

RADIATIVE CORRECTIONS TO THE PHOTOPRODUCTION OF ELECTRON-POSITRON PAIRS

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For the process of photoproduction of an electron-positron pair in the Coulomb field of a nucleus, the S-matrix method is used to calculate radiative corrections to the effective cross section, which are of relative magnitude (compared with the Bethe-Heitler formula) proportional to the first power of the fine-structure constant  $\alpha$ . It is shown that in the ultrarelativistic region  $\omega \gg m$ , for a pair emerging symmetrically relative to the momentum of the photon and making with it small angles  $\theta_+ = \theta_- = m/\omega$ , the fractional correction is proportional to  $(\omega/m)^2$ . The energetic and angular widths of this region of anomalously large radiative corrections are investigated.

INTRODUCTION

THE usual assumption is that radiative corrections to scattering processes are insignificant, with a relative weight of the order of  $\alpha/\pi \cong 0.25$  percent, where  $\alpha$  is the fine-structure constant. Generally speaking this is true, but in some cases the probability of the main process can have a singularity, and then the radiative corrections can be important. In particular, under definite conditions (to be discussed later) this situation can occur for the process of production of an electron-positron pair by a photon in the Coulomb field of an atomic nucleus.

As is well known,<sup>[1]</sup> the use of only the basic fourth-order process represented by the Feynman diagrams of Fig. 1 leads to the Bethe-Heitler formula for the effective cross section,

$$d\sigma^{(2)} = \frac{Z^2 \alpha^3 |\mathbf{p}_+| |\mathbf{p}_-|}{\pi^2 \omega^3 m^4 q^4} \left\{ \frac{2\omega^2}{\kappa_1 \kappa_2} [\mathbf{k}, \mathbf{p}_+ + \mathbf{p}_-]^2 - 4 \left[ \mathbf{k}, \frac{\boldsymbol{\varepsilon}_- \mathbf{p}_+}{\kappa_1} + \frac{\boldsymbol{\varepsilon}_+ \mathbf{p}_-}{\kappa_2} \right]^2 + q^2 \left[ \mathbf{k}, \frac{\mathbf{p}_+}{\kappa_1} - \frac{\mathbf{p}_-}{\kappa_2} \right]^2 \right\} d\varepsilon_+ d\Omega_+ d\Omega_- \tag{1)*}$$

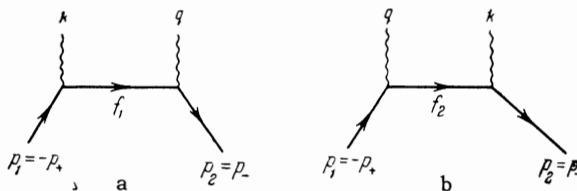


FIG. 1

\* $[\mathbf{k}, \mathbf{p}_+ + \mathbf{p}_-] = \mathbf{k} \times (\mathbf{p}_+ + \mathbf{p}_-)$ .

where  $\mathbf{k}$ ,  $\mathbf{p}_+$ , and  $\mathbf{p}_-$  are the four-momenta of the photon, positron, and electron,  $\mathbf{q} = \mathbf{k} - (\mathbf{p}_+ + \mathbf{p}_-)$  is the recoil momentum of the nucleus ( $q_4 = 0$ ), and  $\kappa_1$  and  $\kappa_2$  are defined by the equations

$$m^2 \kappa_1 = f_1^2 + m^2 = -2k p_+, \quad m^2 \kappa_2 = f_2^2 + m^2 = -2k p_- \tag{2}$$

The remaining notations are the same as in [1] ( $c = \hbar = 1$ ).

In the ultrarelativistic case ( $\omega \gg m$ ) the largest effect is found at small angles ( $\theta_{\pm} \cong m/\omega$ ) between the momenta of the photon and electron (positron). Then the main terms in the Bethe-Heitler formula (1) are the first two, and the third is small, of order  $(m/\omega)^4$  (relative to the first two terms). If furthermore the particles of the pair emerge symmetrically relative to the momentum of the photon (Fig. 2), with momenta equal in magnitude (and energies  $\epsilon_+ = \epsilon_- = \omega/2$ ) and at the same angle ( $\theta_+ = \theta_-$ ), then the main terms in the Bethe-Heitler formula vanish and the cross section is anomalously small. In this case the radiative corrections can be relatively large. Therefore it is interesting to calculate these radiative corrections.

CALCULATION OF THE RADIATIVE CORRECTIONS

The lowest-order radiative corrections come from the following fourth-order processes, which are represented by the Feynman diagrams shown in Fig. 3.

To the processes represented by diagrams a and b there corresponds the matrix element

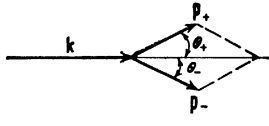


FIG. 2

$$S_{c+d+e+f}^{(4)} = \frac{ie^2}{\sqrt{2\omega}} \bar{u} \left\{ \hat{a}(q) \frac{i\hat{f}_1 - m}{f_1^2 + m^2} e_\mu \Lambda_\mu^{(3)}(p_1 f_1; k) \right. \\ \left. + \Lambda_\mu^{(3)}(f_2 p_2; k) e_\mu \frac{i\hat{f}_2 - m}{f_2^2 + m^2} \hat{a}(q) \right. \\ \left. + \Lambda_\mu^{(3)}(f_1 p_2; q) a_\mu(q) \frac{i\hat{f}_1 - m}{f_1^2 + m^2} \hat{e} \right. \\ \left. + \hat{e} \frac{i\hat{f}_2 - m}{f_2^2 + m^2} \Lambda_\mu^{(3)}(p_1 f_2; q) a_\mu(q) \right\} v, \quad (6)$$

where the vertex part is

$$\Lambda_\mu^{(3)}(p_1 p_2; k) = \frac{ie^2}{(2\pi)^4} \int \gamma_\nu \frac{i(\hat{p}_2 - \hat{r}) - m}{(p_2 - r)^2 + m^2} \gamma_\mu \frac{i(\hat{p}_1 - \hat{r}) - m}{(p_1 - r)^2 + m^2} \gamma_\nu \frac{d^4 r}{r^2 + \lambda^2}, \\ p_2 = p_1 + k. \quad (7)$$

