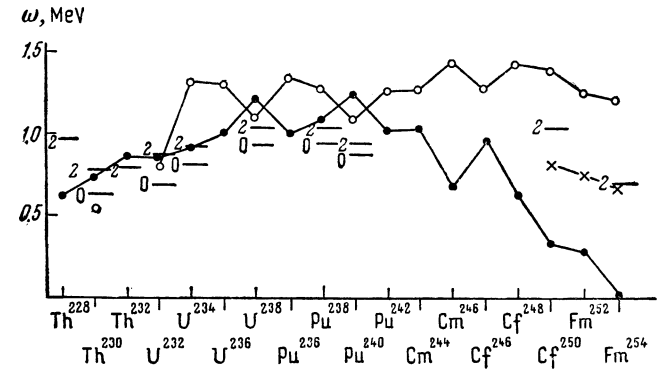

FIG. 2. Energies of β- and γ-vibrational states for $\kappa^{(2)} = 12A^{-4/3}\hbar\omega_0^0 = 480A^{-5/3}$ MeV. Notation the same as in Fig. 1.

ment in two, and eleven iterations were used to find the root with accuracy 10^{-4}.

5. Calculations of the energies of the β- and γ-vibrational states and of the reduced probabilities of the electromagnetic transitions were made in both regions of strongly-deformed nuclei; $152 \leq A \leq 186$ and $228 \leq A \leq 254$. The wave functions and the Nilsson potential single-particle level schemes were used in the calculations [10]. All the calculations in the region $152 \leq A \leq 186$ were made with the wave functions for a deformation $\delta = 0.3$, while in the region $228 \leq A \leq 254$ —for $\delta = 0.2$ and with the same single-particle level scheme in the neutron (proton) system for each nucleus in each region. In order to make the calculations most unambiguous, no account was taken of the changes in the nuclear deformations. Consequently the accuracy of the calculations became worse near the boundaries of the regions of the strongly deformed nuclei, because the equilibrium deformation changed but not the behavior of the single-particle levels of the average field. The single-particle level schemes of the average fields, the values of the correlation functions, and the chemical potentials used in the present calculations are given in [8]. We note that the correlation functions and the chemical potentials were obtained earlier in [11,12].

The quadrupole-quadrupole interaction constant κ is chosen such as to obtain the best agreement between the calculated energies of the states with $K\pi = 2^+$ and the corresponding experimental data. The value of κ depends on the number of transitions in (4). We took into account all the transitions between the single-particle states given in [8] —approximately 100—120 transitions each. We note that the calculated energies of the β- and γ-vibrational states do not depend on the number of transitions in Eqs. (4) or (9) if a sufficiently large number of transitions is taken. If, following Belyaev [3], we assume that

$$\varkappa = kA^{-4/3}\hbar\omega_0{}^0 = k'A^{-5/3} \text{ MeV},$$

then we cannot choose the same value of k for both regions of strongly-deformed nuclei. Thus, in the region $152 \leq A \leq 186$ we have $k = 10$, while in the region $228 \leq A \leq 254$ we have $k = 12$. There is no doubt that by increasing the number of transitions in the region to $228 \leq A \leq 254$ we can obtain the same value of k for both regions. We note that when $\kappa = 1.8A^{-1}\hbar\omega_0^0$ a sufficiently good agreement is obtained between the calculated energies of the collective states and the experimental data in both regions.

If the β-vibrational state energies are calculated in accordance with (8), then to obtain even crude agreement with experiment it is necessary to introduce two values of κ: one to describe the energies of the β oscillations, and the other for the γ oscillations.

6. Let us discuss the results of the energy calculations for the lowest states with $IK\pi = 00^+$ and $IK\pi = 22^+$. The calculated energies of these states and the corresponding experimental data are shown in Figs. 1 and 2. The experimental values of energies were taken from [11,13,14], etc.