The matrix element for the processes of diagrams g and h is

$$S_{g+h}^{(4)} = \frac{ie^2}{\sqrt{2\omega}} a_4(q) \Pi_{v_4}^{(2)}(q) \bar{u} \left\{ \gamma_\nu \frac{i\hat{f}_1 - m}{f_1^2 + m^2} \hat{e} + \hat{e} \frac{i\hat{f}_2 - m}{f_2^2 + m^2} \gamma_\nu \right\} v, \quad (8)$$

where the photon self-energy part is

$$S_{a+b}^{(4)} = -\frac{e^2}{\sqrt{2\omega}} \bar{u} \left\{ \hat{a}(q) \frac{i\hat{f}_1 - m}{f_1^2 + m^2} \Sigma^{(2)}(f_1) \frac{i\hat{f}_1 - m}{f_1^2 + m^2} \hat{e} \right. \\ \left. + \hat{e} \frac{i\hat{f}_2 - m}{f_2^2 + m^2} \Sigma^{(2)}(f_2) \frac{i\hat{f}_2 - m}{f_2^2 + m^2} \hat{a}(q) \right\} v, \quad (3)$$

where the Fourier transform of the Coulomb potential of the nucleus is

$$a_4(q) = 2\pi i \frac{Ze}{q^2} \delta(q_4); \quad a(q) = 0; \quad \hat{a}(q) = 2\pi i \gamma_4 \frac{Ze}{q^2} \delta(q_4), \quad (4)$$

and the electron self-energy part is

$$\Sigma^{(2)}(f) = \frac{e^2}{(2\pi)^4} \int \gamma_\mu \frac{i(\hat{f} - \hat{r}) - m}{(f - r)^2 + m^2} \gamma_\mu \frac{d^4 r}{r^2 + \lambda^2}, \quad (5)$$

where  $\lambda$  is a "photon mass" introduced to remove the so-called "infrared divergence."

The matrix element corresponding to the processes represented by diagrams c-f is

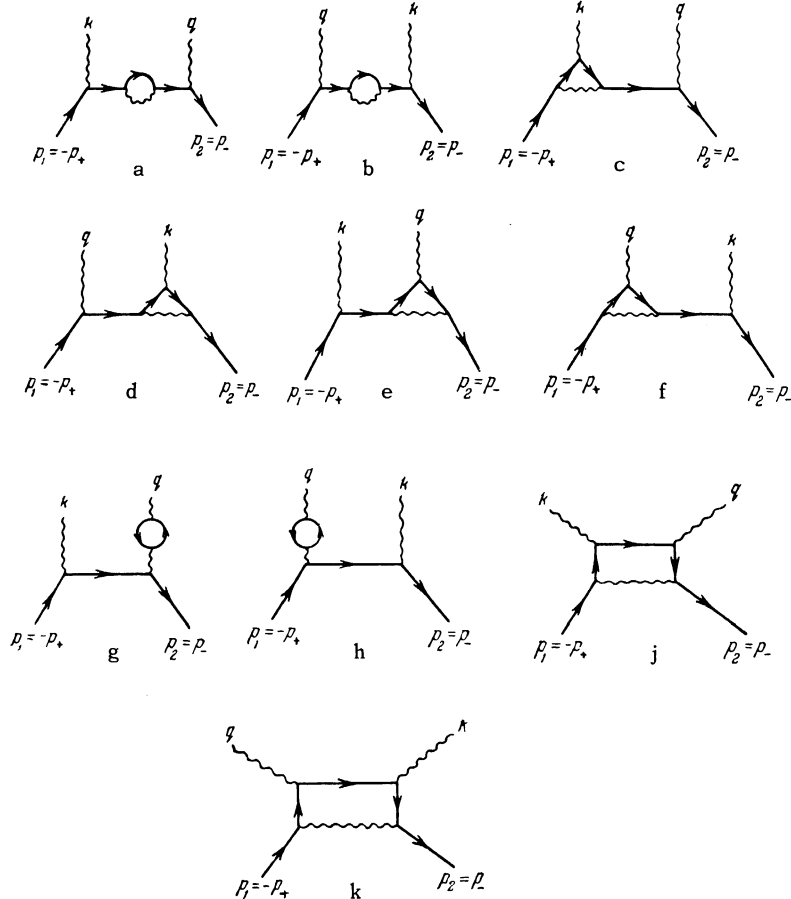


FIG. 3

$$\Pi_{\nu}^{(2)}(q) = -\frac{ie^2}{(2\pi)^4 q^2} \int \text{Sp} \left\{ \gamma_{\nu} \frac{i(\hat{f} + \hat{q}) - m}{(f+q)^2 + m^2} \gamma_4 \frac{i\hat{f} - m}{f^2 + m^2} \right\} d^4 f. \quad (9)$$

Finally, the matrix element for the processes of diagrams j and k is

$$S_{j+k}^{(4)} = -\frac{e^4}{(2\pi)^4 \sqrt{2\omega}} \bar{u} \gamma_{\nu} \frac{i(\hat{p}_2 - \hat{r}) - m}{(p_2 - r)^2 + m^2} \times \left\{ \hat{a}(q) \frac{i(\hat{f}_1 - \hat{r}) - m}{(f_1 - r)^2 + m^2} \hat{e} + \hat{e} \frac{i(\hat{f}_2 - \hat{r}) - m}{(f_2 - r)^2 + m^2} \hat{a}(q) \right\} \times \frac{i(\hat{p}_1 - \hat{r}) - m}{(p_1 - r)^2 + m^2} \frac{d^4 r}{r^2} \gamma_{\nu} v. \quad (10)$$

When we include these fourth-order processes, the complete matrix element for photoproduction of a pair is

$$S_{i \rightarrow j} = S_{a+b}^{(2)} + S_{a+b}^{(4)} + S_{c+d+e+f}^{(4)} + S_{g+h}^{(4)} + S_{j+k}^{(4)} = \bar{u} Q v \delta(q_4); \quad (11)$$

$$S_{a+b}^{(2)} = \frac{ie^2}{\sqrt{2\omega}} \bar{u} \left\{ \hat{a}(q) \frac{i\hat{f}_1 - m}{f_1^2 + m^2} \hat{e} + \hat{e} \frac{i\hat{f}_2 - m}{f_2^2 + m^2} \hat{a}(q) \right\} v \quad (12)$$

is the matrix element corresponding to the basic second-order process (Fig. 1).

Knowing the matrix Q defined by Eq. (11), one can calculate the effective differential cross section of the process from the formula<sup>[1]</sup>

$$d\sigma = \frac{1}{8(2\pi)^7} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \times \text{Sp} \{ Q' (i\hat{p}_1 - m) \bar{Q}' (i\hat{p}_2 - m) \}, \quad (13)$$

where the prime means that  $\hat{e}$  is to be replaced by the matrix  $\gamma_{\mu}$  and then there is a summation over the four values of the index  $\mu$ . It follows from (11) that the matrix Q can be put in the form of a sum

$$Q = Q^{(2)} + \sum_n Q_n^{(4)}, \quad (14)$$

in which the individual terms correspond to individual processes we have considered. We have  $Q^{(2)} \sim \alpha^{3/2}$  and  $Q_n^{(4)} \sim \alpha^{5/2}$ , so that, neglecting corrections that are proportional to higher powers of  $\alpha = e^2/4\pi$  and using the fact that

$$\text{Sp} \{ Q^{(2)} (i\hat{p}_1 - m) \bar{Q}^{(2)} (i\hat{p}_2 - m) \} = \text{Sp} \{ Q^{(4)} (i\hat{p}_1 - m) \bar{Q}^{(2)} (i\hat{p}_2 - m) \},$$

we finally get from (13)

$$d\sigma = d\sigma^{(2)} + \sum_n d\sigma_n^{(4)}, \quad (15)$$

where

$$d\sigma^{(2)} = \frac{1}{8(2\pi)^7} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_-$$

$$\times \text{Sp} \{ Q^{(2)'} (i\hat{p}_1 - m) \bar{Q}^{(2)'} (i\hat{p}_2 - m) \} \quad (16a)$$

is the Bethe-Heitler cross section, and

$$d\sigma_n^{(4)} = \frac{1}{4(2\pi)^7} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \times \text{Sp} \{ Q_n^{(4)'} (i\hat{p}_1 - m) \bar{Q}^{(2)'} (i\hat{p}_2 - m) \} \quad (16)$$

is the desired radiative correction corresponding to any of the fourth-order processes we have been considering.

In the calculation of the radiative corrections from (16) we must allow for the fact that the matrices  $Q_n^{(4)}$  contain divergences, which must be replaced by their regularized values. The following results have been obtained in this way:

1. The radiative correction caused by the electron self-energy part. Using the fact that the regularized value of the electron self-energy part is of the form

$$\Sigma_R^{(2)}(f) = \frac{\alpha}{2\pi} (Af - imB), \quad (17)$$

$$A = \frac{\kappa}{2(1-\kappa)} \left( \frac{\kappa-2}{\kappa-1} \ln \kappa + 1 \right) - 1 - \ln \frac{\lambda^2}{m^2},$$

$$B = \frac{2\kappa}{1-\kappa} \ln \kappa - 1 - \ln \frac{\lambda^2}{m^2}, \quad (18)$$

we find from (3), (4) and (11) that

$$Q_a^{(4)} = -\frac{\alpha Z e^3}{m^4 q^2 \sqrt{2\omega} \kappa_1^2} \gamma_4 (i\hat{f}_1 - m) (iA_1 \hat{f}_1 + mB_1) (i\hat{f}_1 - m) \hat{e}. \quad (19)$$

From (11) and (12) we have

$$\bar{Q}^{(2)} = \frac{2\pi Z e^3}{m^2 q^2 \sqrt{2\omega}} \left\{ \frac{1}{\kappa_1} \hat{e} (i\hat{f}_1 - m) \gamma_4 + \frac{1}{\kappa_2} \gamma_4 (i\hat{f}_2 - m) \hat{e} \right\}. \quad (20)$$