In the region $154 \leq A \leq 182$, the calculated energies of the states with $K\pi = 2^+$ agree sufficiently well with the experimental data. An exception is

W^{182}, owing to the change in the deformation. However, if we take $\kappa = 8A^{-4/3}\hbar\omega_0^2$ we obtain 1.2 MeV for the energy of the state W^{182} with $K\pi = 2^+$ and approximately 1 MeV for W^{184}. If we choose $\kappa = 11A^{-4/3}\hbar\omega_0^2$, then the agreement between theory and experiment for Er^{166}, Er^{168}, and all the isotopes of Yb is very good, but for the Dy isotopes the calculated energies of the 2^+ states lie much lower than the experimental values.

The agreement between the calculated energies of the 0^+ states with the experimental data is satisfactory. Exceptions are the isotopes of Hf and W, in which the energies of the 0^+ states are quite close to the values of the corresponding poles, so that account must be taken of the blocking effect[15], which in this case plays a large role. If we take this effect into account, good agreement with experiment can be obtained also for the Hf and W isotopes.

An interesting question is that of the relative behavior of the energies of the β- and γ-vibrational states. According to our calculations, the energies of the states with $K\pi = 2^+$ are considerably lower than the energies of the states with $K\pi = 0^+$ for the Dy and Er isotopes. The drop in the energies of the γ-vibrational states below the energies of the β-vibrational states is determined by the behavior of the levels of the average field and the corresponding wave functions. This drop is in full agreement with the experimental data and is explained theoretically here for the first time.

An investigation of the structure of the collective state will be dealt with in the next paper. However, getting somewhat ahead of ourselves, we note that most of the lowest states with $K\pi = 2^+$ possess clearly pronounced collective properties. Thus, for Er^{166}, each of the four two-quasiparticle states contributes from 16 to 32% to the collective state with $K\pi = 2^+$. In many cases the lowest states with $K\pi = 2^+$ are close to two-quasiparticle states. Thus, for Yb^{172} the wave function of the lowest state with $K\pi = 2^+$ contains a contribution equal to 97.6% of the two-quasiparticle neutron state $5/2^+[512] - 1/2^+[521]$. Inasmuch as the admixtures of the other states make up less than 3%, this state is very close to the two-quasiparticle one, thus confirming the results previously obtained in [11]. However, Yb^{172} should have a second state with $K\pi = 2^+$, lying 200—400 keV higher than the first and having collective properties.

The results of the calculations of the energies of the β- and γ-vibrational states in the region $228 \leq A \leq 254$ and the corresponding experimental data are shown in Fig. 2. The calculated energies of the states with $K\pi = 2^+$ are in sufficiently good agreement with the corresponding experimental data. The discrepancy in Pu^{240} is connected with the fact that no account is taken of the blocking effect, and if account is taken of this effect, the two-quasiparticle state with $K\pi = 2^+$ has an energy 0.9 MeV, in accordance with [12]. The excessively strong decrease in the states with $K\pi = 2^+$ for Cf^{250} and Fm^{254} is due to the very large matrix element of the transition between the states $7/2^+[613]$ and $3/2^+[611]$. If we increase the number of transitions, the positions of the 2^+ levels in Cf^{220} and Fm^{254} and the behavior of the energies of the states with $K\pi = 0^+$ should be more correct. The cross in Fig. 2 denotes the values of the energies of the states with $K\pi = 2^+$ for Cf^{250}, Fm^{254}, and Fm^{252}, calculated with $\kappa = 11A^{-4/3}\hbar\omega_0^2$. The agreement between the calculations and the experimental data can be improved by foregoing the condition $\kappa_n = \kappa_p = \kappa_{np}$.