Substitution in (16) gives

$$d\sigma_a^{(4)} = \frac{Z^2 \alpha^4}{(2\pi)^3 m^6 q^4 \omega} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \text{Sp} F_a, \quad (21)$$

$$F_a = \kappa_1^{-2} \gamma_4 (i\hat{f}_1 - m) (iA_1 \hat{f}_1 + mB_1) (i\hat{f}_1 - m) \gamma_{\mu} (i\hat{p}_1 - m) \times [\kappa_1^{-1} \gamma_{\mu} (i\hat{f}_1 - m) \gamma_4 + \kappa_2^{-1} \gamma_4 (i\hat{f}_2 - m) \gamma_{\mu}] (i\hat{p}_2 - m). \quad (22)$$

Calculating  $\text{Sp} F_a$  by the usual methods, we get

$$d\sigma_a^{(4)} = \frac{Z^2 \alpha^4}{2\pi^3 m^6 q^4 \omega} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \left\{ 2m^2 \left( \frac{\epsilon_+}{\kappa_2} - \frac{\epsilon_-}{\kappa_1} \right) \left[ \omega A_1 + \frac{2\epsilon_-}{\kappa_1} (3A_1 - B_1) - \frac{8\epsilon_-}{\kappa_1^2} (A_1 - B_1) \right] - m^4 (\kappa_1 + \kappa_2) \times \frac{A_1 - B_1}{\kappa_1^2} + m^4 \frac{\kappa_2}{\kappa_1} A_1 + q^2 \left[ \frac{m^2}{\kappa_1 \kappa_2} (3A_1 - B_1) + \frac{m^2}{\kappa_1^2} (5A_1 - 3B_1) - m^2 \left( \frac{1}{\kappa_1} + \frac{1}{\kappa_2} \right) \times \left( 4 \frac{A_1 - B_1}{\kappa_1^2} + A_1 \right) - \frac{\epsilon_+^2 + \epsilon_-^2}{\kappa_1 \kappa_2} \left( 2A_1 - 4 \frac{A_1 - B_1}{\kappa_1} \right) \right] - \frac{q^4}{\kappa_1 \kappa_2} \left[ \frac{2}{\kappa_1} (A_1 - B_1) - A_1 \right] \right\}. \quad (23)$$

$d\sigma_b^{(4)}$  can be obtained from (23) if in the curly brackets we make the replacements

$$p_1 \rightarrow p_2; \quad p_2 \rightarrow p_1; \quad k \rightarrow -k \quad (24)$$

(and consequently,  $\omega \rightarrow -\omega$ ;  $\epsilon_+ \rightarrow -\epsilon_-$ ;  $\epsilon_- \rightarrow -\epsilon_+$ ;  $f_1 \rightarrow f_2$ ;  $f_2 \rightarrow f_1$ ;  $\kappa_1 \rightarrow \kappa_2$ ;  $\kappa_2 \rightarrow \kappa_1$ ;  $q \rightarrow -q$ ;  $A_1 \rightarrow A_2$ ;  $B_1 \rightarrow B_2$ ). For the special case in which we are interested (see Fig. 2),

$$\epsilon_+ = \epsilon_- = \omega / 2, \quad \theta_+ = \theta_- = m / \omega \ll 1,$$

$$\widehat{(p_+, p_-)} = 2m / \omega,$$

$$\kappa_1 = \kappa_2 = 2,5 \left(1 + \frac{23 m^2}{60 \omega^2}\right), \quad q^2 = \frac{25 m^4}{4 \omega^2} \left(1 + \frac{23 m^2}{30 \omega^2}\right). \quad (25)$$

From (18) we then find (omitting terms containing the "photon mass")

$$A_1 = A_2 = -2.09 - 0.15 m^2 / \omega^2;$$

$$B_1 = B_2 = -4.05 - 0.5 m^2 / \omega^2.$$

Substitution in (23) then gives

$$d\sigma_a^{(4)} = d\sigma_b^{(4)} = \frac{1.83 Z^2 \alpha^4 m^2}{\pi^3 q^4 \omega^3} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_-, \quad (26)$$

and from the Bethe-Heitler formula (1) we find

$$d\sigma^{(2)} = \frac{Z^2 \alpha^3 m^2}{\pi^2 q^4 \omega^3} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_-. \quad (27)$$

From this we see that the fractional radiative correction is

$$\delta_{a+b} = \frac{d\sigma_{a+b}^{(4)}}{d\sigma^{(2)}} = 3.66 \frac{\alpha}{\pi} = 0.85\%. \quad (28)$$

2. The radiative correction caused by vertex parts. When the photon line at a vertex part corresponds to a free photon ( $k^2 = 0$ ) and one of the electron lines to a free electron ( $p^2 = -m^2$ ) (see diagrams c and d of Fig. 3), the regularized value of the vertex part is of the form<sup>1)</sup>

$$\begin{aligned} \Lambda_{\mu R}^{(3)}(p, p+k; k) &= \frac{\alpha}{2\pi} \left( C\gamma_\mu + \frac{i}{m} D\hat{k}\gamma_\mu + \frac{i}{m} E p_\mu + \frac{1}{m^2} G\hat{k}p_\mu \right), \\ C &= \frac{\kappa-2}{2(\kappa-1)} \ln \kappa - 2 - \ln \frac{\lambda^2}{m^2} + \frac{1}{\kappa} [F(\kappa-1) - F(-1)], \\ D &= \frac{1}{\kappa-1} \ln \kappa; \quad E = \frac{3\kappa-2}{(\kappa-1)^2} \ln \kappa - \frac{1}{\kappa-1}, \\ G &= \frac{2}{\kappa} - \frac{1}{\kappa-1} - \frac{(\kappa-2)(2\kappa-1)}{\kappa(\kappa-1)^2} \\ &\quad \times \ln \kappa - \frac{2}{\kappa^2} [F(\kappa-1) - F(-1)], \end{aligned} \quad (29)$$

<sup>1)</sup>In [1] this formula is given incorrectly, with an extra factor  $m^2$ , see Eq. (47.56).

$$F(x) = \int_0^\infty \frac{\ln|1+u|}{u} du; \quad F(-1) = -\frac{\pi^2}{6}. \quad (30)$$

From (6), (4), (16) and (20) we find

$$\begin{aligned} d\sigma_c^{(4)} &= \frac{Z^2 \alpha^3}{(2\pi)^2 \omega m^4 q^4} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \text{Sp } F_c, \\ F_c &= \frac{1}{\kappa_1} \gamma_4 (i\hat{f}_1 - m) \Lambda_{\mu R}^{(3)}(p_1 f_1; k) (i\hat{p}_1 - m) \\ &\quad \times \left[ \frac{1}{\kappa_1} \gamma_\mu (i\hat{f}_1 - m) \gamma_4 + \frac{1}{\kappa_2} \gamma_4 (i\hat{f}_2 - m) \gamma_\mu \right] (i\hat{p}_2 - m). \end{aligned}$$

On calculating  $\text{Sp } F_c$  we get

$$\begin{aligned} d\sigma_c^{(4)} &= \frac{Z^2 \alpha^4}{2\pi^3 \omega m^4 q^4} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \left\{ m^2 \left( \frac{\epsilon_-}{\kappa_1} - \frac{\epsilon_+}{\kappa_2} \right) \right. \\ &\quad \times \left[ 2\omega \left( C_1 - D_1 + \frac{1}{2} E_1 \right) + 2G_1 (\epsilon_- - \epsilon_+) \right. \\ &\quad \left. \left. + 2\epsilon_- \left( E_1 + 4 \frac{C_1 - E_1}{\kappa_1} \right) + \kappa_1 G_1 \epsilon_+ \right] \right. \\ &\quad \left. + m^4 \left[ \frac{\kappa_2}{\kappa_1} \left( D_1 - C_1 + \frac{1}{2} G_1 \right) + D_1 + \frac{1}{2} \frac{\kappa_1}{\kappa_2} G_1 \right. \right. \\ &\quad \left. \left. - \left( \frac{\kappa_1 + \kappa_2}{2} - 1 \right) G_1 \right] + q^2 \left[ m^2 \left( \frac{1}{\kappa_1} + \frac{1}{\kappa_2} \right) \right. \right. \\ &\quad \left. \left. \times \left( 2 \frac{E_1 - C_1}{\kappa_1} + C_1 - \frac{E_1}{2} - G_1 \right) \right. \right. \\ &\quad \left. \left. + m^2 \left( G_1 + \frac{1}{2} \frac{\kappa_1}{\kappa_2} G_1 - \frac{E_1}{\kappa_1} \right) \right. \right. \\ &\quad \left. \left. - \frac{E_1}{\kappa_1 \kappa_2} (\epsilon_+ - \epsilon_-)^2 + (\epsilon_+ - \epsilon_-) \frac{\epsilon_+}{\kappa_2} G_1 - \frac{2\omega^2}{\kappa_1 \kappa_2} D_1 \right. \right. \\ &\quad \left. \left. + \frac{2C_1}{\kappa_1 \kappa_2} (\epsilon_+^2 + \epsilon_-^2) \right] + q^4 \left( \frac{E_1 - C_1}{\kappa_1 \kappa_2} - \frac{G_1}{2\kappa_2} \right) \right\}. \quad (31) \end{aligned}$$

We can get  $d\sigma_d^{(4)}$  from this by the substitutions (24). For our special case (25) we find from (30)

$$C_1 = -0.74 + 0.07 \frac{m^2}{\omega^2}; \quad D_1 = 0.61 - 0.13 \frac{m^2}{\omega^2};$$

$$E_1 = 1.57 - 0.32 \frac{m^2}{\omega^2}; \quad G_1 = -1.08 + 0.23 \frac{m^2}{\omega^2}.$$

Substitution of these values in (31) gives

$$d\sigma_c^{(4)} = d\sigma_d^{(4)} = 0.27 \frac{Z^2 \alpha^4}{\pi^3 \omega q^4} |\mathbf{p}_+| |\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_-. \quad (32)$$

From (32) and (27) we find the fractional radiative correction

$$\delta_{c+d} = \frac{d\sigma_{c+d}^{(4)}}{d\sigma^{(2)}} = 0.54 \frac{\alpha \omega^2}{\pi m^2}. \quad (33)$$

Thus for a photon energy  $\omega \cong 30m = 15 \text{ MeV}$  the fractional radiative correction becomes equal to unity, and it increases without limit with further increase of the photon energy.