7. The reduced probabilities of the E2 transitions from the states $K\pi = 2^+$ to the ground states are calculated from the formula

$$B(E2) = \frac{1.024A^{2/3}\cdot 10^{-53}}{Y_n + \bar{Y}_n + Y_p + \bar{Y}_p}\left\{e_{eff}\sum_{ss'}\frac{[f(ss')^2 + \bar{f}(ss')^2]\,u_{ss'}^2}{\Omega(ss')}\right.$$
$$\left. + (e + e_{eff})\sum_{\nu\nu'}\frac{[f(\nu\nu')^2 + \bar{f}(\nu\nu')^2]\,u_{\nu\nu'}^2}{\Omega(\nu\nu')}\right\}^2. \qquad (12)$$

The reduced probabilities B(E2) were calculated with $e_p = e + e_{eff}$, $e_n = e_{eff}$, and $e_{eff} = 0.8e$. In the cases for which there are experimental data[14], the ratio of the probability of the E2 transition to its single-particle value, $B(E2)/B(E2)_{s.p}$, was found to range from 1 to 4. Thus, for example, the ratio $B(E2)/B(E2)_{s.p}$ is equal to 2.8 for Dy^{160}, 3.2 for Dy^{162}, 3.6 for Dy^{164}, 1.7 for Th^{230}, 1.4 for U^{238}, etc. However, for the 1468-keV 2^+ state $B(E2)/B(E2)_{s.p} = 0.04$ for Yb^{172}, that is, 30—50 times smaller than for the neighboring nuclei. This circumstance is an additional indication of the two-quasiparticle nature of this state, and the experimental determination of B(E2), as well as the determination of the second state with $K\pi = 2^+$ in Yb^{172}, would be highly desirable.

The values we obtained for the energies of the states with $K\pi = 2^+$ and for the ratios $B(E2)/B(E2)_{s.p}$ differ from the results of the calculations of Marshalek and Rasmussen[9] because our calculations are consistently based on the superfluid model of the nucleus, whereas in[9] the phenomenological treatment of the unified model is combined with the microscopic approach of the superfluid model. This difference in approach is manifest, for example, in the choice of the quadrupole-interaction constant. Further, the dif-

ference is connected with the fact that in [9] they investigated the asymptotic wave functions of the Nilsson potential, whereas we took the exact wave functions of the Nilsson potential. This difference lies not so much in the fact that the matrix elements with the asymptotic wave functions have not been calculated with sufficient accuracy, but principally in the fact that a large number of poles of the secular equation (4) is lost when working with asymptotic wave functions. We note that in many cases nearby poles with even relatively small values of the matrix element $f(ss')$ and $\bar{f}(ss')$ play a very important role. The difference in the values of $B(E2)/B(E2)_{s.p}$ is connected principally with the fact that the calculations of [9] were carried out with a formula that follows from the phenomenological theory.

8. Thus, the calculated values of the energies of the β- and γ-vibrational states are in sufficiently good agreement with the corresponding experimental data for the same values of the quadrupole-interaction constants. The calculations show that in many cases the energies of the γ-vibrational states lie below the β-vibrational states, as for example in the middle of the $154 \leq A \leq 182$ region. The behavior of the energies of the quadrupole states is determined by the average field of the nucleus. In most cases the lowest states with $K\pi = 2^+$ and $K\pi = 0^+$ possess clearly pronounced collective properties, and the values of the energies of the states drop considerably relative to the nearest poles in the secular equations. In many cases the energies of the states with $K\pi = 2^+$ and $K\pi = 0^+$ lie near the lowest poles, and their wave functions are close to two-quasiparticle ones. This was pointed out earlier [7] in a study of the behavior of the energies of octopole states with $K\pi = 0^-$.

It must be noted that the calculation results depend greatly on the Nilsson potential wave functions. It is perfectly possible that some errors in the calculations are connected with shortcomings inherent in the Nilsson potential wave functions. However, the sufficiently good general agreement between the theory and experiment with respect to the energies of the states with $K\pi = 2^+$ and $K\pi = 0^+$ is one more indication that it is possible to describe the average field of strongly deformed nuclei sufficiently well with the aid of the Nilsson potential.

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