To investigate the widths in energy and angle of the region of anomalously large radiative correction, we have used instead of (25) the values

$$\epsilon_\pm = \frac{\omega}{2} (1 \pm \beta); \quad \theta_\pm = \frac{m}{\omega} (1 + \rho_\pm), \quad \varphi_+ = 0, \quad \varphi_- = \pi + \psi, \quad (34)$$

taking  $\beta$ ,  $\rho_{\pm}$ , and  $\psi \ll 1$ ; here  $\theta_{\pm}$  and  $\varphi_{\pm}$  are the polar angles of the directions of the momenta  $\mathbf{p}_{\pm}$  of the positron and electron (the direction of the photon momentum  $\mathbf{k}$  is taken as polar axis). Then

$$\begin{aligned}\kappa_1 &= 2.5 \left( 1 - 0.6\beta + 0.4\rho_+ + \frac{23m^2}{60\omega^2} \right), \\ \kappa_2 &= 2.5 \left( 1 + 0.6\beta + 0.4\rho_- + \frac{23m^2}{60\omega^2} \right), \\ q^2 &= m^2 \left\{ \left( \beta + \frac{\rho_+ - \rho_-}{2} \right)^2 + \frac{1}{4}\psi^2 + \frac{25m^2}{4\omega^2} \right\}.\end{aligned}$$

Substitution of these values in (1) and (31) gives

$$\begin{aligned}d\sigma^{(2)} &= \frac{Z^2\alpha^3\omega}{\pi^2 m^2 q^4} |\mathbf{p}_+||\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \\ &\times \left\{ \frac{m^4}{\omega^4} + 0.26 \left( \beta + \frac{\rho_+ - \rho_-}{4} \right)^2 + 0.04\psi^2 \right\},\end{aligned}\quad (27a)$$

$$\begin{aligned}d\sigma_{c+d}^{(4)} &= \frac{Z^2\alpha^4\omega}{\pi^3 m^2 q^4} |\mathbf{p}_+||\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \\ &\times \left\{ 0.54 \frac{m^2}{\omega^2} - (2\beta + \rho_+ - \rho_-) \right. \\ &\times \left. [0.6\beta + 0.7(\rho_+ - \rho_-)] - 0.08\psi^2 \right\}.\end{aligned}\quad (32a)$$

From this it can be seen that if

$$|\beta|, |\rho_{\pm}| \text{ and } |\psi| \lesssim \sqrt{\alpha/\pi} m/\omega, \quad (35)$$

we have for the fractional radiative correction  $\delta_{c+d} \gtrsim 1$ . If, on the other hand,  $|\beta|$ ,  $|\rho_{\pm}|$ , or  $|\psi|$  is  $\gtrsim m/\omega$ , then the radiative correction takes its "normal" value ( $\delta_{c+d} \approx \alpha/\pi$ ).

For the case in which the photon line at the vertex part corresponds to the external field, and one of the electron lines to a free electron ( $p^2 = -m^2$ ), we have calculated the regularized value of the vertex part. The result found by the usual methods is that

$$\begin{aligned}\Lambda_{4R}^{(3)}(p_1, p_1 + q; q) \\ = \frac{\alpha}{2\pi} \left( 2 \frac{L}{m} \epsilon_+ + 2i \frac{M}{m^2} \epsilon_+ \hat{q} + N \gamma_4 + i \frac{R}{m} \gamma_4 \hat{q} \right); \quad (36)\end{aligned}$$

$$L = 1 + \frac{1}{2} \ln \frac{m^2}{\lambda^2} + m^2 (J_1 - J_x),$$

$$M = m^2 (J_1 - J_x - J_{xy} + J_{x^2y}),$$

$$N = \frac{2qp_1}{m^2} M + m^2 (2J_x - J_x^2) + q^2 (J_{x^2y^2} - J_{xy}) + J_0 - 1,$$

$$R = m (J_x - J_1),$$

$$J_{1, x, xy, x^2, x^2y, x^2y^2} = \int_0^1 \int_0^1 \frac{(1, x, xy, x^2, x^2y, x^2y^2) dx dy}{m^2 \kappa_2 y - (p_1 + qy)^2 x},$$

$$J_0 = \int_0^1 \int_0^1 x dx dy \ln \frac{m^2 x}{m^2 \kappa_2 y - (p_1 + qy)^2 x}. \quad (37)$$

From (6), (11), (16) and (20) we have

$$d\sigma_e^{(4)} = \frac{Z^2\alpha^3}{(2\pi)^2 \omega m^4 q^4} |\mathbf{p}_+||\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \text{Sp } F_e,$$

$$\begin{aligned}F_e &= \kappa_2^{-1} \gamma_{\mu} (i\hat{f}_2 - m) \Lambda_{4R}^{(3)}(p_1 f_2; q) (i\hat{p}_1 - m) [\kappa_1^{-1} \gamma_{\mu} (i\hat{f}_1 - m) \gamma_4 \\ &+ \kappa_2^{-1} \gamma_4 (i\hat{f}_2 - m) \gamma_{\mu}] (i\hat{p}_2 - m).\end{aligned}$$

Calculating  $\text{Sp } F_e$ , we get

$$\begin{aligned}d\sigma_e^{(4)} &= \frac{Z^2\alpha^4}{2\pi^3 \omega m^4 q^4} |\mathbf{p}_+||\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_- \\ &\times \left\{ 2m^2 \left( \frac{\epsilon_+}{\kappa_2} - \frac{\epsilon_-}{\kappa_1} \right) \left[ 2L\epsilon_+ \left( 1 - \frac{4}{\kappa_2} \right) \right. \right. \\ &+ (\kappa_1 + \kappa_2 + 4) M\epsilon_+ - \omega N - 4N \frac{\epsilon_+}{\kappa_2} \left. \right] \\ &- m^4 \frac{\kappa_1}{\kappa_2} (N + R) - m^4 R + q^2 \\ &\times \left[ \frac{4(\epsilon_- - \epsilon_+)}{\kappa_1 \kappa_2} L\epsilon_+ - 2\omega \left( \frac{1}{\kappa_1} - \frac{1}{\kappa_2} \right) M\epsilon_+ \right. \\ &- m^2 \left( \frac{1}{\kappa_1} + \frac{1}{\kappa_2} \right) \left( \frac{2N}{\kappa_2} - N + \frac{4R}{\kappa_2} - R \right) \\ &\left. \left. + 2N \frac{\epsilon_+^2 + \epsilon_-^2}{\kappa_1 \kappa_2} + 2m^2 \frac{R}{\kappa_2} + \frac{4\omega}{\kappa_1 \kappa_2} R\epsilon_- \right] - q^4 \frac{N + 2R}{\kappa_1 \kappa_2} \right\}.\end{aligned}\quad (38)$$

By means of the substitution (24) we can get from this the expression for  $d\sigma_f^{(4)}$ . For the case considered in (25) we get from (37)

$$L = 1.79, \quad M = 0.54, \quad N = 0.76, \quad R = -0.61,$$

and substitution in (38) gives

$$d\sigma_e^{(4)} = d\sigma_f^{(4)} = -0.23 \frac{Z^2\alpha^4 m^2}{\pi^3 q^4 \omega^3} |\mathbf{p}_+||\mathbf{p}_-| d\epsilon_+ d\Omega_+ d\Omega_-. \quad (39)$$

From (39) and (27) we find the fractional radiative correction

$$\delta_{e+f} = \frac{d\sigma_{e+f}^{(4)}}{d\sigma^{(2)}} = -0.46 \frac{\alpha}{\pi} = -0.11\%. \quad (40)$$

3. The radiative correction caused by the photon self-energy part. The regularized value of the photon self-energy part is

$$\begin{aligned}\Pi_{4R}^{(2)}(q) &= \frac{\alpha}{\pi} \left\{ \frac{1}{9} + \frac{4m^2 - 2q^2}{3q^2} (1 - \Phi \text{cth } \Phi) \right\} \delta_{v_4}, \\ \text{sh}^2 \Phi &= \frac{q^2}{4m^2}.\end{aligned}\quad (41)^*$$

From (8), (4), and (12) it follows that

$$S_{g+h}^{(4)} = \frac{\alpha}{\pi} \left\{ \frac{1}{9} + \frac{4m^2 - 2q^2}{3q^2} (1 - \Phi \text{cth } \Phi) \right\} S_{a+b}^{(2)}$$

From this and Eqs. (11), (16) and (16a) we find the fractional radiative correction

$$\delta_{g+h} = \frac{d\sigma_{g+h}^{(4)}}{d\sigma^{(2)}} = \frac{2\alpha}{\pi} \left\{ \frac{1}{9} + \frac{4m^2 - 2q^2}{3q^2} (1 - \Phi \text{cth } \Phi) \right\}. \quad (42)$$

For our case (25) we find from Eq. (41)

\*cth = coth, sh = sinh.

$$\Phi = \frac{5}{4} \frac{m}{\omega} \left( 1 + \frac{103}{160} \frac{m^2}{\omega^2} \right)$$

and substitution in (42) gives

$$\delta_{g+th} = \frac{\alpha}{\pi} \frac{65}{108} \frac{m^2}{\omega^2}. \quad (43)$$

4. The radiative correction corresponding to the matrix element  $S_{j+k}^{(4)}$ . From (4), (10), (11), (16) and

(20) we find

$$\begin{aligned} d\sigma_j^{(4)} &= \frac{2iZ^2\alpha^4}{(2\pi)^5\omega m^2 q^4} |\mathbf{p}_+||\mathbf{p}_-| d\varepsilon_+ d\Omega_+ d\Omega_- \\ &\times \int \frac{\text{Sp } F_{\mathbb{N}} d^4r}{r^2 [(p_1 - r)^2 + m^2] [(p_2 - r)^2 + m^2] [(f_1 - r)^2 + m^2]}, \\ F_j &= \gamma_\nu [i(\hat{p}_2 - \hat{r}) - m] \gamma_4 [i(\hat{f}_1 - \hat{r}) - m] \gamma_\mu \\ &\times [i(\hat{p}_1 - \hat{r}) - m] \gamma_\nu (i\hat{p}_1 - m) \\ &\times \left[ \frac{1}{\kappa_1} \gamma_\mu (if_1 - m) \gamma_4 + \frac{1}{\kappa_2} \gamma_4 (if_2 - m) \gamma_\mu \right] (i\hat{p}_2 - m). \end{aligned} \quad (44)$$

On calculating  $\text{Sp } F_j$  we can put (44) in the form

$$\begin{aligned} d\sigma_j^{(4)} &= \frac{2Z^2\alpha^4}{(2\pi)^5\omega m^2 q^4} |\mathbf{p}_+||\mathbf{p}_-| d\varepsilon_+ d\Omega_+ d\Omega_- \{a^{(0)}I^{(0)} + a_\sigma^{(0)}I_\sigma^{(0)} \\ &+ a_{\sigma_4}^{(0)}I_{\sigma_4}^{(0)} + a^{(1)}I^{(1)} + a_\sigma^{(1)}I_\sigma^{(1)} + a_{\sigma_4}^{(1)}I_{\sigma_4}^{(1)} + a^{(2)}I^{(2)} \\ &+ a_\sigma^{(2)}I_\sigma^{(2)} + a_{\sigma_4}^{(2)}I_{\sigma_4}^{(2)} + a^{(3)}I^{(3)} \\ &+ a_\sigma^{(3)}I_\sigma^{(3)} + a_{\sigma_4}^{(3)}I_{\sigma_4}^{(3)} + aI + a_4I_4 + a_{44}I_{44}\}, \end{aligned} \quad (45)$$

$$\begin{aligned} I_{\sigma, \sigma_4} &= \int \frac{(1, r_\sigma, r_\sigma r_4) d^4r}{r^2 [(p_1 - r)^2 + m^2] [(p_2 - r)^2 + m^2] [(f_1 - r)^2 + m^2]}. \end{aligned} \quad (46)$$

The upper index on I indicates that in the denominator of the integrand one of the factors is omitted, according to the scheme

$$\begin{aligned} (0) &\rightarrow r^2, \\ (1) &\rightarrow (p_1 - r)^2 + m^2 = r^2 - 2rp_1, \\ (2) &\rightarrow (p_2 - r)^2 + m^2 = r^2 - 2rp_2, \\ (3) &\rightarrow (f_1 - r)^2 + m^2 = r^2 - 2rf_1 + m^2\kappa_1. \end{aligned}$$

In the general case the calculation of the integrals (46) and their coefficients (a) is a very cumbersome operation. Therefore we have confined ourselves to the special case (25) in which we are interested. The matter of primary interest is whether or not this matrix element gives a fractional radiative correction proportional to  $(\omega/m)^4$ . For this it is enough to use only the main terms in the curly brackets in (45), which are proportional to  $\omega^2$ . In this approximation the integrals (46) have been calculated by well known methods.<sup>[1]</sup> The following are the values obtained for their coefficients:

$$\begin{aligned} a^{(0)} &= 0, \quad a_\sigma^{(0)} = -6.4i\omega^2 k_\sigma + 16m^2\omega\delta_{4\sigma}, \\ a_{\sigma_4}^{(0)} &= 6.4\omega(p_\sigma^{(1)} - p_\sigma^{(2)} + 2k_\sigma) + 32im^2\delta_{\sigma_4}; \\ a^{(1)} &= 0, \quad a_\sigma^{(1)} = 3.2i\omega^2(p_\sigma^{(1)} + p_\sigma^{(2)} + k_\sigma) - 8m^2\omega\delta_{4\sigma}, \\ a_{\sigma_4}^{(1)} &= -6.4\omega(p_\sigma^{(1)} + k_\sigma) - 16im^2\delta_{\sigma_4}; \quad a^{(2)} = 3.2im^2\omega^2, \\ a_\sigma^{(2)} &= -6.4i\omega^2(p_\sigma^{(2)} - k_\sigma) - 22.4m^2\omega\delta_{4\sigma}, \\ a_{\sigma_4}^{(2)} &= 6.4\omega(p_\sigma^{(2)} - k_\sigma) - 16im^2\delta_{\sigma_4}; \\ a^{(3)} &= -3.2im^2\omega^2, \quad a_\sigma^{(3)} = 6.4m^2\omega\delta_{4\sigma}, \\ a_{\sigma_4}^{(3)} &= 0, \quad a = 8im^4\omega^2; \\ a_4 &= -32m^4\omega, \quad a_{44} = -32im^4. \end{aligned} \quad (47)$$

Substitution of these values in (45) gave the result that all of the terms in the curly brackets exactly cancelled. Thus the matrix element  $S_{j+k}^{(4)}$  does not give a fractional correction proportional to  $(\alpha/\pi)(\omega/m)^4$ . It can give only a correction proportional to  $(\alpha/\pi)(\omega/m)^2$  (of the type of  $\delta_C$ ), or else a correction independent of the photon energy (proportional to  $\alpha/\pi$ ). A more precise treatment of this question would involve cumbersome calculations and is of no importance in principle, because the corrections  $\delta_C$  and  $\delta_i$  are independent quantities which cannot cancel each other completely.

## CONCLUSION

In deriving (15) from (13) we have neglected terms proportional to  $\alpha^2$ . It cannot be excluded that the terms we have not taken into account may give a fractional radiative correction proportional to  $(\alpha/\pi)^2(\omega/m)^4$ , which at sufficiently high energy exceeds the correction (33). Independently of this, however, and independently of the results of any improved accuracy in the correction  $\delta_{j+k}$ , we can make the following assertion with complete confidence.

In the ultrarelativistic region (for photons with energies 10 MeV and more) the differential cross section for photoproduction of an electron-positron pair emerging symmetrically relative to the momentum of the photon (Fig. 2) and making very small angles with it ( $\theta_+ = \theta_- \cong m/\omega$ ) differs considerably from the Bethe-Heitler cross section of Eq. (1).

The widths in energy and angle of this region of anomalously large radiative cross sections are given by the inequalities (35). In particular, such effects occur when the energy of the electron (or positron) is in the range between the values  $\frac{1}{2}[\omega \pm m(\alpha/\pi)^{1/2}]$ . This result can be subjected to experimental test already at comparatively moderate energies.

It must be noted that Guzenko and Fomin<sup>[2,3]</sup>

have calculated the radiative corrections to the photoproduction of an electron-positron pair by a different method. Application of their general formulas to our special case (25) does not give our result (33); one finds a normal fractional correction of the order  $\alpha/\pi$ . Owing to this there have been careful and repeated checks of our calculations, including one by P. I. Fomin, and these have confirmed the correctness of (33). Therefore it can be supposed that our calculations are not in error.

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<sup>1</sup>A. I. Akhiezer and V. B. Berestetskiĭ, *Kvantovaya elektrodinamika* (Quantum Electrodynamics), 2d Ed. Fizmatgiz, 1959.

<sup>2</sup>S. Ya. Guzenko and P. I. Fomin, *JETP* **38**, 513 (1960), *Soviet Phys. JETP* **11**, 372 (1960).

<sup>3</sup>P. I. Fomin, *JETP* **35**, 707 (1958), *Soviet Phys. JETP* **8**, 491 (1959).

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